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On existence of certain delta fractional difference models

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ABSTRACT

The discretization of initial and boundary value problems and their existence behaviors are of great significance in various fields. This paper explores the existence of a class of self-adjoint delta fractional difference equations. The study begins by demonstrating the uniqueness of an initial value problem of delta Riemann–Liouville fractional operator type. Based on this result, the uniqueness of the self-adjoint equation will be examined and determined. Next, we define the Cauchy function based on the delta Riemann–Liouville fractional differences. Accordingly, the solution of the self-adjoint equation will be investigated according to the delta Cauchy function. Furthermore, the research investigates the uniqueness of the self-adjoint equation including the component of Green's functions of and examines how this equation has only a trivial solution. To validate the theoretical analysis, specific examples are conducted to support and verify our results

1. Introduction

One of the common areas of applied and pure mathematics is discrete fractional calculus with many applications, which deals with non-integer sums and differences. It has been viewed for a very long time as a purely theoretically interesting subject but later, several applications in engineering and physics modeled by discrete fractional calculus. Discrete fractional calculus has become a discretization field of fractional calculus that supported by computational and sum representations; see e.g. Goodrich and Peterson (2015), Wu and Baleanu (2015) and Mozyrska et al. (2019).

Discrete fractional analyses are always positioned to work on common models of applied and pure mathematics due to their unique potentials to identify memory effects. These are related to real-world applications and included theories regarding signal processing, dynamical system, chaos, financial perspectives, impulsive perturbations, and further different aspects; see e.g. Ostalczyk (2015) and Atici et al. (2017).

Recent literature explores diverse computational methodologies for fractional boundary and initial value models across different physical domains. Also, with the development of discrete fractional operators and the discrete fractional analysis, the extension of discrete initial value problems (see e.g. Goodrich (2012), Wang et al. (2020b), Ahrendt et al. (2012)) and boundary value problems of fractional difference models have brought great convenience to researchers (see e.g. Wang et al. (2020a), Almusawa and Mohammed (2023), Baleanu et al. (2023), Chen et al. (2019)). For a discrete system, the initial value problem (IVP) and boundary value problem (BVP) of a fractional difference equation (FDE) model can be regarded as the problem of finding the stability analysis of the discrete system when the initial time conditions and the function in the right sides are known. Evidently, the stability analysis, existence and uniqueness of the solution to the IVP of fractional difference types are important when analyzing fractional difference equations; see e.g. Mohammed and Abdeljawad (2020), Brackins (2014) and Gholami and Ghanbari (2016). Furthermore, different fractional and discrete fractional self-adjoint models,

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and Green's function for boundary value problems involving fractional difference models are analyzed by the scholars which are available in the literature as Brackins (2014), Kilbas et al. (2006), Cabada et al. (2021) and Ahrendt and Kissler (2019).

In this paper, we first apply existence and uniqueness theorem of the fractional IVP (in Lemma 2.1) to show the uniqueness and regularity of the other results. Using the variation of constants formula, we will continue to introduce a Cauchy function an solve the self-adjoint problem. In addition, our focus is in fact mainly on the Green's function in the sense of discrete fractional operators and novel Cauchy functions.

The rest of the paper is structured as follows: The literature of delta fractional operators has been reviewed in Section 2, and then an essential lemma has been stated and proved. Section 3 is reserved for a presentation of main results regarding fractional self-adjoint problems. In Section 4, the Cauchy function considering the falling function is defined. Then, in the same section, we analyze the self-adjoint problem to get the uniqueness and triviality of the function. Finally, in Section 5, we summarize the content of the paper.

2. Preliminaries

Let $\alpha > 0$, $\mathbb{N}_{p_0} := p_0 + \mathbb{N}$ and ${}_m\mathbb{N} := m - \mathbb{N}$, for $p_0, m \in \mathbb{R}$, where $\mathbb{N}$ represents the natural numbers. Further, let $\mathbb{T} := \{p_0, p_0+1, \dots, m\}$ such that $m = p_0 + k$, for some $k \in \mathbb{N}_0$. Then, it is defined in Goodrich and Peterson (2015, Definition 2.25) the Δ−fractional sum operator as follows:

$$\left({}^{\mathrm{RL}}_{p_0}\Delta^{-\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \sum_{r=p_0}^{x-\alpha} (x-r-1)^{\underline{\alpha-1}} f(r), \quad \text{for } x \text{ in } \mathbb{N}_{p_0+\alpha}, \tag{2.1}$$

and it is defined in Guirao et al. (2022, Theorem 2.2) the Δ−fractional difference operator as follows:

$$\left({}^{\mathrm{RL}}_{p_0}\Delta^{\alpha} f\right)(x) = \frac{1}{\Gamma(-\alpha)} \sum_{r=p_0}^{x+\alpha} (x-r-1)^{\underline{-\alpha-1}} f(r), \quad \text{for } x \text{ in } \mathbb{N}_{p_0+\ell-\alpha}, \tag{2.2}$$

for $\alpha \in (\ell-1, \ell)$ and f is defined on $\mathbb{N}_{p_0}$. Above, we have

$$x^{\underline{\alpha}} = \frac{\Gamma(x+1)}{\Gamma(x+1-\alpha)}, \ \Delta x^{\underline{\alpha}} = \alpha\, x^{\underline{\alpha-1}}. \tag{2.3}$$

A major property of the composition of delta fractional sum and difference is proved in Abdeljawad (2018), which is given by

$$\left({}^{\mathrm{RL}}_{p_0+\ell-\alpha}\Delta^{-\alpha}\, {}^{\mathrm{RL}}_{p_0}\Delta^{\alpha} f\right)(x) = f(x), \tag{2.4}$$

for $x \in \mathbb{N}_{p_0+\ell}$.

Definition 2.1 (see Brackins (2014)). For the homogeneous FDE

$$\left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\alpha}(z\nabla y)\right)(x) = 0,$$

we say $\varphi(x, \bar{\sigma}(r))$, where $\varphi : \mathbb{N}_{p_0+1} \times \mathbb{N}_{p_0+1} \to \mathbb{R}$, as a Cauchy function such that $\varphi(\cdot, \bar{\sigma}(r))$ is the unique solution of

$$\begin{cases} \left({}^{\mathrm{RL}}_{r}\Delta^{\alpha}(z\nabla\varphi)\right)(x) = 0, & x \in \mathbb{N}_{r+1}, \\ \varphi(r-1) = 0, \ (\nabla\varphi)(r) = \frac{1}{z(r)}, \end{cases} \tag{2.5}$$

for each fixed $r \in \mathbb{N}_{p_0+1}$. Also, it is expressed by

$$\varphi(x, \bar{\sigma}(r)) = \sum_{s=r}^{x} \frac{(s+\alpha-\sigma(r))^{\underline{\alpha-1}}}{\Gamma(\alpha)\, z(s)}, \quad x \in \mathbb{N}_{p_0+1}, \tag{2.6}$$

where $\sigma(\tau) = \tau + 1$ and $\bar{\sigma}(\tau) = \tau - 1$.

Here, we will state and prove our main Lemma that will be an essential tool for the next results.

Lemma 2.1. *Let $0 < \theta < 1$, $A_0 \in \mathbb{R}$, and h, g be defined on $\mathbb{N}_{p_0}$. Then the fractional IVP*

$$\begin{cases} \left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\theta} y\right)(x) = g(x+\theta), & x \in \mathbb{N}_{p_0+1}, \\ y(p_0+1) = A_0, \end{cases} \tag{2.7}$$

has the unique solution

$$y(x) = \left({}^{\mathrm{RL}}_{p_0+1}\Delta^{-\theta} g\right)(x+\theta) + \Big[A_0 - g(p_0+1)\Big] \frac{(x-p_0-2+\theta)^{\underline{\theta-1}}}{\Gamma(\theta)},$$

$$s.t.\ x \in \mathbb{N}_{p_0+2}.$$

Proof. By taking ${}^{\mathrm{RL}}_{p_0+2-\theta}\Delta^{-\theta}$ on both sides of (2.7), we have

$$\left({}^{\mathrm{RL}}_{p_0+2-\theta}\Delta^{-\theta}\left({}^{\mathrm{RL}}_{a+1}\Delta^{\theta} y\right)\right)(x) = \left({}^{\mathrm{RL}}_{p_0+2-\theta}\Delta^{-\theta} g(x+\theta)\right)(x). \tag{2.8}$$

Computing the left side of (2.8), we see that

$$\begin{aligned}
&\left({}^{\mathrm{RL}}_{p_0+2-\theta}\Delta^{-\theta}\left({}^{\mathrm{RL}}_{a+1}\Delta^{\theta} y\right)\right)(x) := \left({}^{\mathrm{RL}}_{p_0+2-\theta}\Delta^{-\theta} h\right)(x) \\
&= \frac{1}{\Gamma(\theta)} \sum_{r=p_0+2-\theta}^{x-\theta} (x-\sigma(r))^{\underline{\theta-1}} h(r) \\
&= \frac{1}{\Gamma(\theta)} \sum_{r=p_0+1-\theta}^{x-\theta} (x-\sigma(r))^{\underline{\theta-1}} h(r) \\
&\quad - \frac{1}{\Gamma(\theta)} \left(x-(p_0+1-\theta)-1\right)^{\underline{\theta-1}} h(p_0+1-\theta) \\
&= \left({}^{\mathrm{RL}}_{p_0+1-\theta}\Delta^{-\theta}\ {}^{\mathrm{RL}}_{p_0+1}\Delta^{\theta} y\right)(x) - \frac{1}{\Gamma(\theta)} \left(x-p_0-2+\theta\right)^{\underline{\theta-1}} h(p_0+1-\theta) \\
&= y(x) - \frac{1}{\Gamma(\theta)} (x-p_0-2+\theta)^{\underline{\theta-1}} A_0,
\end{aligned} \tag{2.9}$$

where it is used that

$$\begin{aligned}
h(p_0+1-\theta) &= \frac{1}{\Gamma(-\theta)} \sum_{r=p_0+1}^{p_0+1} (p_0-\theta-r)^{\underline{\theta-1}} y(r) \\
&= \frac{1}{\Gamma(-\theta)} (-\theta-1)^{\underline{-\theta-1}} y(p_0+1) = y(p_0+1) = A_0.
\end{aligned}$$

Next, by computing the right side of (2.8), we have

$$\begin{aligned}
\left({}^{\mathrm{RL}}_{p_0+2-\theta}\Delta^{-\theta} g(x+\theta)\right)(x) &= \frac{1}{\Gamma(\theta)} \sum_{r=p_0+2-\theta}^{x-\theta} (x-\sigma(r))^{\underline{\theta-1}} g(r+\theta) \\
&= \frac{1}{\Gamma(\theta)} \sum_{r=p_0+1-\theta}^{x-\theta} (x-\sigma(r))^{\underline{\theta-1}} g(r+\theta) \\
&\quad - \frac{1}{\Gamma(\theta)} \left(x-(p_0+1-\theta)-1\right)^{\underline{\theta-1}} g(p_0+1) \\
&= \left({}^{\mathrm{RL}}_{p_0+1}\Delta^{-\theta} g\right)(x+\theta) - \frac{1}{\Gamma(\theta)} (x-p_0-2+\theta)^{\underline{\theta-1}} g(p_0+1).
\end{aligned} \tag{2.10}$$

Having the left and right sides of (2.8) as in (2.9) and (2.10), respectively, can give the desired result. □

Example 2.1. Consider the fractional IVP

$$\left({}^{\mathrm{RL}}_{1}\Delta^{\mu} y\right)(x) = 2,$$
$$y(1) = \pi.$$

By considering Lemma 2.1, we have

$$\begin{aligned}
y(x) &= \left({}^{\mathrm{RL}}_{p_0+1}\Delta^{-\mu} g\right)(x+\mu) + (\pi-2) \frac{(x-p_0+\mu-2)^{(\mu-1)}}{\Gamma(\mu)} \\
&= \frac{2}{\Gamma(\mu+1)} (x+\mu-p_0-1)^{(\mu)} + (\pi-2) \frac{(x-p_0+\mu-2)^{(\mu-1)}}{\Gamma(\mu)} \\
&= \frac{2}{\Gamma(\mu+1)} \frac{\Gamma(x+\mu-p_0)}{\Gamma(x-p_0)} + \frac{\pi-2}{\Gamma(\mu)} \frac{\Gamma(x-p_0+\mu-1)}{\Gamma(x-p_0)},
\end{aligned}$$

for $x \in \mathbb{N}_2$. In addition, we have represented the values of $y(x)$ for $x \in \{2, 3, \dots, 10\}$ and different values of μ in Fig. 1.

3. Discrete self-adjoint problems

In this section, we examine the existence and uniqueness of the delta fractional self-adjoint IVP.

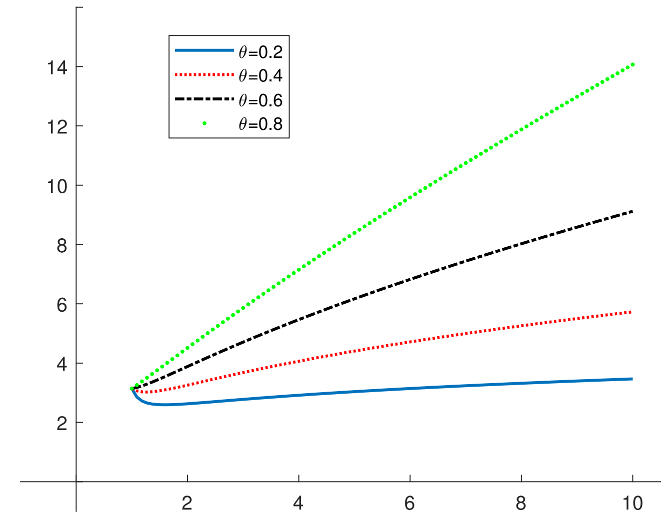

Fig. 1. Graph of the function outcome for some values of θ.

Theorem 3.1. *Let $0<\mu<1$, $z:\mathbb{N}_{p_0+1}\to(0,\infty)$, and $g_1,g_2:\mathbb{N}_{p_0+2}\to\mathbb{R}$. Then, the fractional IVP*

$$\begin{cases}\left({}^{\mathrm{RL}}_{a+1}\Delta^{\mu}\left(z\nabla y\right)\right)(x)+g_1(x+\mu)\,y(x+\mu-1)=g_2(x+\mu), & x\in\mathbb{N}_{p_0+2},\\ y(p_0)=A_0,\quad y(p_0+1)=B_0,\end{cases}\tag{3.1}$$

has a unique solution for $y:\mathbb{N}_{p_0}\to\mathbb{R}$.

Proof. Rewriting (3.1) by using (2.2), we have

$$\frac{1}{\Gamma(-\mu)}\sum_{r=p_0+1}^{x+\mu}(x-r-1)^{\underline{-\mu-1}}\left[z\,\nabla y\right](r)+g_1(x+\mu)y(x+\mu-1)=g_2(x+\mu).\tag{3.2}$$

With $x=p_0+2-\mu$, the above equality becomes

$$\begin{aligned}g_2(p_0+2)&=\frac{1}{\Gamma(-\mu)}\sum_{r=p_0+1}^{p_0+2}(p_0+1-\mu-r)^{\underline{-\mu-1}}\left[z\nabla y\right](r)\\&\quad+g_1\left(p_0+2\right)y\left(p_0+1\right)\\&=\frac{1}{\Gamma(-\mu)}\left[(-\mu)^{\underline{-\mu-1}}\left[z\nabla y\right](p_0+1)+(-\mu-1)^{\underline{-\mu-1}}\left[z\Delta y\right](p_0+2)\right]\\&\quad+g_1(p_0+2)y(p_0+1)\\&=z(p_0+2)y(p_0+2)+A_0\mu z(p_0+1)+B_0\left[g_1(p_0+2)-\mu z(p_0+1)-z(p_0+2)\right].\end{aligned}$$

We solve this equation for $y(p_0+2)$ to get

$$y(p_0+2)=\frac{1}{z(p_0+2)}\left[g_2(p_0+2)-A_0\mu z(p_0+1)-B_0\left(g_1(p_0+2)-\mu z(p_0+1)-z(p_0+2)\right)\right].$$

This implies that $y(p_0+2)$ can be determined uniquely by considering $y(p_0)=A_0$ and $y(p_0+1)=B_0$ and the given values of z, g_1, and g_2.

We are continuing by using induction and we have to demonstrate that $y(x)$ is uniquely determined on $\mathbb{N}_{p_0}$. For this, let $y(x)$ be the unique solution to (3.1), for $x\in\mathbb{N}_{p_0}^{x_0}$ and $x_0\in\mathbb{N}_{p_0+2}$. Then, we have to prove that $y(x_0+1)$ is also the unique solution of (3.1).

To do this, we use $x=x_0+1-\mu$ in (3.2), we get

$$\begin{aligned}g_2(x_0+1)&=\sum_{r=p_0+1}^{x_0+1}\frac{(x_0-\mu-r)^{\underline{-\mu-1}}}{\Gamma(-\mu)}\left[z\,\nabla y\right](r)+g_1\left(x_0+1\right)y\left(x_0\right)\\&=\sum_{r=p_0+1}^{x_0}\frac{(x_0-\mu-r)^{\underline{-\mu-1}}}{\Gamma(-\mu)}\left[z\,\nabla y\right](r)+g_1\left(x_0+1\right)y\left(x_0\right)\\&\quad+z(x_0+1)y(x_0+1)-z(x_0+1)y(x_0).\end{aligned}$$

This can be solved for $y(x_0+1)$,

$$\begin{aligned}y(x_0+1)&=\frac{1}{z(x_0+1)}\left[g_2(x_0+1)-\sum_{r=p_0+1}^{x_0}\frac{(x_0-\mu-r)^{\underline{-\mu-1}}}{\Gamma(-\mu)}\left[z\,\nabla y\right](r)\right.\\&\qquad\left.-g_1(x_0+1)y(x_0)+z(x_0+1)y(x_0)\right].\end{aligned}$$

Thus, by considering the hypothesis, each $y(x)$, for x in $\mathbb{N}_{p_0}^{x_0}$, are known. Therefore, we can say that $y(x_0+1)$ is also the unique solution of (3.1). Consequently, $y(x)$ is the unique solution of (3.1) on $\mathbb{N}_{p_0}^{x_0+1}$. This give us our proof. □

Next, we consider the fractional self-adjoint IVP:

$$\begin{cases}\left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\mu}\left(z\nabla y\right)\right)(x)=f(x+\mu), & x\in\mathbb{N}_{p_0+2},\\ y(p_0)=y(p_0+1)=0,\end{cases}\tag{3.3}$$

where $0<\mu<1$, $z:\mathbb{N}_{p_0+1}\to(0,\infty)$, and $f:\mathbb{N}_{p_0+2}\to\mathbb{R}$. Then, the following can be deduced as a variation of constants formula.

Theorem 3.2. *The solution to (3.3) can be expressed by*

$$y(x)=\sum_{r=p_0+2}^{x}\varphi(x,\bar{\sigma}(r))f(r),$$

where $\varphi(x,\bar{\sigma}(r))$ is as defined in Definition 2.1.

Proof. Assume that $h(x):=\left(z\nabla y\right)(x)$ and $y(x)$ is a solution of the fractional IVP (3.3). Therefore, $h(x)$ is a solution of

$$\left({}^{\mathrm{RL}}_{p_0}\Delta^{\mu}h\right)(x)=f(x+\mu),\quad h(p_0+1)=z(p_0+1)\nabla y(p_0+1)=0.$$

Moreover, by Lemma 2.1, its solution can be represented as

$$\begin{aligned}h(x)&=\left({}^{\mathrm{RL}}_{p_0+1}\Delta^{-\upsilon}f\right)(x+\mu)-\frac{(x-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)}f(p_0+1)\\&=\sum_{r=p_0+1}^{x}\left[\frac{\left[x-\sigma(r)+\mu\right]^{\underline{\mu-1}}}{\Gamma(\mu)}f(r)\right]-\frac{(x-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma\mu}f(p_0+1).\end{aligned}$$

It can be divided both sides by $z(x)$ to get

$$(\nabla y)(x)=\sum_{r=p_0+1}^{x}\frac{(x+\mu-\sigma(r))^{\underline{\mu-1}}}{\Gamma(\mu)\,z(x)}f(r).$$

By summing both sides $\sum_{s=p_0+2}^{x}$, we have

$$y(x)-y(p_0+1)=\sum_{s=p_0+2}^{x}\left[\sum_{r=p_0+2}^{s}\frac{(s+\mu-\sigma(r))^{\underline{\mu-1}}}{\Gamma(\mu)\,z(s)}f(r)\right],\tag{3.4}$$

where we have used that

$$\sum_{s=p_0+2}^{x}(\nabla y)(x)=y(x)-y(p_0+1).$$

Now, we should interchange the order of the sums in (3.4) and use $y(p_0+1)=0$ to obtain

$$\begin{aligned}y(x)&=\sum_{r=p_0+2}^{x}\left[\sum_{s=r}^{x}\frac{(s+\mu-\sigma(r))^{\underline{\mu-1}}}{\Gamma(\mu)\,z(s)}f(r)\right]\\&=\sum_{r=p_0+2}^{x}\left[\sum_{s=r}^{x}\frac{(s+\mu-\sigma(r))^{\underline{\mu-1}}}{\Gamma(\mu)\,z(s)}\right]f(r)\\&=\sum_{r=p_0+2}^{x}\varphi(x,\bar{\sigma}(r))\,f(r).\end{aligned}$$

This ends our proof. □

Corollary 3.1. *Let $z:\mathbb{N}_{p_0+1}\to(0,\infty)$ and $w_1(x),w_2(x)$ can satisfy*

$$\left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\mu}\left(z\nabla w_1\right)\right)(x) \geq \left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\mu}\left(z\nabla w_2\right)\right)(x), \quad x \in \mathbb{N}_{p_0+2},$$
$$w_1(p_0) = w_2(p_0), \quad w_1(p_0+1) = w_2(p_0+1).$$

Then, $w_1(x) \geq w_2(x)$, for $x \in \mathbb{N}_{p_0}$.

Proof. Let us set $w(x) = w_1(x) - w_2(x)$, and

$$g(x) := \left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\mu}\left(z\nabla w\right)\right)(x) = \left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\mu}\left(z\nabla w_1\right)\right)(x) - \left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\mu}\left(z\nabla w_2\right)\right)(x) \geq 0, \quad x \in \mathbb{N}_{p_0+2}.$$

Therefore, w solves the IVP

$$\left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\mu}\left(z\nabla w\right)\right)(x) = g(x), \quad x \in \mathbb{N}_{p_0+2},$$
$$w(p_0+1) = 0, \quad (\nabla w)(p_0+1) = 0.$$

Thus, by using Theorem 3.2, we see that

$$w(x) = \sum_{r=p_0+2}^{x} g(r)\varphi(x,\bar{\sigma}(r)) = \sum_{r=p_0+2}^{x} g(r) \sum_{s=r}^{x} \frac{(s-\sigma(r))^{\underline{\mu-1}}}{\Gamma(\mu)z(s)} \geq 0,$$

and this implies that $w(x) = w_1(x) - w_2(x) \geq 0$. Thus, the proof has been done. □

4. BVPs with Green's function

This section is dedicated to examine the Green's function for homogeneous and nonhomogeneous fractional BVPs with homogeneous BCs.

Theorem 4.1. *Let p_0, m be two real numbers such that $m - p_0 \in \mathbb{N}_1$, $h, z : \mathbb{N}_{p_0+1}^{m} \to \mathbb{R}$, and $z(x) > 0$. Then, the fractional BVP*

$$\begin{cases} -\left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\mu}\left(z\nabla y\right)\right)(x) = h(x+\mu), & x \in \mathbb{N}_{p_0+1}^{m}, \\ y(p_0) = 0, \quad y(m) = 0, \end{cases} \tag{4.1}$$

has the unique solution

$$y(x) = \sum_{x=p_0+1}^{m} G(x,r)\,h(r), \tag{4.2}$$

where

$$G(x,r) = \begin{cases} \frac{\varphi(m,\bar{\sigma}(r))}{\varphi(m,p_0)}\varphi(x,p_0), & x \leq r-1, \\ \frac{\varphi(m,\bar{\sigma}(r))}{\varphi(m,p_0)}\varphi(x,p_0) - \varphi(x,\bar{\sigma}(r)), & x \geq r, \end{cases} \tag{4.3}$$

and $\varphi(\cdot,\cdot)$ is as defined in Definition 2.1.

Proof. Let $\varphi(x) = (z\nabla y)(x)$, and let $A_0 := \varphi(p_0+1) = z(p_0+1)(\nabla y)(p_0+1)$. Then, by using Lemma 2.1, the solution $\varphi(x)$ of the fractional IVP

$$\begin{cases} -\left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\mu}\varphi\right)(x) = h(x+\mu), \\ \varphi(p_0+1) = A_0, \end{cases}$$

is given by

$$\varphi(x) = -\left({}^{\mathrm{RL}}_{p_0+1}\nabla^{-\mu}h\right)(x+\mu) - \left[A_0 - h(p_0+1)\right]\frac{(x-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)}.$$

By setting $k_0 := A_0 - h(p_0+1)$ and using $(\nabla y)(x) = \varphi(x)/z(x)$, we have

$$\begin{aligned}(\nabla y)(x) &= \frac{-1}{z(x)}\left[\left({}^{\mathrm{RL}}_{p_0+1}\Delta^{-\mu}h\right)(x+\mu) - k_0\frac{(x-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)}\right] \\ &= \frac{-1}{z(x)}\left[\frac{1}{\Gamma(\mu)}\sum_{r=p_0+1}^{x}(x+\mu-\sigma(r))^{\underline{\mu-1}}h(r) + k_0\frac{(x-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)}\right].\end{aligned}$$

By summing both sides $\sum_{s=p_0+1}^{x}$ we have

$$y(x) = -\sum_{s=p_0+1}^{x}\left[\sum_{r=p_0+1}^{s}\frac{(s+\mu-\sigma(r))^{\underline{\mu-1}}}{\Gamma(\mu)}h(r) + k_0\frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)}\right].$$

We change the order of sums to have

$$\begin{aligned} y(x) &= -\sum_{r=p_0+1}^{x} h(x)\sum_{s=r}^{x}\frac{(s+\mu-\sigma(r))^{\underline{\mu-1}}}{\Gamma(\mu)z(s)} - k_0\sum_{s=p_0+1}^{x}\frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)} \\ &= -\sum_{r=p_0+1}^{x} h(r)\varphi(x,\bar{\sigma}(r)) - k_0\varphi(x,p_0). \end{aligned} \tag{4.4}$$

If we let $x = m$ and solve it for k_0, then we have

$$k_0 = \frac{-\sum_{r=p_0+1}^{m} h(r)\varphi(m,\bar{\sigma}(r))}{\varphi(m,p_0)}.$$

Substituting k_0 into (4.4) to obtain

$$\begin{aligned} y(x) &= -\sum_{r=p_0+1}^{x} h(r)\varphi(x,\bar{\sigma}(r)) + \frac{\varphi(x,p_0)}{\varphi(m,p_0)}\sum_{r=p_0+1}^{m} h(r)\varphi(m,\bar{\sigma}(r)) \\ &= -\sum_{r=p_0+1}^{x} h(r)\varphi(x,\bar{\sigma}(r)) + \frac{\varphi(x,p_0)}{\varphi(m,p_0)}\sum_{r=p_0+1}^{x} h(r)\varphi(m,\bar{\sigma}(r)) \\ &\quad + \frac{\varphi(x,p_0)}{\varphi(m,p_0)}\sum_{r=x+1}^{m} h(r)\varphi(m,\bar{\sigma}(r)) \\ &= \sum_{r=p_0+1}^{x} h(r)\left[\frac{\varphi(m,\bar{\sigma}(r))}{\varphi(m,p_0)}\varphi(x,p_0) - \varphi(x,\bar{\sigma}(r))\right] \\ &\quad + \sum_{r=x+1}^{m} h(r)\left[\frac{\varphi(m,\bar{\sigma}(r))}{\varphi(m,p_0)}\varphi(x,p_0)\right] \\ &= \sum_{r=p_0+1}^{m} h(r)G(x,r), \end{aligned}$$

where

$$G(x,r) = \begin{cases} \frac{\varphi(m,\bar{\sigma}(r))}{\varphi(m,p_0)}\varphi(x,p_0), & x \leq r-1, \\ \frac{\varphi(m,\bar{\sigma}(r))}{\varphi(m,p_0)}\varphi(x,p_0) - \varphi(x,\bar{\sigma}(r)), & x \geq r. \end{cases}$$

Consequently, any solution y of the BVP (4.1) is necessarily given by (4.2) as we just established. Furthermore, the uniqueness of y can follow from Theorem 3.1. □

Next, we generalize the above Green's function theorem to the following fractional self-adjoint BVP:

$$\begin{cases} -\left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\mu}\left(z\nabla y\right)\right)(x) = 0, & x \in \mathbb{N}_{p_0+2}^{m}, \\ \xi_0 y(p_0+1) - \xi_1\nabla y(p_0+1) = 0, \\ \xi_2 y(m) + \xi_3\nabla y(m) = 0. \end{cases} \tag{4.5}$$

where $z : \mathbb{N}_{p_0+1}^{m} \to (0,\infty)$, $\xi_0^2 + \xi_1^2 > 0$, and $\xi_2^2 + \xi_3^2 > 0$.

Lemma 4.1. *The fractional self-adjoint BVP (4.5) has only the trivial solution* iff

$$\Lambda = \frac{\xi_1\xi_2}{z(p_0+1)} + \xi_0\xi_2\sum_{s=p_0+2}^{m}\frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)} + \frac{\xi_0\xi_3(m-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)} \neq 0.$$

Proof. If we consider $-\left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\xi_0}\left(z\nabla y\right)\right)(x) = 0$, then it follows from Lemma 2.1 that

$$(z\nabla y)(x) = k_0\frac{(x-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)},$$

that is,

$$(\nabla y)(x) = k_0\frac{(x-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(x)}. \tag{4.6}$$

It can be rewritten as follows

$$y(x) - y(p_0) = \sum_{s=p_0+1}^{x} k_0\frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)}.$$

Let us set $y(p_0) = k_1$ to have

$$y(x) = k_0 \sum_{s=p_0+1}^{x} \frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)} + k_1. \tag{4.7}$$

We can find the values of k_0 and k_1 by considering BCs. By computing both Eqs. (4.6) and (4.7) at $x = p_0+1$, we have

$$y(p_0+1) = \frac{k_0}{z(p_0+1)} + k_1,$$
$$(\nabla y)(p_0+1) = \frac{k_0}{z(p_0+1)}.$$

Thus, the BC $\xi_0 y(p_0+1) - \xi_1 \nabla y(p_0+1) = 0$ gives

$$k_0\left(\frac{\xi_0-\xi_1}{z(p_0+1)}\right) + k_1\xi_0 = 0. \tag{4.8}$$

Again, by evaluating Eqs. (4.6) and (4.7) at $x = m$, we see that

$$y(m) = \sum_{s=p_0+1}^{m} k_0 \frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)} + k_1,$$
$$\nabla y(m) = k_0 \frac{(m-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}.$$

Therefore, the BC $\xi_2 y(m) + \xi_3 \nabla y(m) = 0$ becomes

$$k_0\left(\xi_2 \sum_{s=p_0+1}^{m} \frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)} + \xi_3 \frac{(m-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right) + k_1\xi_2 = 0. \tag{4.9}$$

It is known that the system of Eqs. (4.8) and (4.9) (for k_0 and k_1) has only the trivial solution **iff** the determinant of the system

$$\Lambda := -\begin{vmatrix} \frac{\xi_0-\xi_1}{z(p_0+1)} & \xi_0 \\ \xi_2 \sum_{s=p_0+1}^{m} \frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)} + \xi_3 \frac{(m-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)} & \xi_2 \end{vmatrix},$$

is not equal to 0. This equals to

$$\begin{aligned}\Lambda &= \frac{\xi_0\xi_2}{z(p_0+1)} - \frac{\xi_1\xi_2}{z(p_0+1)} - \xi_0\xi_2 \sum_{s=p_0+1}^{m} \frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)} \\ &\quad - \xi_0\xi_2 \frac{(m-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)} \\ &= -\frac{\xi_1\xi_2}{z(p_0+1)} - \xi_0\xi_2 \sum_{s=p_0+2}^{m} \frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)} - \frac{\xi_0\xi_3(m-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}.\end{aligned}$$

Consequently, we found that k_0 and k_1 are not both equal to 0. That is, y is nontrivial **iff** $\Lambda \neq 0$. The proof is done. □

Theorem 4.2. *Suppose that Λ be a quantity as in Lemma 4.1. Then the Green's function for the BVP (4.5) is expressed by*

$$G(x,r) = \begin{cases} u(x,r), & x \le r-1, \\ v(x,r), & x \ge r, \end{cases} \tag{4.10}$$

$$\begin{aligned}u(x,r) = \frac{1}{\Lambda}\Bigg(&\xi_0\xi_2\varphi(x,p_0)\varphi(m,\bar{\sigma}(r)) + \xi_0\xi_3\varphi(x,p_0)\frac{[m+\mu-\sigma(r)]^{\underline{\mu-1}}}{\Gamma(\mu)z(m)} \\ &+ \frac{\xi_2(\xi_1-\xi_0)}{z(p_0+1)}\varphi(m,\bar{\sigma}(r)) + \frac{\xi_3(\xi_1-\xi_0)[m+\mu-\sigma(r)]^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\Bigg),\end{aligned} \tag{4.11}$$

and

$$v(x,r) = u(x,r) - \varphi(x,\bar{\sigma}(r)), \tag{4.12}$$

where $\varphi(\cdot,\cdot)$ is as defined in Definition 2.1.

Proof. Suppose that $y(x)$ is a solution of the BVP

$$\begin{cases} -\left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\mu}(z\nabla y)\right)(x) = h(x+\mu), & x \in \mathbb{N}^m_{p_0+2}, \\ \xi_0 y(p_0+1) - \xi_1(\nabla y)(p_0+1) = 0, \\ \xi_2 y(m) + \xi_3(\nabla y)(m) = 0. \end{cases} \tag{4.13}$$

Therefore, $\phi(x) := (z\nabla y)(x)$ can solve the IVP

$$\begin{cases} -\left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\mu}\phi\right)(x) = h(x+\mu), & x \in \mathbb{N}_{p_0+2}, \\ \phi(p_0+1) = z(p_0+1)\left[y(p_0+1) - y(p_0)\right]. \end{cases}$$

By considering Lemma 2.1, the solution of this IVP will be

$$\phi(x) = -\left({}^{\mathrm{RL}}_{p_0+1}\Delta^{\mu}h\right)(x+\mu) - k_0\frac{(x-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)}.$$

That is,

$$(\nabla y)(x) = -\sum_{r=p_0+1}^{x} \frac{(x+\mu-\sigma(r))^{\underline{\mu-1}}}{\Gamma(\mu)z(t)}h(r) - k_0\frac{(x-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(x)}. \tag{4.14}$$

By summing both sides $\sum_{s=p_0+1}^{x}$ to obtain

$$y(x) - y(p_0) = -\sum_{s=p_0+1}^{x}\left(\sum_{r=p_0+1}^{s} \frac{(s+\mu-\sigma(r))^{\underline{\mu-1}}}{\Gamma(\mu)z(s)}h(r) - k_0\frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)}\right).$$

We let $y(p_0) = k_1$ and interchange the sums to obtain

$$\begin{aligned}y(x) &= -\sum_{r=p_0+1}^{x}\sum_{s=r}^{x}\left(\frac{(s+\mu-\sigma(r))^{\underline{\mu-1}}}{\Gamma(\mu)z(s)}h(r)\right) \\ &\quad - k_0\sum_{s=p_0+1}^{x}\frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)} + k_1 \\ &= -\sum_{r=p_0+1}^{x}\varphi(x,\bar{\sigma}(r))\,h(r) - k_0\varphi(x,p_0) + k_1.\end{aligned} \tag{4.15}$$

Therefore, we see at $x = p_0+1$ that

$$y(p_0+1) = \frac{-k_0}{z(p_0+1)} - \frac{h(p_0+1)}{z(p_0+1)} + k_1,$$

and

$$(\nabla y)(p_0+1) = \frac{-k_0}{z(p_0+1)} - \frac{h(p_0+1)}{z(p_0+1)}.$$

Then, by using the first condition we have

$$\xi_0\left(\frac{-k_0}{z(p_0+1)} - \frac{h(p_0+1)}{z(p_0+1)} + k_1\right) - \xi_1\left(\frac{-k_0}{z(p_0+1)} - \frac{h(p_0+1)}{z(p_0+1)}\right) = 0.$$

Since h is a function defined on $\mathbb{N}_{p_0+2}$, we can extend the domain of h by setting $h(p_0+1) = 0$. So, we recast the last equation as follows

$$k_0\left(\frac{\xi_1-\xi_0}{z(p_0+1)}\right) + k_1\xi_0 = 0 \implies k_1 = \frac{(\xi_0-\xi_1)k_0}{\xi_0\, z(p_0+1)}. \tag{4.16}$$

Besides, since we have

$$y(m) = -\sum_{r=p_0+1}^{m} h(r)\varphi(m,\bar{\sigma}(r)) - k_0\varphi(m,p_0) + k_1,$$

and by using (4.14),

$$(\nabla y)(m) = -\sum_{r=p_0+1}^{m} h(r)\left(\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right) - k_0\frac{(m-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)},$$

we can say by applying the second condition that

$$\begin{aligned}0 &= \xi_2\left(-\sum_{r=p_0+1}^{m} h(r)\varphi(m,\bar{\sigma}(r)) - k_0\varphi(m,p_0) + k_1\right) \\ &\quad + \xi_3\left(-\sum_{r=p_0+1}^{m} h(r)\left(\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right) - k_0\frac{(m-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right),\end{aligned}$$

or equivalently, we have

$$k_0\left(-\xi_2\varphi(m,p_0)-\xi_3\frac{(m-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right)+k_1\xi_2 = \xi_2\sum_{r=p_0+1}^{m}h(r)\varphi(m,\bar{\sigma}(r))+\xi_3\sum_{r=p_0+1}^{m}h(r)\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)},$$

$$k_0\left(-\xi_2\varphi(m,p_0)-\xi_3\frac{(m-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right)+k_1\xi_2 = \sum_{r=p_0+2}^{m}h(r)\left[\xi_2\varphi(m,\bar{\sigma}(r))+\xi_3\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right]. \tag{4.17}$$

By substituting k_1 (in (4.16)) into Eq. (4.17), we obtain

$$k_0\left(-\xi_2\varphi(m,p_0)-\xi_3\frac{(m-p_0)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}+\frac{\xi_2(\xi_0-\xi_1)k_0}{\xi_0 z(p_0+1)}\right) = \sum_{r=p_0+2}^{m}h(r)\left[\xi_2\varphi(m,\bar{\sigma}(r))+\xi_3\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right].$$

By using the definition of $\varphi(m,p_0)$ and multiplying both sides of it by ξ_0, one can have

$$\sum_{r=p_0+2}^{m}h(r)\left[\xi_0\xi_2\varphi(m,\bar{\sigma}(r))+\xi_0\xi_3\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right]=k_0\left(-\xi_0\xi_2\frac{1}{z(p_0+1)} -\xi_0\xi_2\sum_{s=p_0+2}^{m}\frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)}-\xi_0\xi_2\frac{(m-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}+\frac{\xi_0\xi_2}{z(p_0+1)}-\frac{\xi_1\xi_2}{z(p_0+1)}\right).$$

It follows that

$$\sum_{r=p_0+2}^{m}h(r)\left[\xi_0\xi_2\varphi(m,\bar{\sigma}(r))+\xi_0\xi_3\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right]=k_0\left[-\left(\frac{\xi_1\xi_2}{z(p_0+1)} +\xi_0\xi_2\sum_{s=p_0+2}^{m}\frac{(s-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(s)}+\frac{\xi_0\xi_3(m-p_0-2+\mu)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right)\right]=-\Lambda k_0.$$

This implies that

$$k_0=-\frac{1}{\Lambda}\sum_{r=p_0+2}^{m}h(r)\left[\xi_0\xi_2\varphi(m,\bar{\sigma}(r))+\xi_0\xi_3\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right].$$

By using k_1 in (4.16), we find

$$k_1=\frac{(\xi_0-\xi_1)}{\xi_0 z(p_0+1)}\left(-\frac{1}{\Lambda}\sum_{r=p_0+2}^{m}h(r)\left[\xi_0\xi_2\varphi(m,\bar{\sigma}(r))+\xi_0\xi_3\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right]\right) = \frac{(\xi_1-\xi_0)}{\Lambda z(p_0+1)}\left(\sum_{r=p_0+2}^{m}h(r)\left[\xi_2\varphi(m,\bar{\sigma}(r))+\xi_3\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right]\right).$$

We know that $z(x)>0$ and $\Lambda\neq 0$. Therefore, we substitute k_0 and k_1 in (4.15), we have

$$\begin{aligned} y(x)&=-\sum_{r=p_0+1}^{x}h(r)\varphi(x,\bar{\sigma}(r))-k_0\varphi(x,p_0)+k_1\\ &=-\sum_{r=p_0+1}^{x}h(r)\varphi(x,\bar{\sigma}(r))\\ &\quad+\varphi(x,p_0)\left(\frac{1}{\Lambda}\sum_{r=p_0+2}^{m}h(r)\left[\xi_0\xi_2\varphi(m,\bar{\sigma}(r))+\xi_0\xi_3\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right]\right)\\ &\quad+\frac{(\xi_1-\xi_0)}{\Lambda z(p_0+1)}\left(\sum_{r=p_0+2}^{m}h(r)\left[\xi_2\varphi(m,\bar{\sigma}(r))+\xi_3\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right]\right).\end{aligned}$$

The terms can be combined to have

$$\begin{aligned} y(x)&=-\sum_{r=p_0+1}^{x}h(r)\varphi(x,\bar{\sigma}(r))\\ &\quad+\sum_{r=p_0+1}^{m}h(r)\left[\frac{1}{\Lambda}\left(\xi_0\xi_2\varphi(x,p_0)\varphi(m,\bar{\sigma}(r))+\xi_0\xi_3\varphi(x,p_0)\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right.\right.\\ &\qquad\left.\left.+\frac{\xi_2(\xi_1-\xi_0)}{z(p_0+1)}\varphi(m,\bar{\sigma}(r))+\frac{\xi_3(\xi_1-\xi_0)}{z(p_0+1)}\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right)\right].\end{aligned}$$

By using $\sum_{r=p_0+1}^{m}(\cdot)=\sum_{r=p_0+1}^{x}(\cdot)+\sum_{r=x+1}^{m}(\cdot)$, it follows that

$$\begin{aligned} y(x)&=\sum_{r=p_0+1}^{x}h(r)\left[\frac{1}{\Lambda}\left(\xi_0\xi_2\varphi(x,p_0)\varphi(m,\bar{\sigma}(r))+\xi_0\xi_3\varphi(x,p_0)\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right.\right.\\ &\qquad\left.\left.+\frac{\xi_2(\xi_1-\xi_0)}{z(p_0+1)}\varphi(m,\bar{\sigma}(r))+\frac{\xi_3(\xi_1-\xi_0)}{z(p_0+1)}\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right)-\varphi(x,\bar{\sigma}(r))\right]\\ &\quad+\sum_{r=x+1}^{m}h(r)\left[\frac{1}{\Lambda}\left(\xi_0\xi_2\varphi(x,p_0)\varphi(m,\bar{\sigma}(r))+\xi_0\xi_3\varphi(x,p_0)\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right.\right.\\ &\qquad\left.\left.+\frac{\xi_2(\xi_1-\xi_0)}{z(p_0+1)}\varphi(m,\bar{\sigma}(r))+\frac{\xi_3(\xi_1-\xi_0)}{z(p_0+1)}\frac{(m+\mu-r-1)^{\underline{\mu-1}}}{\Gamma(\mu)z(m)}\right)-\varphi(x,\bar{\sigma}(r))\right].\end{aligned}$$

Consequently, it leads to

$$\begin{aligned} y(x)&=\sum_{r=p_0+1}^{x}h(r)u(x,r)+\sum_{r=x+1}^{m}h(r)v(x,r)\\ &=\sum_{r=p_0+1}^{m}h(r)G(x,r).\end{aligned}$$

This completes the proof. □

5. Concluding remarks

To conclude, our study focuses on the existence, uniqueness and trivial solutions in the classes of self-adjoint equations with delta Riemann–Liouville fractional operators. The uniqueness of the initial value problem on delta fractional difference operators is represented by applying delta fractional sum to both sides of the equation and using some discrete delta properties. By applying this uniqueness theorem, we have analyzed and derived the existence and uniqueness of the proposed self-adjoint delta fractional difference equation. In the other part of our study, the Cauchy function based on the delta Riemann–Liouville fractional differences has been introduced and the self-adjoint problem has been solved accordingly. In addition, the uniqueness of the self-adjoint problem including the component of Green's functions has been determined. Then, we have examined how this problem has only a trivial solution and the condition under which it has only a trivial solution has been found. Throughout the study, we have presented some examples and through these extensive examples, we have validated the theoretical results.

CRediT authorship contribution statement

Pshtiwan Othman Mohammed: Investigation, Methodology, Writing – original draft. **Hari Mohan Srivastava:** Conceptualization, Investigation, Writing – original draft. **Rebwar Salih Muhammad:** Conceptualization, Investigation, Visualization, Writing – review & editing. **Eman Al-Sarairah:** Data curation, Project administration, Validation. **Nejmeddine Chorfi:** Funding acquisition, Software, Writing – review & editing. **Dumitru Baleanu:** Funding acquisition, Investigation, Software, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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