Original article

# Sine-Gordon expansion method for exact solutions to conformable time fractional equations in RLW-class

Alper Korkmaz [a,*], Ozlem Ersoy Hepson [b], Kamyar Hosseini [c], Hadi Rezazadeh [d], Mostafa Eslami [e]

[a] Çankırı Karatekin University, Department of Mathematics, 18200 Çankırı, Turkey
[b] Eskişehir Osmangazi University, Department of Computer and Mathematics, Eskişehir, Turkey
[c] Department of Mathematics, Rasht Branch, Islamic Azad University, Rasht, Iran
[d] Faculty of Engineering Technology, Amol University of Special Modern Technologies, Amol, Iran
[e] Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

A B S T R A C T

The Sine-Gordon expansion method is implemented to construct exact solutions some conformable time fractional equations in Regularized Long Wave (RLW)-class. Compatible wave transform reduces the governing equation to classical ordinary differential equation. The homogeneous balance procedure gives the order of the predicted polynomial-type solution that is inspired from well-known Sine-Gordon equation. The substitution of this solution follows the previous step. Equating the coefficients of the powers of predicted solution leads a system of algebraic equations. The solution of resultant system for coefficients gives the necessary relations among the parameters and the coefficients to be able construct the solutions. Some solutions are simulated for some particular choices of parameters.

© 2018 The Authors. Production and hosting by Elsevier B.V. on behalf of King Saud University. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

## 1. Introduction

Even though some nonlinear partial differential equations are integrable, it may be not easy to integrate them. Instead, a predicted solution with parameters are assumed to be a solution of governing equations and the relations among the parameters are investigated. The logic is simply based on the similarity with exponential-type solutions to the ordinary differential equations with constant coefficients. These predicted solutions are of various forms covering exponential, hyperbolic, trigonometric or rational functions, and more. Moreover the interaction of various types of waves were also determined by using analytical techniques (Ma and Zhou, 2018; Zhang and Ma, 2017; Zhao and Ma, 2017; Yang et al., 2017; Ma et al., 2018). Parallel to the recent developments in computer algebra in the last four decades, a tendency has been observed to determine exact solutions to nonlinear PDEs by following the procedure that starts with a predicted solution. Recently, this tendency has focused on exact solutions to fractional nonlinear partial differential equations. Many of techniques implemented to nonlinear PDEs to find exact solutions have been adapted for fractional nonlinear PDES (Korkmaz and Hosseini, 2017; Hosseini and Ansari, 2017; Kumar et al., 2017; Korkmaz, 2017a,b; Hosseini et al., 2017a,b; Kumar et al., 2018a,b; Rezazadeh et al., 2018; Singh et al., 2017a,b; Guner et al., 2017a,b). We also derive exact solutions to some conformable fractional equations in RLW-class modeling various wave phenomena both in nature or technology implementations. Different from previous studies, we adapt Sine-Gordon expansion approach to determine exact solutions to governing equations in fractional RLW-class.

The first equation considered in this study is the RLW equation

$$D_t^\alpha u + p u_x + q u u_x + r D_t^\alpha u_{xx} = 0, \quad t \geqslant 0 \tag{1}$$

* Corresponding author.
  E-mail address: alperkorkmaz7@gmail.com (Alper Korkmaz).
Peer review under responsibility of King Saud University.

**Production and hosting by Elsevier**

where $p$, $q$ and $r$ real coefficients, $D_t^\alpha$ conformable fractional differential operator and $u = u(x, t)$. The integer ordered form of the RLW equation describes formation and development of undular bore by a long wave in shallow water (Peregrine, 1966). The same study investigates the transition and interaction between still water and a uniform flow. The origins and similarities to the KdV Equation of the RLW equation are discussed in general terms (Benjamin et al., 1972). Moreover, Benjamin et al. developed an exact theory to the RLW equation and existence of classical solutions were proved. The Lagrangian density for the RLW equation was defined in Morrison et al. (1984). Significant concept covering peak positions, amplitude and widths of two solitary waves were studied by trial function approach in the same paper.

The second equation to discuss exact solutions here is the modified RLW (mRLW) equation of the form

$$D_t^\alpha u + p u_x + q u^2 u_x + r D_t^\alpha u_{xx} = 0, \quad t \geqslant 0 \tag{2}$$

where $D_t^\alpha$ conformable fractional differential operator. This equation seemed in Gardners' study (Gardner et al., 1997). B-spline finite elements based approximate solutions defining motion of solitary waves were investigated in that study. In the following decades, various numerical and exact solutions to different problems constructed on the mRLW equation are solved using diverse techniques (Soliman, 2017; Khalifa et al., 2008; Santarelli, 1978; Karakoc and Geyikli, 2013; Karakoc et al., 2013; Mei and Chen, 2012).

The last equation to derive the exact solutions is the symmetric RLW (sRLW) equation

$$D_{tt}^{2\alpha} u + p u_{xx} + q u D_t^\alpha u_x + q u_x D_t^\alpha u + r D_{tt}^{2\alpha} u_{xx} = 0, \quad t \geqslant 0 \tag{3}$$

where $u = u(x, t)$, $p$, $q$ and $r$ real parameters, $D_{tt}^{2\alpha}$ is the fractional differential operator of order $2\alpha$ in conformable sense. This equation is an interesting model to describe ion-acoustic and space charge waves with weak non linearity (Seyler and Fenstermacher, 1984).

Before starting the solution procedure, we should give some significant properties of conformable fractional derivative. Thus, the next section focuses on conformable fractional derivative definition and some important properties. We explain the solution procedure in the third section. The following sections cover implementations of the proposed procedure to some conformable fractional PDEs in RLW-class.

## 2. Conformable fractional derivative

The conformable derivative of order $\alpha$ with respect to the independent variable $t$ is defined as

$$D_t^\alpha(y(t)) = \lim_{\tau \to 0} \frac{y(t + \tau t^{1-\alpha}) - y(t)}{\tau}, \quad t > 0, \ \alpha \in (0, 1]. \tag{4}$$

for a function $y = y(t) : [0, \infty) \to \mathbb{R}$ (Khalil et al., 2014). This newly defined fractional derivative is capable of satisfying some well known required properties.

**Theorem 1.** *Assume that the order of the derivative $\alpha \in (0, 1]$, and suppose that $u = u(t)$ and $y = y(t)$ are $\alpha$-differentiable for all positive $t$. Then,*

- $D_t^\alpha(c_1 u + c_2 y) = c_1 D_t^\alpha(u) + c_2 D_t^\alpha(y)$
- $D_t^\alpha(t^k) = k t^{k-\alpha}, \forall k \in \mathbb{R}$
- $D_t^\alpha(\lambda) = 0$, for all constant function $u(t) = \lambda$
- $D_t^\alpha(uy) = u D_t^\alpha(y) + y D_t^\alpha(u)$
- $D_t^\alpha \left( \frac{u}{y} \right) = \frac{y D_t^\alpha(u) - u D_t^\alpha(y)}{y^2}$
- $D_t^\alpha(u)(t) = t^{1-\alpha} \frac{du}{dt}$

*for $\forall c_1, c_2 \in \mathbb{R}$ (Atangana et al., 2015).*

Conformable fractional differential operator satisfies some critical fundamental properties like the chain rule, Taylor series expansion and Laplace transform (Abdeljawad, 2015).

**Theorem 2.** *Let $u = u(t)$ be an $\alpha$-conformable differentiable function and assume that $y$ is differentiable and defined in the range of $u$. Then,*

$$D_t^\alpha(u \circ y)(t) = t^{1-\alpha} y'(t) u'(y(t)) \tag{5}$$

## 3. Sine-Gordon expansion method

The compatible fractional form $u(x, t) = U(\xi)$ with $\xi = a(x - vt^\alpha/\alpha)$ of the classical wave transform (Ma and Fuchssteiner, 1996; Ma and Lee, 2009; Ma et al., 2010; Korkmaz, 2018) reduces the fractional Sine–Gordon equation in one dimension of the form

$$\frac{\partial^2 u}{\partial x^2} - D_t^{2\alpha} u = m^2 \sin u, \quad m \text{ is constant} \tag{6}$$

to the ODE

$$\frac{d^2 U}{d\xi^2} = \frac{m^2}{a^2(1 - v^2)} \sin U \tag{7}$$

where $v$ represents velocity of the traveling wave defined in the transform (Yan, 1996). Some simplifications lead

$$\left( \frac{d(U/2)}{d\xi} \right)^2 = \frac{m^2}{a^2(1 - v^2)} \sin^2 U/2 + C \tag{8}$$

where $C$ is constant of integration. $C$ is assumed zero for simplicity. Let $w(\xi) = U(\xi)/2$ and $b^2 = m^2/(a^2(1 - v^2))$. Then, (8) is converted to

$$\frac{d(w)}{d\xi} = b \sin w \tag{9}$$

Set $b = 1$ in (9). Then, (9) yields two significant relations

$$\sin w(\xi) = \frac{2de^\xi}{d^2 e^{2\xi} + 1} \bigg|_{d=1} = \operatorname{sech} \xi \tag{10}$$

or

$$\cos w(\xi) = \frac{d^2 e^{2\xi} - 1}{d^2 e^{2\xi} + 1} \bigg|_{d=1} = \tanh \xi \tag{11}$$

where $d$ is nonzero integral constant. The fractional PDE of the form

$$P(u, D_t^\alpha u, u_x, D_{tt}^{2\alpha} u, u_{xx}, \ldots) = 0 \tag{12}$$

can be reduced to an ODE

$$\tilde{P}(U, U', U'', \ldots) = 0 \tag{13}$$

by using a compatible wave transform $u(x, t) = U(\xi)$ where the transform variable $\xi$ is defined as $a(x - vt^\alpha/\alpha)$. Then, the predicted solution to (13) of the form

$$U(\xi) = A_0 + \sum_{i=1}^{s} \tanh^{i-1}(\xi)(B_i \operatorname{sech} \xi + A_i \tanh \xi) \tag{14}$$

can be written as

$$U(w) = A_0 + \sum_{i=1}^{s} \cos^{i-1}(w)(B_i \sin w + A_i \cos w) \tag{15}$$

owing to (10)–(11). The procedure starts by determining index limit $s$ by the assistance of homogenous balance of the terms in (13). Following the substitution of the predicted solution (15) into (13) the coefficients of powers of $\sin w \cos w$ are assumed as zero. Next, the

resultant algebraic system is tried to be solved for the coefficients $A_0, A_1, B_1, \ldots, a, v$. Then, the solutions are constructed, if exists, by using (10)–(11) and $\xi$

## 4. Solutions to the conformable time fractional RLW equation

The traveling wave transform $u(x,t) \rightarrow U(\xi)$, $\xi = a(x - vt^\alpha/\alpha)$ reduces the time fractional RLW equation to

$$(-av + ap)U + 1/2\,qaU^2 - rva^3U'' = C \tag{16}$$

where $C$ is constant of integration, and $''$ denotes differentiation wrt $\xi$. Here, we assume that $C = 0$ to reduce the complexity of the solutions. The balance between $U^2$ and $U''$ gives $s = 2$. Thus, the predicted solution takes the form

$$U(w) = A_0 + B_1 \sin w + A_1 \cos w + B_2 \cos w \sin w + A_2 \cos^2 w \tag{17}$$

Substituting this solution into (16) yields

$$
\begin{aligned}
&(-a^3 v r B_2 + aq A_2 B_2) \sin(w(\xi))(\cos(w(\xi)))^3 + (\cos(w(\xi)))^3 aq A_1 A_2 \\
&+ \left(ap A_2 - av A_2 + 1/2\,aq A_1^2 + aq A_0 A_2\right)(\cos(w(\xi)))^2 \\
&+ (-a^3 v r B_1 + aq A_1 B_2 + aq A_2 B_1) \sin(w(\xi))(\cos(w(\xi)))^2 \\
&+ \left(1/2\,aq B_2^2 + 4a^3 v r A_2 - 1/2\,aq A_2^2\right)(\sin(w(\xi)))^2 (\cos(w(\xi)))^2 \\
&+ 5(\sin(w(\xi)))^3 \cos(w(\xi))a^3 v r B_2 \\
&+ (2a^3 v r A_1 + aq B_1 B_2)(\sin(w(\xi)))^2 \cos(w(\xi)) \\
&+ (aq A_0 B_2 + aq A_1 B_1 - av B_2 + ap B_2) \sin(w(\xi)) \cos(w(\xi)) \\
&+ (aq A_0 A_1 - av A_1 + ap A_1) \cos(w(\xi)) + 1/2\,aq A_2^2 \\
&+ 1/2\,aq A_0^2 - av A_0 + ap A_0 + (aq A_0 B_1 - av B_1 + ap B_1) \sin(w(\xi)) \\
&+ (\sin(w(\xi)))^3 a^3 v r B_1 - 2(\sin(w(\xi)))^4 a^3 v r A_2 \\
&+ \left(1/2\,aq B_1^2 - 1/2\,aq A_2^2\right)(\sin(w(\xi)))^2 = 0 \tag{18}
\end{aligned}
$$

Using some trigonometric identities and simplifications we find the following algebraic system of equations:

$$
\begin{aligned}
&1/2\,a\left(-4a^2 v r A_2 - q A_2^2 + q B_1^2\right) + 1/2\,aq A_2^2 + 1/2\,aq A_0^2 - av A_0 \\
&\quad + ap A_0 = 0 \\
&a^3 v r B_1 + aq A_0 B_1 - av B_1 + ap B_1 = 0 \\
&\quad - av A_2 + ap A_2 + 1/2\,aq A_1^2 \\
&\quad + aq A_0 A_2 + 6a^3 v r A_2 + 1/2\,aq B_2^2 - 1/2\,aq A_2^2 \\
&\quad - 1/2\,a\left(-4a^2 v r A_2 - q A_2^2 + q B_1^2\right) = 0 \\
&\quad - 6a^3 v r A_2 - 1/2\,aq B_2^2 + 1/2\,aq A_2^2 = 0 \\
&\quad - 2a^3 v r A_1 + aq A_1 A_2 - aq B_1 B_2 = 0 \\
&2a^3 v r A_1 + aq A_0 A_1 + aq B_1 B_2 - av A_1 + ap A_1 = 0 \\
&\quad - 6a^3 v r B_2 + aq A_2 B_2 = 0 \\
&\quad - 2a^3 v r B_1 + aq A_1 B_2 + aq A_2 B_1 = 0 \\
&5a^3 v r B_2 + aq A_0 B_2 + aq A_1 B_1 - av B_2 + ap B_2 = 0
\end{aligned}
\tag{19}
$$

Solution of this system for $a$, $v$, $A_0$, $A_1$, $A_2$, $B_1$ and $B_2$ gives various solution sets with $a \neq 0$ and $v \neq 0$.

Set 1:

$$v = \frac{p}{ra^2 + 1}, \quad A_0 = -\frac{6pa^2 r}{(ra^2 + 1)q}, \quad A_1 = 0,$$

$$A_2 = \frac{6pa^2 r}{(ra^2 + 1)q}, \quad B_1 = 0, \quad B_2 = \frac{6ra^2 pi}{(ra^2 + 1)q}, \quad i = \sqrt{-1} \tag{20}$$

Set 2:

$$v = \frac{p}{ra^2 + 1}, \quad A_0 = -\frac{6pa^2 r}{(ra^2 + 1)q}, \quad A_1 = 0, \quad A_2 = \frac{6pa^2 r}{(ra^2 + 1)q},$$

$$B_1 = 0, \quad B_2 = \frac{-6ra^2 pi}{(ra^2 + 1)q}, \quad i = \sqrt{-1} \tag{21}$$

Set 3:

$$v = -\frac{p}{ra^2 - 1}, \quad A_0 = \frac{4pa^2 r}{(ra^2 - 1)q}, \quad A_1 = 0, \quad A_2 = -\frac{6pa^2 r}{(ra^2 - 1)q},$$

$$B_1 = 0, \quad B_2 = \frac{6ra^2 pi}{(ra^2 - 1)q}, \quad i = \sqrt{-1} \tag{22}$$

Set 4:

$$v = -\frac{p}{ra^2 - 1}, \quad A_0 = \frac{4pa^2 r}{(ra^2 - 1)q}, \quad A_1 = 0, \quad A_2 = -\frac{6pa^2 r}{(ra^2 - 1)q},$$

$$B_1 = 0, \quad B_2 = \frac{-6ra^2 pi}{(ra^2 - 1)q}, \quad i = \sqrt{-1} \tag{23}$$

Set 5:

$$v = \frac{p}{4ra^2 + 1}, \quad A_0 = -\frac{\frac{(8ra^2 - 1)p}{4ra^2 + 1} + p}{q}, \quad A_1 = 0, \quad A_2 = \frac{12pa^2 r}{(4ra^2 + 1)q},$$

$$B_1 = 0, \quad B_2 = 0 \tag{24}$$

Set 6:

$$v = -\frac{p}{4ra^2 - 1}, \quad A_0 = -\frac{-\frac{(8ra^2 - 1)p}{4ra^2 - 1} + p}{q}, \quad A_1 = 0,$$

$$A_2 = -\frac{12pa^2 r}{(4ra^2 - 1)q}, \quad B_1 = 0, \quad B_2 = 0 \tag{25}$$

Using these solution sets of algebraic equations, we construct the solutions to (1) as

$$
\begin{aligned}
u_1(x,t) = {}& -\frac{6a^2 rv}{q} + \frac{6a^2 rv}{q} i \tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) \operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) \\
&+ \frac{6a^2 rv}{q} \tanh^2\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
u_2(x,t) = {}& -\frac{6a^2 rv}{q} + \frac{6a^2 rv}{q} i \tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) \operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) \\
&- \frac{6a^2 rv}{q} \tanh^2\left(a(x - v\frac{t^\alpha}{\alpha})\right)
\end{aligned}
$$

for $v = p/(ra^2 + 1)$,

$$
\begin{aligned}
u_3(x,t) = {}& -\frac{4a^2 rv}{q} + \frac{6a^2 rv}{q} i \tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) \operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) \\
&- \frac{6a^2 rv}{q} \tanh^2\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
u_4(x,t) = {}& -\frac{4a^2 rv}{q} + \frac{6a^2 rv}{q} i \tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) \operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) \\
&+ \frac{6a^2 rv}{q} \tanh^2\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right)
\end{aligned}
$$

for $v = -p/(ra^2 - 1)$,

$$u_5(x,t) = \frac{(8ra^2 - 1)v + p}{q} + \frac{12a^2 rv}{q} \tanh^2\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right)$$

for $v = p/(4ra^2 + 1)$.

$$u_6(x,t) = -\frac{(8ra^2 - 1)v + p}{q} + \frac{12a^2 rv}{q}\tanh^2\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right)$$

for $v = -p/(4ra^2 + 1)$.

A particular form of $u_5(x,t)$ is depicted for various values of $\alpha \in \{0.25, 0.5, 0.75, 1\}$ in a finite domain in Fig. 1(a)–(d). It is observed that the propagation of the initial pulse propagates along the $x$-axis as time proceeds by preserving its shape and amplitude in all cases. $\alpha$ affects only propagation velocity. The propagation is faster in small times but later it gets slower when $\alpha$ is less than 1. $\alpha = 1$ choice gives a constant propagation velocity to the pulse, Fig. 1(d).

## 5. Solutions to the conformable time fractional mRLW equation

The traveling wave transform $u(x,t) \rightarrow U(\xi), \xi = a(x - vt^\alpha/\alpha)$ reduces the mRLW Eq. (2) to

$$(-av + ap)U + \frac{qa}{3}U^3 - rva^3 U'' = C \tag{26}$$

where $C$ is constant of integration. The balance between $U^3$ and $U''$ leads $s = 1$. Thus, the predicted solution takes the form

$$U(w) = A_0 + A_1 \cos w + B_1 \sin w \tag{27}$$

Substituting the predicted solution (27) into (26) gives

$$1/3\,(\cos(w(\xi)))^3 aqA_1^3\left(-a^3 vrB_1 + aqA_1^2 B_1\right)\sin(w(\xi))(\cos(w(\xi)))^2$$
$$+ (\cos(w(\xi)))^2 aqA_0 A_1^2 + \left(2a^3 vrA_1 + aqA_1 B_1^2\right)(\sin(w(\xi)))^2$$
$$\times \cos(w(\xi)) + 2\sin(w(\xi))\cos(w(\xi))aqA_0 A_1 B_1$$
$$+ \left(aqA_0^2 A_1 - avA_1 + apA_1\right)\cos(w(\xi)) - C - avA_0 + apA_0$$
$$+ 1/3\,aqA_0^3 + \left(a^3 vrB_1 + 1/3\,aqB_1^3\right)(\sin(w(\xi)))^3$$
$$+ (\sin(w(\xi)))^2 aqA_0 B_1^2 + \left(aqA_0^2 B_1 - avB_1 + apB_1\right)\sin(w(\xi)) = 0 \tag{28}$$

Following implementation of some trigonometric identities and some simplifications, we equate the coefficients of powers of $\sin, \cos$ and multiplications of them to zero to give the algebraic system of equations

$$\begin{aligned}
&aqA_0 B_1^2 - C - avA_0 + apA_0 + 1/3\,aqA_0^3 = 0\\
&aqA_0^2 B_1 - avB_1 + apB_1 + a^3 vrB_1 + 1/3\,aqB_1^3 = 0\\
&aqA_0 A_1^2 - aqA_0 B_1^2 = 0\\
&-2a^3 vrB_1 + aqA_1^2 B_1 - 1/3\,aqB_1^3 = 0\\
&-2a^3 vrA_1 - aqA_1 B_1^2 + 1/3\,aqA_1^3 = 0\\
&2\,aqA_0 A_1 B_1 = 0\\
&2a^3 vrA_1 + aqA_0^2 A_1 + aqA_1 B_1^2 - avA_1 + apA_1 = 0
\end{aligned} \tag{29}$$



(a) $\alpha = 0.25$

(b) $\alpha = 0.50$
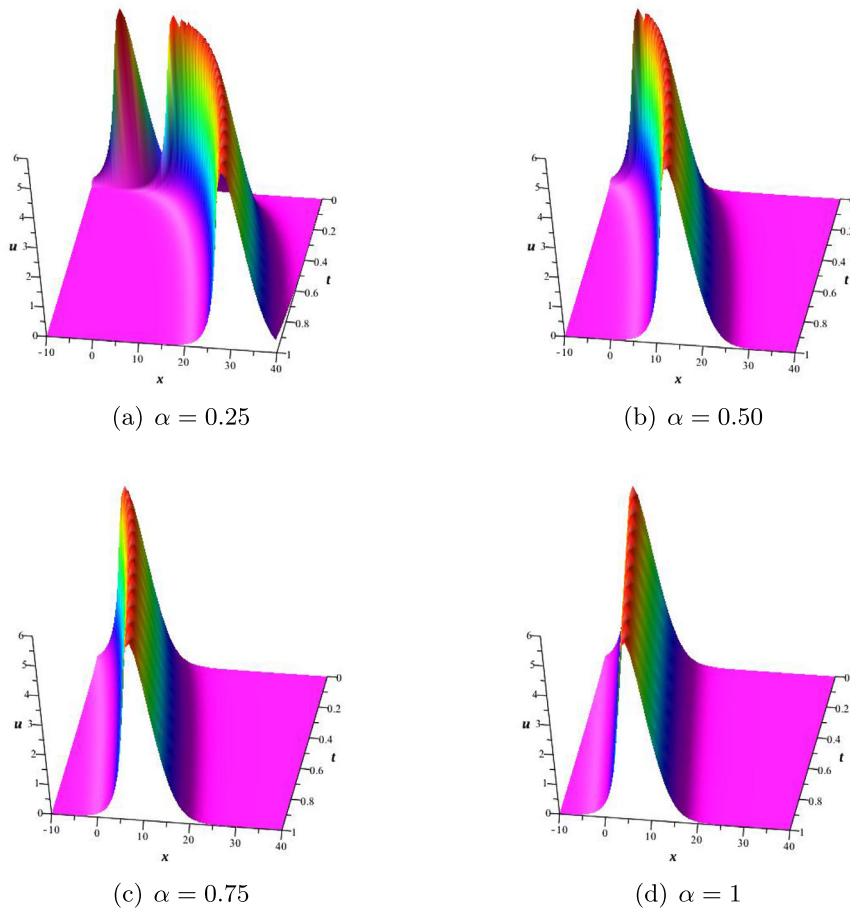
(c) $\alpha = 0.75$

(d) $\alpha = 1$

**Fig. 1.** The solution $u_5(x,t)$ for $\{a = 1/4, q = -1, r = 1, p = 10\}$.

The solution of this system for $v$, $A_0$, $A_1$, $B_1$ and $C$ forces $C$ to be zero. Thus, the solutions of this system can be summarized as

$$v = -\frac{2pq}{qra^2 - 2q}, \quad A_0 = 0, \quad A_1 = \sqrt{\frac{3rv}{2q}}a, \quad B_1 = \sqrt{\frac{-3rv}{2q}}a$$

$$v = -\frac{2pq}{qra^2 - 2q}, \quad A_0 = 0, \quad A_1 = \sqrt{\frac{3rv}{2q}}a, \quad B_1 = -\sqrt{\frac{-3rv}{2q}}a$$

$$v = -\frac{2pq}{qra^2 - 2q}, \quad A_0 = 0, \quad A_1 = -\sqrt{\frac{3rv}{2q}}a, \quad B_1 = \sqrt{\frac{-3rv}{2q}}a$$

$$v = -\frac{2pq}{qra^2 - 2q}, \quad A_0 = 0, \quad A_1 = -\sqrt{\frac{3rv}{2q}}a, \quad B_1 = -\sqrt{\frac{-3rv}{2q}}a$$

$$v = \frac{p}{ra^2 + 1}, \quad A_0 = 0, \quad A_1 = 0, \quad B_1 = \sqrt{\frac{6rv}{q}}a$$

$$v = \frac{p}{ra^2 + 1}, \quad A_0 = 0, \quad A_1 = 0, \quad B_1 = -\sqrt{\frac{6rv}{q}}a$$

$$v = -\frac{p}{2ra^2 - 1}, \quad A_0 = 0, \quad A_1 = \sqrt{\frac{-6rv}{q}}a, \quad B_1 = 0$$

$$v = -\frac{p}{2ra^2 - 1}, \quad A_0 = 0, \quad A_1 = -\sqrt{\frac{-6rv}{q}}a, \quad B_1 = 0$$

$$(30)$$

for arbitrary $a$. When $a$ is considered as non arbitrary, we have some more solutions such that

$$a = \sqrt{\frac{1}{r}}, \quad v = 2p, \quad A_0 = 0, \quad A_1 = \sqrt{\frac{3p}{q}}, \quad B_1 = \sqrt{\frac{-3p}{q}}$$

$$a = -\sqrt{\frac{1}{r}}, \quad v = 2p, \quad A_0 = 0, \quad A_1 = \sqrt{\frac{3p}{q}}, \quad B_1 = \sqrt{\frac{-3p}{q}}$$

$$a = \sqrt{\frac{1}{r}}, \quad v = 2p, \quad A_0 = 0, \quad A_1 = -\sqrt{\frac{3p}{q}}, \quad B_1 = \sqrt{\frac{-3p}{q}}$$

$$a = -\sqrt{\frac{1}{r}}, \quad v = 2p, \quad A_0 = 0, \quad A_1 = -\sqrt{\frac{3p}{q}}, \quad B_1 = \sqrt{\frac{-3p}{q}}$$

$$(31)$$

$$a = \sqrt{\frac{1}{r}}, \quad v = 2p, \quad A_0 = 0, \quad A_1 = \sqrt{\frac{3p}{q}}, \quad B_1 = -\sqrt{\frac{-3p}{q}}$$

$$a = -\sqrt{\frac{1}{r}}, \quad v = 2p, \quad A_0 = 0, \quad A_1 = \sqrt{\frac{3p}{q}}, \quad B_1 = -\sqrt{\frac{-3p}{q}}$$

$$a = \sqrt{\frac{1}{r}}, \quad v = 2p, \quad A_0 = 0, \quad A_1 = -\sqrt{\frac{3p}{q}}, \quad B_1 = -\sqrt{\frac{-3p}{q}}$$

$$a = -\sqrt{\frac{1}{r}}, \quad v = 2p, \quad A_0 = 0, \quad A_1 = -\sqrt{\frac{3p}{q}}, B_1 = -\sqrt{\frac{-3p}{q}}$$

Thus, the solutions to mRLW Eq. (2) are represented by

$$u_7(x,t) = \sqrt{\frac{3rv}{2q}}a\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) + \sqrt{\frac{-3rv}{2q}}a\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right),$$
$$v = -\frac{2pq}{qra^2 - 2q}$$

$$u_8(x,t) = \sqrt{\frac{3rv}{2q}}a\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) - \sqrt{\frac{-3rv}{2q}}a\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right),$$
$$v = -\frac{2pq}{qra^2 - 2q}$$

$$u_9(x,t) = -\sqrt{\frac{3rv}{2q}}a\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) + \sqrt{\frac{-3rv}{2q}}a\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right),$$
$$v = -\frac{2pq}{qra^2 - 2q}$$

$$u_{10}(x,t) = -\sqrt{\frac{3rv}{2q}}a\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) - \sqrt{\frac{-3rv}{2q}}a\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right),$$
$$v = -\frac{2pq}{qra^2 - 2q}$$

$$u_{11}(x,t) = \sqrt{\frac{6rv}{q}}a\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right), \quad v = \frac{p}{ra^2 + 1}$$

$$u_{12}(x,t) = -\sqrt{\frac{6rv}{q}}a\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right), \quad v = \frac{p}{ra^2 + 1}$$

$$u_{13}(x,t) = \sqrt{\frac{-6rv}{q}}a\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right), \quad v = -\frac{p}{2ra^2 - 1}$$

$$u_{14}(x,t) = -\sqrt{\frac{-6rv}{q}}a\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right), \quad v = -\frac{p}{2ra^2 - 1}$$

$$(32)$$

when $v$ is dependent on $a$. Whenever $v$ and $a$ are independent upon each other, the solution are represented by

$$u_{15}(x,t) = \sqrt{\frac{3p}{q}}\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) + \sqrt{\frac{-3p}{q}}\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right),$$
$$v = 2p, \quad a = \sqrt{\frac{1}{r}}$$

$$u_{16}(x,t) = \sqrt{\frac{3p}{q}}\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) + \sqrt{\frac{-3p}{q}}\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right),$$
$$v = 2p, \quad a = -\sqrt{\frac{1}{r}}$$

$$u_{17}(x,t) = -\sqrt{\frac{3p}{q}}\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) + \sqrt{\frac{-3p}{q}}\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right),$$
$$v = 2p, \quad a = \sqrt{\frac{1}{r}}$$

$$u_{18}(x,t) = -\sqrt{\frac{3p}{q}}\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) + \sqrt{\frac{-3p}{q}}\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right),$$
$$v = 2p, \quad a = -\sqrt{\frac{1}{r}}$$

$$u_{19}(x,t) = \sqrt{\frac{3p}{q}}\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) - \sqrt{\frac{-3p}{q}}\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right),$$
$$v = 2p, \quad a = \sqrt{\frac{1}{r}}$$

$$u_{20}(x,t) = \sqrt{\frac{3p}{q}}\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) - \sqrt{\frac{-3p}{q}}\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right),$$
$$v = 2p, \quad a = -\sqrt{\frac{1}{r}}$$

$$u_{21}(x,t) = -\sqrt{\frac{3p}{q}}\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) - \sqrt{\frac{-3p}{q}}\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right),$$
$$v = 2p, \quad a = \sqrt{\frac{1}{r}}$$

$$u_{22}(x,t) = -\sqrt{\frac{3p}{q}}\tanh\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right) - \sqrt{\frac{-3p}{q}}\operatorname{sech}\left(a\left(x - v\frac{t^\alpha}{\alpha}\right)\right),$$
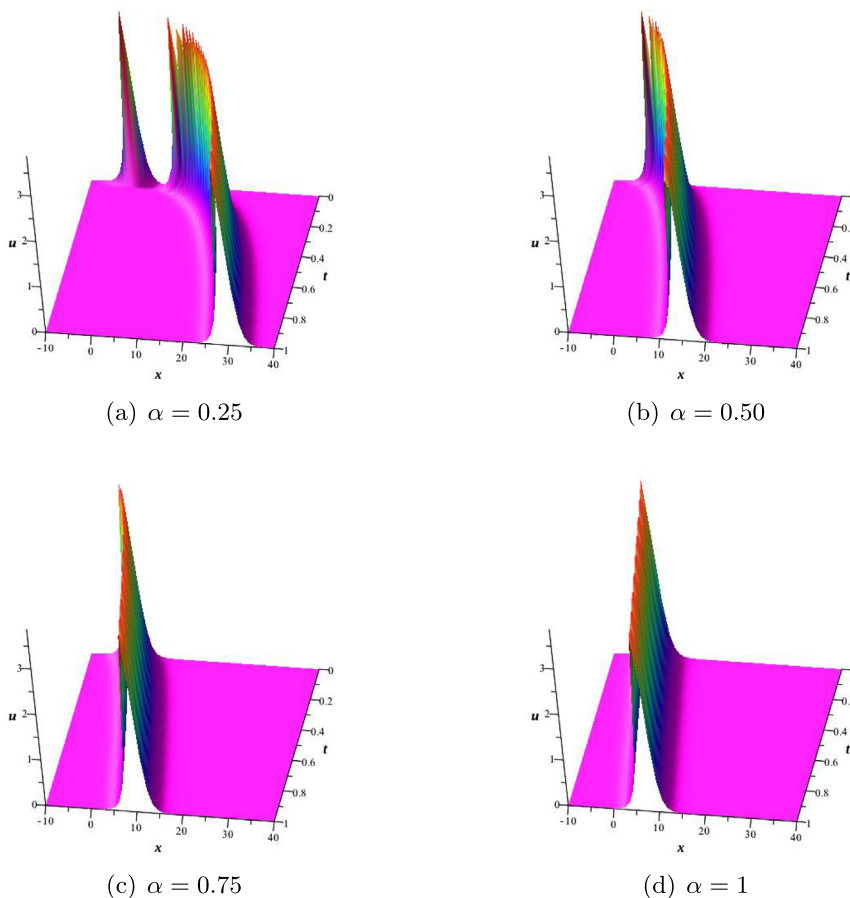$$v = 2p, \quad a = -\sqrt{\frac{1}{r}}$$

$$(33)$$

**Fig. 2.** The solution $u_{11}(x,t)$ for $\{a=1, q=-1, r=1/3, p=10\}$.

The solution $u_{11}(x,t)$ is simulated for various values of $\alpha$ in Fig. 2(a)–(d) by using the parameter set $\{a=1, q=-1, r=1/3, p=10\}$. The solution propagates along $x$-axis as time proceeds with preserved shape and amplitude. The propagation is not linear when $\alpha$ is less than 1, Fig. 2(a)–(c) as we observe a linear propagation with constant speed with $\alpha=1$, Fig. 2(d).

## 6. Solutions to the conformable time fractional sRLW equation

The traveling wave transform $u(x,t) \to U(\xi)$, $\xi = a(x - vt^\alpha/\alpha)$ reduces the sRLW Eq. (2) to

$$(a^2v^2 + a^2p)U' + rv^2a^4U''' - qva^2UU' = C \tag{34}$$

The balance between $U'''$ and $UU'$ gives $s=2$. Thus, we seek a solution of the form

$$U(w) = A_0 + A_1\cos w + B_1\sin w + B_2\cos w\sin w + A_2\cos^2 w \tag{35}$$

Substitution of this solution into (34) gives

$$a^4v^2rB_2\sin(w(\xi))(\cos(w(\xi)))^4$$
$$+ \left(-8a^4v^2rA_2 + 2a^2vqA_2^2 - a^2vqB_2^2\right)(\sin(w(\xi)))^2(\cos(w(\xi)))^3$$
$$+ (a^4v^2rB_1 - a^2vqA_1B_2 - a^2vqA_2B_1)\sin(w(\xi))(\cos(w(\xi)))^3$$
$$+ (-18a^4v^2rB_2 + 4a^2vqA_2B_2)(\sin(w(\xi)))^3(\cos(w(\xi)))^2$$
$$+ (-4a^4v^2rA_1 + 3a^2vqA_1A_2 - 2a^2vqB_1B_2)(\sin(w(\xi)))^2(\cos(w(\xi)))^2$$
$$+ (-a^2vqA_0B_2 - a^2vqA_1B_1 + a^2v^2B_2 + a^2pB_2)\sin(w(\xi))(\cos(w(\xi)))^2$$
$$+ \left(16a^4v^2rA_2 + a^2vqB_2^2\right)(\sin(w(\xi)))^4\cos(w(\xi))$$
$$+ (-5a^4v^2rB_1 + 2a^2vqA_1B_2 + 2a^2vqA_2B_1)(\sin(w(\xi)))^3\cos(w(\xi))$$
$$+ \left(2a^2vqA_0A_2 + a^2vqA_1^2 - a^2vqB_1^2 - 2a^2v^2A_2 - 2a^2pA_2\right)$$
$$\times (\sin(w(\xi)))^2\cos(w(\xi)) + (-a^2vqA_0B_1 + a^2v^2B_1 + a^2pB_1)$$
$$\times \sin(w(\xi))\cos(w(\xi)) + 5(\sin(w(\xi)))^5a^4v^2rB_2$$
$$+ (2a^4v^2rA_1 + a^2vqB_1B_2)(\sin(w(\xi)))^4$$
$$+ (a^2vqA_0B_2 + a^2vqA_1B_1 + a^2vqA_2B_2 - a^2v^2B_2 - a^2pB_2)(\sin(w(\xi)))^3$$
$$+ (a^2vqA_0A_1 - a^2v^2A_1 - a^2pA_1)(\sin(w(\xi)))^2 - a^2vqA_2B_2\sin(w(\xi)) - C = 0 \tag{36}$$

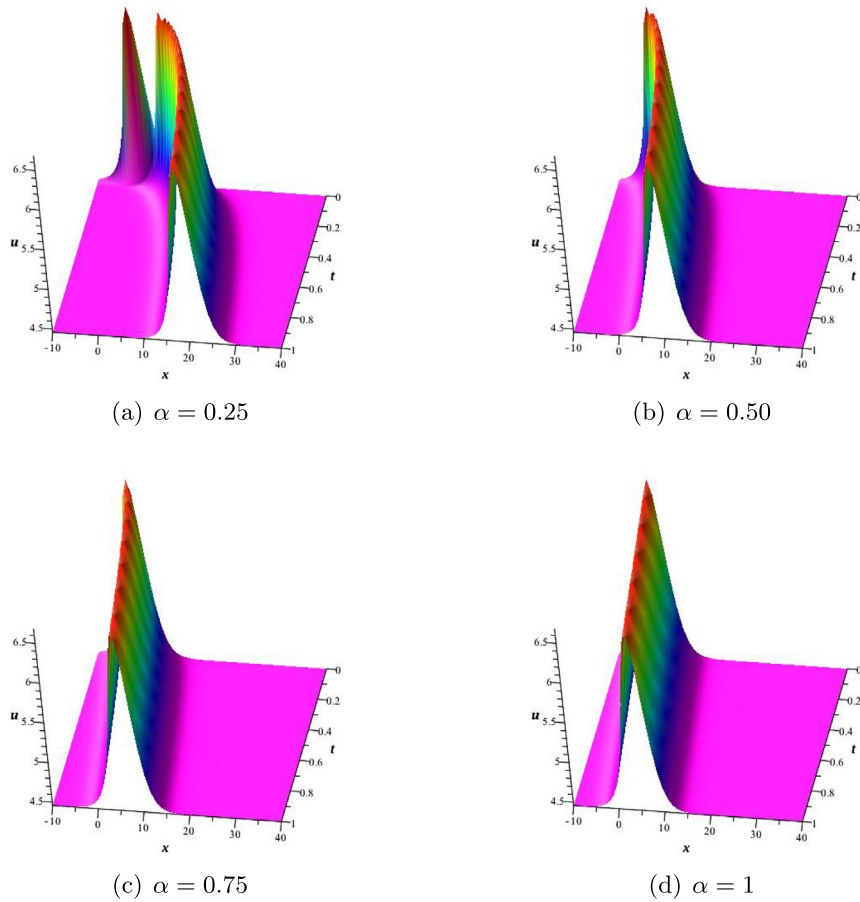**Fig. 3.** The solution $u_{25}(x,t)$ for $\{a=1, q=1, r=-1/3, p=1, v=5\}$.

Substitution of some trigonometric identities and some simplifications leads the following system of algebraic equations from the equating the coefficients of powers of cos, sin functions and their multiplications:

$$-3a^2v\left(2a^2vrA_1-qA_1A_2+qB_1B_2\right)+8a^4v^2rA_1+a^2vqA_0A_1-3a^2vqA_1A_2$$
$$+4a^2vqB_1B_2-a^2v^2A_1-a^2pA_1-C=0$$
$$5a^4v^2rB_2+a^2vqA_0B_2+a^2vqA_1B_1-a^2v^2B_2-a^2pB_2=0$$
$$24a^4v^2rB_2-4a^2vqA_2B_2=0$$
$$6a^4v^2rB_1-3a^2vqA_1B_2-3a^2vqA_2B_1=0$$
$$-28a^4v^2rB_2-2a^2vqA_0B_2-2a^2vqA_1B_1$$
$$+3a^2vqA_2B_2+2a^2v^2B_2+2a^2pB_2=0$$
$$-5a^4v^2rB_1-a^2vqA_0B_1+2a^2vqA_1B_2$$
$$+2a^2vqA_2B_1+a^2v^2B_1+a^2pB_1=0$$
$$a^2\left(24a^2v^2rA_2-2vqA_2^2+2vqB_2^2\right)=0$$
$$6a^4v^2rA_1-3a^2vqA_1A_2+3a^2vqB_1B_2=0$$
$$a^2\left(-16a^2v^2rA_2-2vqA_0A_2-vqA_1^2+vqB_1^2-vqB_2^2+2v^2A_2+2pA_2\right)$$
$$-a^2\left(24a^2v^2rA_2-2vqA_2^2+2vqB_2^2\right)=0$$
$$-14a^4v^2rA_1-a^2vqA_0A_1+6a^2vqA_1A_2-7a^2vqB_1B_2$$
$$+a^2v^2A_1+a^2pA_1+3a^2v\left(2a^2vrA_1-qA_1A_2+qB_1B_2\right)=0$$
$$-a^2\left(-16a^2v^2rA_2-2vqA_0A_2-vqA_1^2+vqB_1^2-vqB_2^2+2v^2A_2+2pA_2\right)=0$$

$$(37)$$

Solution of this system gives

$$A_0=-\frac{5a^2v^2r-v^2-p}{vq},\quad A_1=0,\quad A_2=\frac{12a^2vr}{q},\quad B_1=0, B_2=0$$
$$A_0=-\frac{5a^2v^2r-v^2-p}{vq},\quad A_1=0,\quad A_2=\frac{6a^2vr}{q},\quad B_1=0,\quad B_2=\frac{6a^2rvi}{q}$$
$$A_0=-\frac{5a^2v^2r-v^2-p}{vq},\quad A_1=0,\quad A_2=\frac{6a^2vr}{q},\quad B_1=0, B_2=-\frac{6a^2rvi}{q}$$

$$(38)$$

for arbitrarily chosen $v$, $a$ and $C=0$. One should note that the solution of this system is determinable whenever $C=0$. Thus, the solutions to (3) are constructed as

$$u_{23}(x,t)=-\frac{5a^2v^2r-v^2-p}{vq}+\frac{6a^2rv}{q}i\tanh\left(a(x-v\frac{t^\alpha}{\alpha})\right)\mathrm{sech}\left(a(x-v\frac{t^\alpha}{\alpha})\right)$$
$$+\frac{6a^2rv}{q}\tanh^2\left(a(x-v\frac{t^\alpha}{\alpha})\right)$$
$$u_{24}(x,t)=-\frac{5a^2v^2r-v^2-p}{vq}+\frac{6a^2rv}{q}i\tanh\left(a(x-v\frac{t^\alpha}{\alpha})\right)\mathrm{sech}\left(a(x-v\frac{t^\alpha}{\alpha})\right)$$
$$-\frac{6a^2rv}{q}\tanh^2\left(a(x-v\frac{t^\alpha}{\alpha})\right)$$
$$u_{25}(x,t)=-\frac{8a^2v^2r-v^2-p}{vq}+\frac{12a^2vr}{q}\tanh^2\left(a(x-v\frac{t^\alpha}{\alpha})\right)$$

$$(39)$$

The solution $u_{25}(x,t)$ is depicted in Fig. 3(a)–(d) for various $\alpha$ values in some finite domain of independent variables. The initial pulse propagates along the space axis without changing its shape and amplitude in all cases. $\alpha$ affects only propagation velocity due to being multiplier

of time variable. Even though the pulse propagates non linearly for smaller $\alpha$ values, the propagation is linear when $\alpha$ is 1.

## 7. Conclusion

In the paper, exact solutions of some conformable fractional equations in the RLW-class are investigated by using Sine-Gordon expansion approach. Using compatible wave transform, the equations are reduced to some ODEs. Then, the predicted solutions are substituted into the resultant ODE. Equating the coefficients of cosine and sine functions and their multiplications to zero leads to some algebraic system of equation. Solving this system gives the relations among the parameters. The method differs from many classical method since the terms in the series are multiplication of powers of different hyperbolic functions. In conclusion, some real and complex solutions that are combinations of powers of hyperbolic tangent and hyperbolic secant functions are determined explicitly. Graphical representations of some real valued solutions are depicted in some finite domains to comprehend the effects of $\alpha$.

## References

Abdeljawad, T., 2015. On conformable fractional calculus. J. Comput. Appl. Math. 279, 57–66.

Atangana, A., Baleanu, D., Alsaedi, A., 2015. New properties of conformable derivative. Open Math. 13 (1), 1–10.

Benjamin, T.B., Bona, J.L., Mahony, J.J., 1972. Model equations for long waves in nonlinear dispersive systems. Philos. Trans. R. Soc. London A: Math., Phys. Eng. Sci. 272 (1220), 47–78.

Gardner, L.R.T., Gardner, G.A., Ayoub, F.A., Amein, N.K., 1997. Approximations of solitary waves of the MRLW equation by B-spline finite element. Arab. J. Sci. Eng. 22, 183–193.

Guner, O., Bekir, A., Korkmaz, A., 2017a. Tanh-type and sech-type solitons for some space-time fractional PDE models. Eur. Phys. J. Plus 132 (92), 1–12.

Guner, O., Korkmaz, A., Bekir, A., 2017b. Dark solutions of space-time fractional Sharma-Tasso-Olver and potential Kadomtsev-Petviashvili Equations. Commun. Theor. Phys. 67 (2), 182–188.

Hosseini, K., Ansari, R., 2017. New exact solutions of nonlinear conformable time-fractional Boussinesq equations using the modified Kudryashov method. Waves Random Complex Media, 1–9.

Hosseini, K., Mayeli, P., Ansari, R., 2017a. Bright and singular soliton solutions of the conformable time-fractional Klein-Gordon equations with different nonlinearities. Waves Random Complex Media, 1–9.

Hosseini, K., Mayeli, P., Ansari, R., 2017b. Modified Kudryashov method for solving the conformable time-fractional Klein-Gordon equations with quadratic and cubic nonlinearities. Optik-Int. J. Light Electron Opt. 130, 737–742.

Karakoc, S.B.G., Geyikli, T., 2013. Petrov-Galerkin finite element method for solving the MRLW equation. Math. Sci. 7 (1), 25.

Karakoc, S.B.G., Yagmurlu, N.M., Ucar, Y., 2013. Numerical approximation to a solution of the modified regularized long wave equation using quintic B-splines. Boundary Value Prob. 2013 (1), 27.

Khalifa, A.K., Raslan, K.R., Alzubaidi, H.M., 2008. A collocation method with cubic B-splines for solving the MRLW equation. J. Comput. Appl. Math. 212 (2), 406–418.

Khalil, R., Al Horani, M., Yousef, A., Sababheh, M., 2014. A new definition of fractional derivative. J. Comput. Appl. Math. 264, 65–70.

Korkmaz, A., 2017a. Exact solutions of space-time fractional EW and modified EW equations. Chaos, Solitons Fractals 96, 132–138.

Korkmaz, A., 2017b. Exact solutions to (3 + 1) conformable time fractional Jimbo-Miwa, Zakharov-Kuznetsov and modified Zakharov-Kuznetsov equations. Commun. Theor. Phys. 67 (5), 479–482.

Korkmaz, A., 2018. Complex wave solutions to mathematical biology models I: Newell-Whitehead-Segel and Zeldovich equations 13 (8), 081004–1, 1.

Korkmaz, A., Hosseini, K., 2017. Exact solutions of a nonlinear conformable time-fractional parabolic equation with exponential nonlinearity using reliable methods. Opt. Quant. Electron. 49 (8), 278.

Kumar, D., Singh, J., Baleanu, D., 2017. A new analysis for fractional model of regularized long wave equation arising in ion acoustic plasma waves. Math. Methods Appl. Sci. 40 (15), 5642–5653.

Kumar, D., Singh, J., Baleanu, D., 2018a. Analysis of regularized long-wave equation associated with a new fractional operator with Mittag-Leffler type kernel. Physica A 492, 155–167.

Kumar, D., Singh, J., Baleanu, D., 2018b. A new numerical algorithm for fractional Fitzhugh-Nagumo equation arising in transmission of nerve impulses. Nonlinear Dyn. 91 (1), 307–317.

Ma, W.X., Fuchssteiner, B., 1996. Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation. Int. J. Non-Linear Mech. 31 (3), 329–338.

Ma, W.X., Lee, J.H., 2009. A transformed rational function method and exact solutions to the 3 + 1 dimensional Jimbo-Miwa equation. Chaos, Solitons Fractals 42 (3), 1356–1363.

Ma, W.X., Zhou, Y., 2018. Lump solutions to nonlinear partial differential equations via Hirota bilinear forms. J. Diff. Equ. 264 (4), 2633–2659.

Ma, W.X., Huang, T., Zhang, Y., 2010. A multiple exp-function method for nonlinear differential equations and its application. Phys. Scr. 82 (6). 065003.

Ma, W.X., Yong, X., Zhang, H.Q., 2018. Diversity of interaction solutions to the (2 + 1)-dimensional Ito equation. Comput. Math. Appl. 75, 289–295.

Mei, L., Chen, Y., 2012. Numerical solutions of RLW equation using Galerkin method with extrapolation techniques. Comput. Phys. Commun. 183 (8), 1609–1616.

Morrison, P.J., Meiss, J.D., Cary, J.R., 1984. Scattering of regularized-long-wave solitary waves. Physica D 11 (3), 324–336.

Peregrine, D.H., 1966. Calculations of the development of an undular bore. J. Fluid Mech. 25 (02), 321–330.

Rezazadeh, H., Korkmaz, A., Eslami, M., Vahidi, J., Asghari, R., 2018. Traveling wave solution of conformable fractional generalized reaction Duffing model by generalized projective Riccati equation method. Opt. Quant. Electron. 50 (3), 150.

Santarelli, A.R., 1978. Numerical analysis of the regularized long-wave equation: anelastic collision of solitary waves. Il Nuovo Cimento B (1971–1996) 46 (1), 179–188.

Seyler, C.E., Fenstermacher, D.L., 1984. A symmetric regularized-long-wave equation. Phys. Fluids (1958-1988) 27 (1), 4–7.

Singh, J., Kumar, D., Qurashi, M.A., Baleanu, D., 2017a. A novel numerical approach for a nonlinear fractional dynamical model of interpersonal and romantic relationships. Entropy 19 (7), 375.

Singh, J., Rashidi, M.M., Kumar, D., 2017b. A hybrid computational approach for Jeffery-Hamel flow in non-parallel walls. Neural Comput. Appl., 1–7

Soliman, A.M.A., 2017. Collocation method using quartic B-splines for solving the modified RLW equation. Indian J. Sci. Technol. 10 (31).

Yan, C., 1996. J. Comput. Nonlinear Dyn. 224, 77–84.

Yang, J.Y., Ma, W.X., Qin, Z., 2017. Lump and lump-soliton solutions to the (2 + 1)-dimensional Ito equation. Anal. Math. Phys., 1–10

Zhang, J.B., Ma, W.X., 2017. Mixed lump-kink solutions to the BKP equation. Comput. Math. Appl. 74 (3), 591–596.

Zhao, H.Q., Ma, W.X., 2017. Mixed lump-kink solutions to the KP equation. Comput. Math. Appl. 74 (6), 1399–1405.