Journal of King Saud University – Science (2017) **29**, 133–136

King Saud University
Journal of King Saud University – Science
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SHORT COMMUNICATION

Line with attached segment as a model of Helmholtz resonator: Resonant states completeness

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Received 15 May 2016; accepted 17 July 2016
Available online 8 August 2016

KEYWORDS
Quantum graph;
Scattering;
Resonance

Abstract Quantum graph consisting of a line with attached segment is considered as a simple model of the Helmholtz resonator. Completeness of resonant states in the space of square integrable functions on the segment is proved. Relation between the completeness and the factorization of the characteristic function in Sz.-Nagy model is discussed.

1. Introduction

The problem of resonances and resonant states attracted great attention starting from famous Lord Rayleigh work (Lord Rayleigh, 1916). But rigorous mathematical description of the problem was given at the end of 20-th century. Particularly, it became clear that resonances are eigenvalues of some dissipative operator (Lax and Phillips, 1967, 1976; Adamyan and Arov, 1965). A few models and asymptotic approaches to the problem were developed on the background of this operator treatment (see, e.g., Hislop and Martinez, 1991; Gadyl'shin, 1997; Popov, 1993; Popov, 1992a,b) and references therein). One of the intriguing question in this problem is: What is a domain Ω which gives one the completeness of the resonant states in $L_2(\Omega)$? Our hypothesis is that it is the convex hull of the scatterer. It is not yet proved. There are only some examples of solved particular problems (Shushkov, 1985; Vorobiev and Popov, 2015). There is an interesting relation between the scattering problem and functional model (Sz.-Nagy et al., 2010; Nikol'skii, 2012; Khrushchev et al., 1981; Peller, 2003). More precisely, the completeness is related to the factorization of the scattering matrix characteristic function for the functional model). We use this relation in the present paper. Namely, we consider the simplest, one-dimensional, model of the Helmholtz resonator and investigate the scattering matrix for this quantum graph. This system is, in some sense, close to a waveguide with local perturbation (see, e.g., (Borisov et al., 2001, 2013; Frolov and Popov, 2000; Wulf et al., 2013; Popov and Popova, 1993a,b). The rest of Introduction is devoted to the description of the model.

Let us define the Schrödinger operator on the graph Γ consisting of three edges $\Omega_1 \cup \Omega_2 \cup \Omega_3$ (see Fig. 1) coupled at vertex V. $\partial\Gamma = V_0$.

Definition. The Schrödinger operator H on Γ acts as $-\frac{d^2}{dx^2}$ at each edge Ω_i. The operator has the following domain:

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2016.07.004
1018-3647

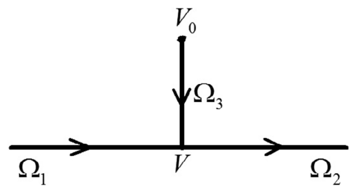

Figure 1 Graph geometry. Arrows show the direction at edges. The origin at Ω_3 is at vertex V_0, the origins at Ω_1, Ω_2 are at vertex V.

$$dom\,H = \{\psi \in C(\Gamma) \cap W_2^2(\Gamma \setminus V); \psi_1(-0) = \psi_2(+0) = \psi_3(L-0),$$
$$-\psi_1'(-0) + \psi_2'(+0) - \psi_3'(L-0) = \alpha\psi_2(0), \psi_3(0) = 0.\} \tag{1}$$

Here W_2^2 is the Sobolev space, $\frac{d\psi_i}{dx_i}(V)$ is the derivative of the solution at the vertex V of edge Ω_i in the outgoing direction from the vertex.

We consider the scattering in the framework of Lax-Phillips approach (Lax and Phillips, 1967, 1976). Let us briefly describe the approach. Consider the Cauchy problem

$$\begin{cases} u_{tt}'' = u_{xx}'', \\ u(x,0) = u_0(x), u_t'(x,0) = u_1(x), x \in \Gamma. \end{cases} \tag{2}$$

Let $\mathcal{E}$ be the Hilbert space of two-component functions (u_0, u_1) on the graph with finite energy

$$\|(u_0, u_1)\|_{\mathcal{E}}^2 = 2^{-1} \int_\Gamma (|u_0'|^2 + |u_1|^2) dx.$$

The pair (u_0, u_1) is called the Cauchy data. Solving operator for problem (2), $U(t)$, $U(t)(u_0, u_1) = (u(x,t), u_t'(x,t))$, is unitary in $\mathcal{E}$. Unitary group $U(t)|_{t\in\mathbb{R}}$ has two orthogonal (in $\mathcal{E}$) subspaces, D_- and D_+, called, correspondingly, incoming and outgoing subspaces.

Lemma 1.1. *Outgoing subspace D_+ has the following properties:*

(a) $U(t)D_+ \subset D_+$, $t > 0$;
(b) $\cap_{t>0} U(t) D_+ = \{0\}$,
(c) $\overline{\cup_{t<0} U(t) D_+} = \mathcal{E}$.

D_- has the analogous properties (with the natural replacement $t > 0 \leftrightarrow t < 0$).

Lemma 1.2. *Subspaces $D_\pm$ can be chosen as follows:*

$$D_+ = \{(u_0, u_1) : -u_1 = u_0', x \in \Omega_1; u_1 = u_0', x \in \Omega_2;$$
$$u_1 = u_0 = 0, x \in \Omega_3\},$$

$$D_- = \{(u_0, u_1) : u_1 = u_0', x \in \Omega_1; -u_1 = u_0', x \in \Omega_2;$$
$$u_1 = u_0 = 0, x \in \Omega_3\}.$$

Lemma 1.3. *There is a pair of isometric maps $T_\pm : \mathcal{E} \to L_2(\mathbb{R}, \mathbb{C}^2)$ having the following properties:*

$$T_\pm U(t) = \exp ikt T_\pm, \quad T_+ D_+ = H_+^2(\mathbb{C}^2), \quad T_- D_- = H_-^2(\mathbb{C}^2),$$

where $H_\pm^2$ is the Hardy space.

It is said that $T_+(T_-)$ gives one the outgoing (incoming) spectral representation of the unitary group $U(t)$. Let $K = \mathcal{E} \ominus (D_+ \oplus D_-)$. Consider a semigroup $Z(t) = P_K U(t)|_K$, $t > 0$, P_K is a projector to K. Let B be the generator of the semigroup $Z(t) : Z(t) = \exp iBt, t > 0$. Data which are eigenvectors of B are called resonant states. Operator $T_- T_+^{-1}$ is called the scattering operator. It acts as a multiplication by a matrix-function $S(k)$ which is the boundary value at the real axis of analytic matrix-function in the upper half-plane k such that $\|S(k)\| \leqslant 1$ for $\Im k > 0$ and $S^* S = I$ almost everywhere on the real axis. This analytic matrix-function $S(k)$ is called the scattering matrix.

2. Scattering matrix

To describe the scattering matrix $S = \{s_{jp}(k)\}$ and related topics, one can consider the whole set of solutions of the scattering problem $\psi_{1,2}^\pm$ having the following form.

$$\psi_1^- = s_{11}(k) \exp(-ikx), x \in \Omega_1,$$

$$\psi_1^+ = \exp(-ikx) + s_{12}(k) \exp(ikx), x \in \Omega_2,$$

$$\psi_2^- = \exp(ikx) + s_{21}(k) \exp(-ikx), x \in \Omega_1,$$

$$\psi_2^+ = s_{22}(k) \exp(ikx), x \in \Omega_2,$$

$$\psi_1^- = \overline{\psi_2^+}, \quad \psi_2^- = \overline{\psi_1^+}.$$

For $x \in \Omega_3$ the solutions have forms $p \sin(kx)$. Here $s_{11}(k) = s_{22}(k) = t$ is the transmission coefficient and $s_{12}(k) = s_{21}(k) = r$ is the reflection coefficient.

Let us determine an isometric map $T_- : \mathcal{E} \to L_2(\mathbb{R}, \mathbb{C}^2)$ as a closure of $\widetilde{T}_-$ defined on the set of smooth functions in $\mathcal{E}$:

$$\widetilde{T}_- \Phi = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \langle \Phi, \Psi_2^-(\cdot, k) \rangle_{\mathcal{E}} \\ \langle \Phi, \Psi_1^-(\cdot, k) \rangle_{\mathcal{E}} \end{pmatrix}, \quad \Psi_{1,2}^- = \begin{pmatrix} (ik)^{-1} \psi_{1,2}^- \\ \psi_{1,2}^- \end{pmatrix},$$

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Lemma 2.1. *Map T_- gives one a spectral representation for the unitary group $U(t)$. The following relations take place.*

$$T_- D_- = H_-^2(\mathbb{C}^2), \quad T_- D_+ = S H_+^2(\mathbb{C}^2),$$
$$T_- U(t) = \exp(ikt) T_-.$$

Matrix-function S is an inner function in $\mathbb{C}_+$ and

$$K_- = T_- K = H_+^2 \ominus S H_+^2, \quad T_- Z(t)|_K = P_{K_-} \exp(ikt) T_-.$$

As an inner function, S can be represented in the form $S = \Pi\Theta$, where Π is the Blaschke-Potapov product and Θ is a singular inner function (Sz.-Nagy et al., 2010; Nikol'skii, 2012; Khrushchev et al., 1981; Peller, 2003). We are interested in the completeness of the system of resonant states. It is related with the factorization of the scattering matrix.

Theorem 2.2 (*(Completeness criterion), (Nikol'skii, 2012)*). *Let S be an inner function, $H_+^2(N) \ominus S H_+^2(N)$, $Z = P_K U|_K$. The following statements are equivalent:*

1. Operator Z is complete;
2. Operator Z^* is complete;
3. S is a Blaschke–Potapov product.

Here N is an auxiliary space (in our case it is $\mathbb{C}^2$).

As for the case of finite-dimensional N (as in our situation), there is simple criterion (for general operator case there is no such criterion) for absence of the singular inner factor (we reformulate the theorem from (Nikol'skii, 2012, p. 99) for the half-plane):

Theorem 2.3. *Let* $\dim N < \infty$. *The following statements are equivalent:*

1. S is a Blaschke-Potapov product;
2. $$\lim_{r\to 1-0}\int_{L_r}\log_e|\det S(z)|\frac{dz}{(z-1)^2}=0. \quad (3)$$

Here L_r *is the image of the curve* $|\zeta| = r < 1$ *under the map* $z = i\frac{1+\zeta}{1-\zeta}$.

One can, immediately, find the scattering matrix for our simple graph (see, e.g., Exner and Seresova, 1994). The coefficients are as follows.

$$t=\frac{2ik}{2ik-\alpha-k\cot(kL)}, \quad (4)$$

$$r=\frac{\alpha+k\cot(kL)}{2ik-\alpha-k\cot(kL)}. \quad (5)$$

The scattering matrix has the form

$$S(k)=\begin{pmatrix} t & r\\ r & t\end{pmatrix}. \quad (6)$$

Correspondingly,

$$\det S(k)=t^2-r^2=\frac{-4k^2-(\alpha+k\cot(kL))^2}{(2ik-\alpha-k\cot(kL))^2}. \quad (7)$$

Remark. One can see that if $k_i = 0, k = k_r + ik_i$, then the natural property is valid: $|\det S| = 1$.

Poles k_* of the scattering matrix are given by roots k_* of the equation:

$$\cot(kL)=2i-\frac{\alpha}{k}, \quad (8)$$

Correspondingly, roots of $S(k)$ are at points $\overline{k_*}$ ($\lambda = \overline{k_*}^2$). Taking into account the expression for $\cot(kL)$, one reduces Eq. (8) to the following system:

$$\frac{\sin(k_rL)\cos(k_rL)}{\sin^2(k_rL)+\sinh^2(k_iL)}=-\frac{\alpha k_r}{k_r^2+k_i^2}, \quad (9)$$

$$\frac{-\sinh(k_iL)\cosh(k_iL)}{\sin^2(k_rL)+\sinh^2(k_iL)}-2=\frac{\alpha k_i}{k_r^2+k_i^2}. \quad (10)$$

If $\alpha = 0$, then one can find the solution in an explicit form:

$$k_r=\frac{\pi n}{L},\ k_i=-\frac{1}{2}\ln 3,\ n\in\mathbb{Z}. \quad (11)$$

Correspondingly, $\det S(k)$ has roots at points $k_n = \frac{\pi n}{L} + i\frac{1}{2}\ln 3,\ n \in \mathbb{Z}$. One can see that $\Im k_n$ does not depend on n in this simple case. It should be noted that $\Re k_0 = 0$ and, correspondingly, $\lambda_0 = k_0^2 \in \mathbb{R}$. It is a negative eigenvalue (i.e. it is not a resonance). The existence of such eigenstates for analogous systems is well-known (see, e.g., (Sols et al., 1989)). The corresponding state is added to the set of resonance states considered below.

If $\alpha \neq 0$, then system (9), (10) has no solution in an analytic form. However, the right hand sides of (9), (10) can be simply estimated. Then, the Rouche's theorem shows that there are only finite number of roots below some line in $\mathbb{C}_+$ parallel to the real axis (it is not difficult to obtain the asymptotics of the resonances in n, $n \to \infty$ or in α, $\alpha \to 0$). To prove the completeness, in accordance with Theorem 2.3, we should estimate the corresponding integral in condition (3). Let us describe briefly the procedure of this estimation. There are two reasons of possible breaking of condition (3): infinite length of the integration path after the limiting procedure and singularities appearing time to time at the integration curve when one transforms the curve in accordance with the limiting procedure.

The integration curve L_r is a circle $\{R(r)\exp(it) + iC(r) | t \in [0, 2\pi)\}$.

$$\lim_{r\to 1-0}\int_0^{2\pi}\log_e|\det(R(r)\exp(it)+iC(r))|\frac{R(r)}{(R(r)\exp(it)+iC(r)+i)^2}dt=0. \quad (12)$$

To estimate the integral in (12), we use the Cauchy inequality

$$\left|\int_0^{2\pi}f(t)\overline{g(t)}dt\right|^2\leqslant\int_0^{2\pi}|f(t)|^2dt\int_0^{2\pi}|g(t)|^2dt, \quad (13)$$

where

$f(t) = \log_e|\det(R(r)\exp(it) + iC(r))|\frac{\sqrt{R(r)}}{R(r)\exp(it)+iC(r)+i}$, $\frac{\sqrt{R(r)}}{R(r)\exp(it)+iC(r)+i}$.

As for $\int_0^{2\pi}|g(t)|^2dt$, it is proved that this integral is bounded by a constant which does not depend on r. The second integral in (13), $\int_0^{2\pi}|f(t)|^2dt$, tends to zero if $r \to 1-0$ (correspondingly, $R \to \infty$, $C \to 0$). To prove this statement, we use the information about the resonances (i.e., singularities) positions (particularly, the resonances asymptotics, see above). Consequently, one can perform the limiting procedure in statement 2 of Theorem 2.3. As a result, one comes to the concluding theorem.

Theorem 2.4. *The system of resonant states of the Schrödinger operator* H *on the graph* Γ *is complete in* $L_2(\Omega_3)$ *and is not complete in* $L_2(\Omega_3 \cup (-b, b))$ *for* $b > 0$.

The first part of the theorem has been proved. To prove the second statement, we change the construction slightly. Subspaces $D_\pm$ can be chosen in another way than in Lemma 1.2:

$$D_+=\{(u_0,u_1):\ -u_1=u_0',x\in\Omega_1\setminus[-b,0];u_1=u_0',x\in\Omega_2\setminus[0,b];\ u_1=u_0=0,x\in\Omega_3\cup(-b,b)\},b>0,$$

$$D_-=\{(u_0,u_1):\ u_1=u_0',x\in\Omega_1\setminus[-b,0];-u_1=u_0',x\in\Omega_2\setminus[0,b];\ u_1=u_0=0,x\in\Omega_3\cup(-b,b)\},b>0.$$

In this case, a factor $\exp(2ikb)$ appears in the expression for the determinant of the scattering matrix, the space $L_2(\Omega_3)$ is replaced by $L_2(\Omega_3 \cup (-b, b))$. One can see that condition 2 of Theorem 2.3 does not take place, hence, there is a non-trivial singular inner factor. Correspondingly, one can conclude that the system of resonant states is complete in $L_2(\Omega_3)$ and is not complete in $L_2(\Omega_3 \cup (-b, b))$ for $b > 0$. QED.

Acknowledgement

This work was partially financially supported by the Government of the Russian Federation (grant 074-U01), by Ministry of Education and Science of the Russian Federation (GOSZADANIE 2014/190, Projects No 14.Z50.31.0031 and No. 1.754.2014/K), by grant MK-5001.2015.1 of the President of the Russian Federation and DFG Grant NE 1439/3-1 (Development of Methods in the Theory of Self-adjoint Extensions), by grant 16-11-10330 of Russian Science Foundation. The authors thanks Referees for useful remarks.

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