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Original article

A reliable numerical method for solving fractional reaction-diffusion equations

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ABSTRACT

The present work aims to solve the fractional reaction–diffusion equation (RDE) using an effective and powerful hybrid analytical scheme, namely q-HASTM. The suggested technique is the combination of Sumudu transform (ST) and HAM technique. The definition of Caputo's fractional derivative has used. The numerical procedure reveals that only few iterations are needed for better approximation of the solution which illustrate the competence and sincerity of the suggested scheme. The impact of the reaction term in the solution of the problem explained through the graph.

1. Introduction

The word diffusion had been consequent of the Latin word "diffundere," which indicates diffused of a substance concerning area of high concentration to low concentration.

It is also well-known as Brownian motion. It was noticed by the Brownian who was the botanist and investigating that the movement of pollen grains under a microscope, and observing their unpredictable jitter in water (Brown, 1828). In 1905, Einstein established the relation between the traditional diffusion equation and Brownian motion.

The process of diffusion may be described in one of two ways, via Langevin's equation (Langevin and Acad, 1908) or a continuum approach. In 1885, Adolf Fick introduced Fick's first and second laws of diffusion (Philbert, 2005). The first law of diffusion describes the relationship between the flux of particles J and the spatial derivative of the probability distribution function of the particles. Fick's second law relates the time and special derivative of the probability density function (
PDF) and flux respectively.

The standard diffusion equation $\frac{\partial \mathfrak{U}}{\partial \xi} = D(\mathfrak{U})\frac{\partial^2 \mathfrak{U}}{\partial \eta^2}$, where $\mathfrak{U}(\xi,\eta)$ is the concentration of chemical, $D(\mathfrak{U})$is coefficients of diffusion. The fundamental solution of the linear diffusion equation can be elucidated as a PDF $\mathfrak{U}(\xi,\eta) = \frac{1}{\sqrt{4\pi D\eta}} exp\left(-\frac{\xi^2}{4D\eta}\right)$ Gaussian type in one dimension, such that the mean-squared displacement $\propto$ time η.

The RDEs are significant type of parabolic equations emerge in many scientific and engineering problems $\frac{\partial \mathfrak{U}}{\partial \xi} = D(\mathfrak{U})\frac{\partial^2 \mathfrak{U}}{\partial \eta^2} + f(\mathfrak{U})$ where $f(\mathfrak{U})$ represents the kinetics.

We know that this equation straightforwardly solved when $f(\mathfrak{U})$ is linear but it becomes more intricated when$f(\mathfrak{U})$ is nonlinear. Fisher equation (Fisher, 1937) is simple form of nonlinear reaction diffusion equation for$f(\mathfrak{U}) = \mathfrak{U}(1-\mathfrak{U})$. Feng (Feng, 2007) studied generalized form of Fisher equation by taking $f(\mathfrak{U}) = \mathfrak{U}(c + d\mathfrak{U} - e\mathfrak{U}^2)$; where c, d, e are real constants. However, it is very intricate to manage the state when both the terms kinetics and diffusion are nonlinear. Pablo and Vazquez (De Pablo and Vazquez, 1990; de Pablo and Vazquez, 1991) considered the strong reaction-slow diffusion equation. Pablo and Sanchez (De Pablo and Sanchez, 1998) analyzed the nature of fisher equation in permeable medium. Witelski (1994) discussed merging dynamics of substance with the help of the perturbation method. Timus and

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Srivastava (1997) suggested and investigated an alternative method to the Epstein-Hubbell generalized elliptic type integral for the energy behavior study of a class of nuclear reaction products.

FDEs have been new applications in areas of science and technology. Some essential results interrelated to solving differential equations with derivatives of fractional order (FO) may be initiate in Kilbas et al. (2006) have provided a very reliable and relevant discussion expressly on the subject of fractional calculus. The fractional differential equations became popular because the FO system eventually converges to the response system of integer-order. During past years the fractional reaction diffusion equations have been proved to be a decent model especially for numerous systems in the several streams of science, engineering and biology due to their practical relevance. The RDEs naturally express the various evolving problems in the physical realm: chemistry, physics, biomechanics, and other fields of knowledge. Generally, the nonlinear problems do not have an exact solution; substantially difficult to get it for the nonlinear problems with fractional order derivatives. So, approximate analytical methods will help determine these kinds of equations. Mainardi et al. (2001) explained the significant solution of time and space fractional diffusion equation. Wazwaz (2001) used the ADM to determine the diffusion equation's solution in the nonlinear case. Hao et al. (2013) examined the Helmholtz and diffusion equations on the Cantor sets involving local fractional derivative operators.

Yang et al. (2015) have been presented the non-differentiable diffusion equation by the local fractional similarity solution. Yang et al. (2016) employed a two-dimensional extended differential transform method for solving the local fractional diffusion equation.

Gao et al. (2016) proposed the coupling of the variational iteration method with the Sumudu transform through the local fractional calculus operator to solve the local fractional diffusion equation in one-dimensional fractal space.

Zhukovsky and Srivastava (2017) implemented an operational method, coupled with integral transforms and extended forms of orthogonal polynomials to achieve exact analytical solutions for heat diffusion equations beyond the Fourier model.

Yang et al. (2017) studied the Riemann-Liouville and Liouville-Caputo fractional time anomalous diffusion models with non-singular power-law kernel.

Mahto et al. (2019) investigated the approximate controllability of an impulsive system's fractional sub-diffusion equation by employing a unique continuation property.

Schot et al. (2007) obtained a solution of the diffusion equation approximately in terms of Fox H-function. Zahran (2009) have derived an exact solution of the generalized linear fractional reaction–diffusion equation in the form of Fox H-function. Silva et al. (2007) and Lenzi et al. (2009) solved a nonlinear diffusion equation that comprises spatial derivatives with fractional order. Das (2009) investigated the fractional diffusion equation's analytical solution in the appearance of outside force via VIM. Das and Gupta (2010) discussed the fractional diffusion equation in the presence of a reaction term and outside force via HPM. Singh et al. (2012) performed HPTM for handling fractional order RDEs. Kumar *et al.* (Kumar et al., 2015) analyzed the fractional-order multi-dimensional diffusion equations by the Laplace transform approach. Tripathi et al. (2016) obtained a solution of higher-order fractional time derivative RDE. In 2017, Tiwana et al. (2017) solved the Lotka-Volterra type fractional non-linear reaction diffusion system via HPTM. Zheng et al. (2017) solved numerically fractional order RDE with moving boundary conditions for the first time. Das et al. (2018) determined the numerical explication for the fractional-order advection-reaction diffusion equation.

Saad et al. (2020) used the numerical method based upon the Lagrange polynomial interpolation to estimate the approximate solutions to the fractional-order quadratic autocatalysis model with linear inhibition.

Singh et al. (2017) proposed a numerical algorithm for the fractional vibration equation solution.

Srivastava et al. (2019) proposed an appropriate novel method based on the Gegenbauer wavelet expansion, coupling with operational matrices of fractional integral and block-pulse functions to solve the Bagley–Torvik equation.

Srivastava et al. (2020) applied the Chebyshev spectral collocation method for approximate solutions of the Ebola virus model with fractional Liouville-Caputo derivative.

Singh and Srivastava (2020) studied numerical Simulation for fractional-order Bloch equation in nuclear magnetic resonance using the Jacobi polynomials.

Srivastava and Saad (2020a) investigated the numerical solutions of the fractal-fractional Ebola virus in the sense of three different kernels based on the power law, the exponential decay, and the generalized Mittag-Leffler function.

Srivastava and Saad (2020b) employed the spectral collocation method with the shifted Legendre polynomials, the two-stage fractional Runge–Kutta method, and the four-stage fractional Runge–Kutta method to find the numerical solutions of the fractional-order clock chemical model.

Das et al. (2011) discussed the explication of a fractional reaction-diffusion equation via HPM. S. J. Liao (Liao, 1995, 2003, 2004) proposed and developed an approximate analytical scheme HAM by adopting the concept of homotopy, which methodically studies the differential equations of nonlinear characteristics. The approach of various nonlinear problems applying HAM confirms its efficiency and effectiveness (Mohyud-Din et al., 2011; Onyejek, 2014; Song and Zhang, 2007; Tan and Abbasbandy, 2008).

Integral transform technique is extensively used and a lot of work has been done. In 1993, the Sumudu transform was developed by Watugala (1993) and applied to solve of ODE in control engineering problems. The fundamental properties of Sumudu transform are deliberate in Belgacem and Karaballi (2006), Chaurasia and Singh (2010).

The fusion scheme q-HASTM has been proposed by Singh et al. (2017). This scheme is a combination of Sumudu transform and q-HAM. El-Tawil and Huseen (2012, 2013) proposed the method q-HAM, the improvement in HAM with the simplification of the embedding parameter q. As we know that HAM comprises an auxiliary parameter $\hbar$ for control and adjustment of the region of convergence, whereas q-HAM contains an $\hbar$ and n in such a way that HAM is a particular instance of q-HAM for $n = 1$.

In this article the powerful mathematical tool q-HASTM is successfully applied to solve fractional order reaction diffusion equation in existence of reaction term $\mathscr{a}$

$$\frac{\partial^{\alpha}\mathfrak{U}}{\partial\eta^{\alpha}} = \frac{\partial}{\partial\xi}\left(\mathfrak{U}^{\lambda}\frac{\partial\mathfrak{U}}{\partial\xi}\right) + \mathscr{a}\mathfrak{U}(1-\mathfrak{U}^{\lambda}), 0 < \alpha \le 1 \tag{1}$$

$$\text{Subjecttocondition}\mathfrak{U}(\xi, 0) = \mathscr{g}(\xi) \tag{2}$$

2. Preliminaries and notations

This section defines the essential definition and characteristics of fractional derivatives, integrals and the ST operator.

Definition 2.1. (*Kilbas et al., 2006*). *A function* $f(\eta) \in \mathscr{C}_{\alpha}, \alpha \in \Re$ *if* $\exists$ *a number* $\mathfrak{p} \in \Re, (\mathfrak{p} > \alpha)$ *s.t.* $f(\eta) = \eta^{\mathfrak{p}} f_1(\eta)$, *where* $f_1(\eta) \in C[0, \infty)$ *and* $f(\eta) \in \mathscr{C}_{\alpha}^{m}$ *if* $f^{(m)} \in \mathscr{C}_{\alpha}$, $m \in \mathbb{N} \cup \{0\}$.

Definition 2.2. *Liouvile-Caputo derivative of fractional order αis describe as*

$$\mathscr{D}_t^\alpha f(\eta)=\mathscr{J}_\eta^{m-\alpha}\mathscr{D}_\eta^m f(\eta)=\frac{1}{\Gamma(m-\alpha)}\int_0^\eta(\eta-\ell)^{m-\alpha-1}f^m(\ell)d\ell, m\in Z^+$$

we have

$$\mathscr{D}_t^\alpha\eta^\ell=0,\ell<\alpha$$

$$\mathscr{D}_t^\alpha\eta^l=\frac{\Gamma(\ell+1)}{\Gamma(\ell-\alpha+1)}\eta^{\ell-\alpha},\ell\geq\alpha.$$

Definition 2.3. *The operator $\mathscr{J}_\eta^\alpha$ of fractional order α of the function $f\in\mathscr{C}_\mu,\mu\geq-1$ in Riemann Liouville (RL) sense is described as*

$$\mathscr{J}_\eta^\alpha f(\eta)=\frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}f(\tau)d\tau$$

we have

$$\mathscr{J}_\eta^\alpha\eta^l=\frac{\Gamma(l+1)}{\Gamma(l-\alpha+1)}\eta^{l+\alpha},l\geq\alpha,\alpha>0$$

Definition 2.4.. *Consider a set of functions*

$$\mathscr{A}=\left\{f(\eta)|\exists\mathscr{M},\zeta_1,\zeta_2>0,|f(\eta)|<Me^{-\frac{|t|}{\zeta_i}}\text{if}\eta\in(-1)^j\times[0,\infty)\right\}$$

where ST described as,

$$\mathscr{S}[f(\eta)]=\mathscr{F}[u]=\int_0^\infty e^{-\eta}f(u\eta)d\eta,u\in(\zeta_1,\zeta_2)$$

Definition 2.5.. *The ST of derivative of fractional order in Caputo' form is*

$$\mathscr{S}\left[D_\eta^\alpha f(\eta)\right]=u^{-\alpha}\mathscr{S}[f(\eta)]-\sum_{k=0}^{m-1}u^{-(\alpha+k)}f^{(k)}(0+),m-1<\alpha\leq m$$

and $\mathscr{S}[1]=1,\mathscr{S}\left[\frac{\eta^{n-1}}{\Gamma(n)}\right]=u^{n-1},n>0.$

3. Solution of problem by q-HASTM

Applying ST operator on Eq. (1), and employing fractional derivative of ST, we obtain

$$u^{-\alpha}\mathscr{S}[\mathfrak{U}(\xi,\eta)]-u^{-\alpha}\mathfrak{U}(\xi,0)-\mathscr{S}\left[\frac{\partial}{\partial\xi}\left(\mathfrak{U}^\lambda\frac{\partial\mathfrak{U}}{\partial\xi}\right)\right]-a\mathscr{S}[\mathfrak{U}(\xi,\eta)]+a\mathscr{S}\left[\mathfrak{U}^{\lambda+1}\right]=0 \tag{3}$$

On simplifying Eq. (3), we obtain

$$\mathscr{S}[\mathfrak{U}(\xi,\eta)]-\mathfrak{U}(\xi,0)-u^\alpha\mathscr{S}\left[\frac{\partial}{\partial\xi}\left(\mathfrak{U}^\lambda\frac{\partial\mathfrak{U}}{\partial\xi}\right)\right]-au^\alpha\mathscr{S}[\mathfrak{U}]+au^\alpha\mathscr{S}\left[\mathfrak{U}^{\lambda+1}\right]=0 \tag{4}$$

Now, we state an operator in nonlinear form

$$\begin{aligned}\mathscr{N}[\varphi(\xi,\eta;q)]=&\mathscr{S}[\mathfrak{U}(\xi,\eta)]-\mathfrak{U}(\xi,0)\\&-u^\alpha\mathscr{S}\left[\frac{\partial}{\partial\xi}\left((\varphi(\xi,\eta;q))^\lambda\frac{\partial}{\partial\xi}\varphi(\xi,\eta;q)\right)\right]\\&-au^\alpha\mathscr{S}[\varphi(\xi,\eta;q)]+au^\alpha\mathscr{S}\left[(\varphi(\xi,\eta;q))^{\lambda+1}\right]\end{aligned} \tag{5}$$

Now, construct the homotopy as

$$(1-nq)\mathscr{S}[\varphi(\xi,\eta,q)-\mathfrak{U}_0(\xi,\eta)]=\hbar\mathscr{H}(\eta)\mathscr{N}[\varphi(\xi,\eta,q)] \tag{6}$$

where $\mathscr{H}(\eta)\neq0$ denote the auxiliary function,

hence q increase from zero to $\frac{1}{n}$ then the solution changes from the primary guess to the desired solution.

Now expansion $\varphi(\xi,\eta;q)$ in the form of Taylor's series w.r.t. q, we have

$$\varphi(\xi,\eta;q)=\mathfrak{U}_0(\xi,\eta)+\sum_{m=1}^\infty q^m\mathfrak{U}_m(\xi,\eta) \tag{7}$$

We can control convergence of above series by adjusting value of $\hbar$. If the primary guess, the auxiliary linear operator, $\hbar$ and $\mathscr{H}(t)$ are appropriately selected, then series (7) at $q=\frac{1}{n}$ is

$$\mathfrak{U}(\xi,\eta)=\mathfrak{U}_0(\xi,\eta)+\sum_{m=1}^\infty\mathfrak{U}_m(\xi,\eta)\left(\frac{1}{n}\right)^m \tag{8}$$

Define the vectors

$$\overrightarrow{\mathfrak{U}_m}=\{\mathfrak{U}_0,\mathfrak{U}_1,\mathfrak{U}_2,\mathfrak{U}_3,\cdots,\mathfrak{U}_n\} \tag{9}$$

Differentiating equation (6) m times w. r. t. q and then taking $q=0$ and divide them by $m!$, we obtain the deformation equation of mthorderis

$$\mathscr{S}\left[\mathfrak{U}_m(\xi,\eta)-\chi_m\mathfrak{U}_{m-1}(\xi,\eta)\right]=\hbar\mathscr{H}(\eta)\mathscr{R}_m\left[\overrightarrow{\mathfrak{U}_{m-1}}(\xi,\eta)\right] \tag{10}$$

Operating inverse ST in Eq. (10) and setting $q=1,\mathscr{H}(t)=1$, we have

$$\mathfrak{U}_m(\xi,\eta)=\chi_m\mathfrak{U}_{m-1}(\xi,\eta)+\hbar\mathscr{S}^{-1}\left(\mathscr{R}_m\left[\overrightarrow{\mathfrak{U}_{m-1}}(\xi,\eta)\right]\right) \tag{11}$$

Where,

$$\chi_m=\begin{cases}0,m\leq1\\n,m>1\end{cases}\text{and}\mathscr{R}_m\left[\overrightarrow{\mathfrak{U}_{m-1}}(\xi,\eta)\right]=\frac{1}{(m-1)!}\left(\frac{\partial\varphi(\xi,\eta,q)}{\partial q^{m-1}}\right)_{q=0} \tag{12}$$

So,

$$\begin{aligned}\mathscr{R}_m\left[\overrightarrow{\mathfrak{U}_{m-1}}(\xi,\eta)\right]=&\mathscr{S}[\mathfrak{U}_{m-1}(\xi,\eta)]-\mathfrak{U}_0(\xi,\eta)\left(1-\frac{\chi_m}{n}\right)\\&-u^\alpha\left[\mathscr{S}\left[B'_{m-1}\right]+a\mathscr{S}(u_{m-1}(\xi,\eta))-a\mathscr{S}\left(\left[B''_{m-1}\right]\right)\right]\end{aligned} \tag{13}$$

Where,

$$B'_m(\xi,\eta)=\frac{1}{\Gamma m}\left[\frac{\partial^m}{\partial q^m}\frac{\partial}{\partial\xi}\left((\varphi(\xi,\eta;q))^\lambda\frac{\partial}{\partial\xi}\varphi(\xi,\eta;q)\right)\right]_{q=0} \tag{14}$$

$$\text{and}B''_m(\xi,\eta)=\frac{1}{\Gamma m}\left[\frac{\partial^m}{\partial q^m}(\varphi(\xi,\eta;q))^{\lambda+1}\right]_{q=0} \tag{15}$$

Now, we obtain the solution of Eq. (11) is

$$\mathfrak{U}_1(\xi,\eta)=\hbar\mathscr{S}^{-1}\left[\mathscr{S}[\mathfrak{U}_0]-g(\xi)-u^\alpha\mathscr{S}\left[\frac{\partial}{\partial\xi}\left((\mathfrak{U}_0)^\lambda\frac{\partial\mathfrak{U}_0}{\partial\xi}\right)\right]-au^\alpha\mathscr{S}[\mathfrak{U}_0]+au^\alpha\mathscr{S}\left[(\mathfrak{U}_0)^{\lambda+1}\right]\right] \tag{16}$$

$$\begin{aligned}U_2(\xi,\eta)=&nU_1+\hbar S^{-1}[S[U_1]-g(\xi)\\&-u^\alpha S\left[\frac{\partial}{\partial\xi}\left({}^\lambda C_1(U_0)^{\lambda-1}U_1\frac{\partial U_0}{\partial\xi}+(U_0)^\lambda\frac{\partial U_1}{\partial\xi}\right)\right]-au^\alpha S[U_1]\\&+au^\alpha S\left[{}^{\lambda+1}C_1(U_0)^\lambda U_1\right]]\end{aligned} \tag{17}$$

$$U_3(\xi,\eta) = nU_2 + \hbar S^{-1}\Big[S[U_2] - g(\xi) - a^{\alpha}S\Big[\frac{\partial}{\partial\xi}\Big({}^{\lambda}C_1(U_0)^{\lambda-1}U_2\frac{\partial U_0}{\partial\xi} + {}^{\lambda}C_2(U_0)^{\lambda-2}(U_1)^2\frac{\partial U_0}{\partial\xi}\Big)\Big] + {}^{\lambda}C_1(U_0)^{\lambda-1}U_1\frac{\partial U_1}{\partial\xi} + (U_0)^{\lambda}\frac{\partial U_2}{\partial\xi} - a^{\alpha}S\Big[{}^{\lambda+1}C_1(U_0)^{\lambda}U_2 - {}^{\lambda+1}C_2(U_0)^{\lambda-1}(U_1)^2\Big]\Big] \tag{18}$$

Thus,

$$\mathfrak{U}_0(\xi,\eta) = g(\xi) \tag{19}$$

$$\mathfrak{U}_1(\xi,\eta) = -n\,\hbar\left[a g(\xi) - a(g(\xi))^{\lambda+1} + \lambda\ (g(\xi))^{\lambda-1}(g'(\xi))^2 + \ (g(\xi))^{\lambda}\ g''(\xi)\right]\frac{\eta^{\alpha}}{\Gamma\alpha+1} \tag{20}$$

$$\mathfrak{U}_2(\xi,\eta) = -(n+\hbar)\hbar\Big[a g(\xi) - a\big(g(\xi)\big)^{\lambda+1} + \lambda\big(g(\xi)\big)^{\lambda-1}\big(g'(\xi)\big)^2 + \big(g(\xi)\big)^{\lambda}g''(\xi)\Big]\frac{\eta^{\alpha}}{\Gamma(\alpha+1)}$$
$$+\hbar^2\Big[a^2 g(\xi) - a^2(2\lambda+3)\big(g(\xi)\big)^{\lambda+1} + 2a^2(\lambda+1)\big(g(\xi)\big)^{2\lambda+1} + a\lambda(\lambda+2)\big(g(\xi)\big)^{\lambda-1}\big(g'(\xi)\big)^2 - 2a\lambda(3\lambda+2)\big(g(\xi)\big)^{2\lambda-1}\big(g'(\xi)\big)^2 - 2\lambda(3\lambda-1)\big(g(\xi)\big)^{2\lambda-3}\big(g'(\xi)\big)^4 + a(\lambda+2)\big(g(\xi)\big)^{\lambda}g''(\xi) - a(4\lambda+3)\big(g(\xi)\big)^{2\lambda}g''(\xi) + 7\lambda(2\lambda-1)\big(g(\xi)\big)^{2\lambda-2}\big(g'(\xi)\big)^2 g''(\xi) + 4\lambda\big(g(\xi)\big)^{2\lambda-1}\big(g''(\xi)\big)^2 + 6\lambda\big(g(\xi)\big)^{2\lambda-1}g'(\xi)g''(\xi) + \big(g'(\xi)\big)^{2\lambda}g'^{v}(\xi)\Big]\frac{\eta^{2\alpha}}{\Gamma(2\alpha+1)} \tag{21}$$

$$\mathfrak{U}_3(\xi,\eta) = \hbar\Big[(n+\hbar)^2 - a\big(g(\xi)\big)^2 + a\big(g(\xi)\big)^{\lambda+2} - \lambda\Big\{\lambda(\lambda-1) + (n+\hbar)^2\big(g(\xi)\big)^{\lambda} - \lambda(\lambda-1) + (n+\hbar)^2\big(g(\xi)\big)^{\lambda}g''^{(\xi)}\Big\}\Big]\frac{\eta^{\alpha}}{\Gamma(\alpha+1)}$$
$$+\hbar^2(n+\hbar)\big(g(\xi)\big)^{\lambda-3}\Big[a(1+\lambda)\big(g(\xi)\big)^2\big(a\big(g(\xi)\big)^2 + 3\lambda\big(g'(\xi)\big)^2 + 3g(\xi)g''(\xi) + \big(g(\xi)\big)^{\lambda}\big(a^2(\lambda+1)\big(g(\xi)\big)^4 + 6\lambda(\lambda-1)(2\lambda-1) + \big(g'(\xi)\big)^4 + 21\lambda(2\lambda-1)g(\xi)\big(g'(\xi)\big)^2 g''(\xi) + \lambda\big(g(\xi)\big)^2\big(-a(7+13\lambda)\big(g'(\xi)\big)^2 + 12\big(g''(\xi)\big)^2 + 18g'(\xi)g''(\xi) + \big(g(\xi)\big)^3\big(-a(4+7\lambda)g''(\xi) + 3g'^{v}(\xi)\big)\big)\big)\Big]\frac{\eta^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+\hbar^3\Big[-a^3 g(\xi) + a^3(4+3\lambda)\big(g(\xi)\big)^{\lambda+1} - a^3(\lambda+1)(5+2\lambda)\big(g(\xi)\big)^{2\lambda+1} + 2a^3(\lambda+1)^2\big(g(\xi)\big)^{3\lambda+1} + a^2\lambda^2\big(g(\xi)\big)^{\lambda-1}\big(g'(\xi)\big)^2 + 2a\lambda(4\lambda^3-7\lambda+3)\big(g(\xi)\big)^{2\lambda-3}\big(g'(\xi)\big)^4 + 12\lambda(\lambda-1)^2(4+\lambda(6\lambda-11))\big(g(\xi)\big)^{3\lambda-5}\big(g'(\xi)\big)^6\Big]$$

$$+\hbar^3[a^2\lambda\big(g(\xi)\big)^{\lambda}g''(\xi) + 7a\lambda(4\lambda-3)(\lambda+1)\big(g(\xi)\big)^{2\lambda-2}\big(g'(\xi)\big)^2 g''(\xi) + 6\lambda(\lambda-1)(2\lambda-1)(39\lambda-32)\big(g(\xi)\big)^{3\lambda-4}\big(g'(\xi)\big)^4 g''(\xi) - a\lambda\big(g(\xi)\big)^{2\lambda-1}\big(a(3\lambda+1)(5\lambda+6)\big(g'(\xi)\big)^2$$

$$-4(2\lambda+3)\big(g''(\xi)\big)^2 - 6(2\lambda+3)g'(\xi)g''(\xi) - 2\lambda\big(g(\xi)\big)^{3\lambda-3}\big(g'(\xi)\big)^2\big(a(7+\lambda(-22+\lambda(-17+52\lambda)))\big(g'(\xi)\big)^2 + (-102+(353-294\lambda)\lambda)\big(g'(\xi)\big)^2 - 2(24+11\lambda)(7\lambda-8)\big)g'(\xi)g''(\xi) - a\big(g(\xi)\big)^{2\lambda}\big(a(1+\lambda(9+7\lambda))g''(\xi) - (3+2\lambda)g'^{v}(\xi) - \lambda\big(g(\xi)\big)^{3\lambda-2}\big(4(9-20\lambda)\big(g''(\xi)\big)^3 + 4(38-93\lambda)$$

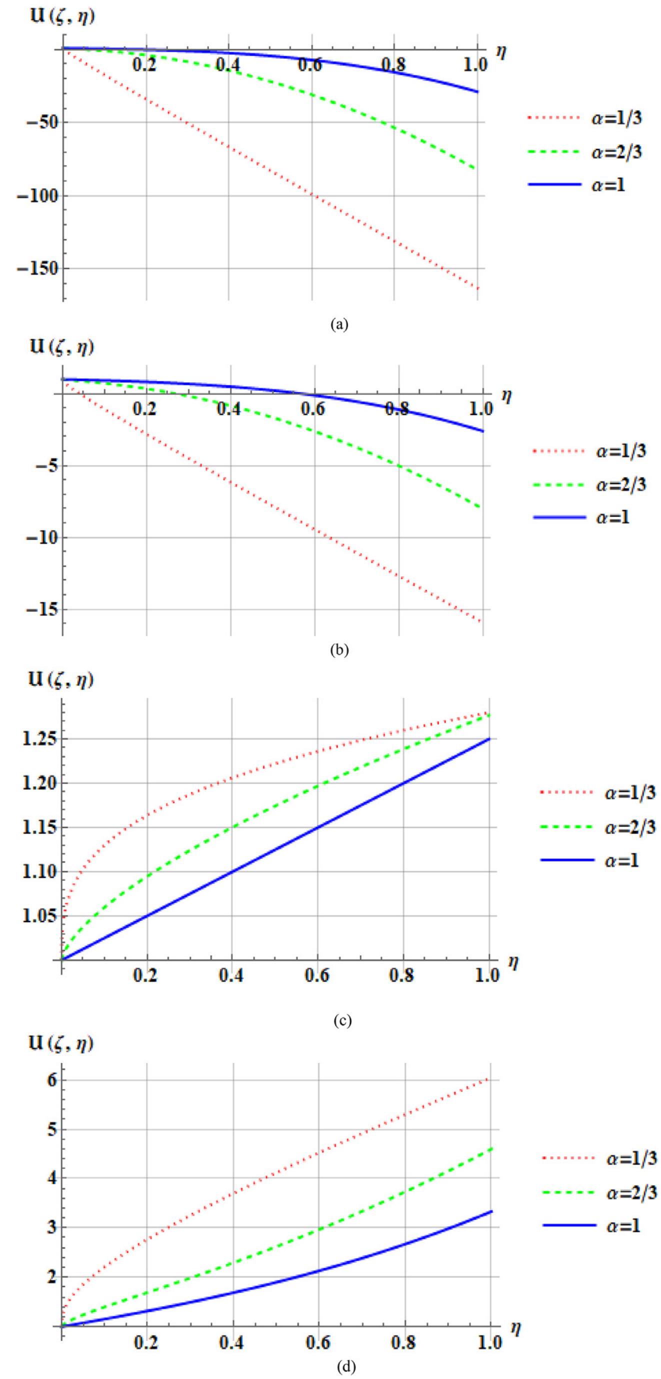

Fig. 1. Plots of $\mathfrak{U}(\xi,\eta)$ vs. $\eta$when (a) $\lambda=-2$ (b) $\lambda=-1$ (c) $\lambda=1(d)\lambda=2$at $\xi=1$, $a=0$for different values of $\alpha=\frac{1}{3},\frac{2}{3},1$.

$$g'(\xi)g''(\xi)g'''(\xi) + \big(g'(\xi)\big)^2\big(a(-51+\lambda(83+238\lambda))g''(\xi) + 2(22-59\lambda)g'^{v}(\xi)\big) + 2\lambda\big(g(\xi)\big)^{3\lambda-1}\big(a^2(\lambda+1)(4+15\lambda)\big(g'(\xi)\big)^2 + 14\big(g''(\xi)\big)^2 + g'''(\xi)\big(-a(15+22\lambda)g''(\xi) + 23g'^{v}(\xi)\big) + g'(\xi)\big(-a(15+22\lambda)g''(\xi)$$

$$(-a(23+33\lambda)g''(\xi)+12g'^{v}(\xi)))+(g(\xi))^{3\lambda}(a^2(\lambda+1)(8\lambda+1)g''(\xi) - a(5+7\lambda)g'^{v}(\xi)+2g'^{v}(\xi))]\frac{\eta^{3\alpha}}{\Gamma(3\alpha+1)}+\lambda\hbar^3(g(\xi))^{\lambda-5}[a^3(\lambda+1)(g(\xi))^6-2a^3(\lambda+1)(g(\xi))^{\lambda+6}+a^3(\lambda+1)(g(\xi))^{6+2\lambda}+2a^2(\lambda-1)\lambda$$

$$(g(\xi))^4(g'(\xi))^2+4a(\lambda-1)^2\lambda^2(g(\xi))^{\lambda+2}(g'(\xi))^4+2(\lambda-1)^2\lambda^2(3\lambda-4)(g(\xi))^{2\lambda}(g'(\xi))^6+2a^2(\lambda-1)(g(\xi))^5g''(\xi)+2a(\lambda^2+(9\lambda-10))(g(\xi))^{\lambda+3}(g'(\xi))^2g''(\xi)+2(\lambda-1)^2\lambda(17\lambda-3)(g(\xi))^{2\lambda+1}$$

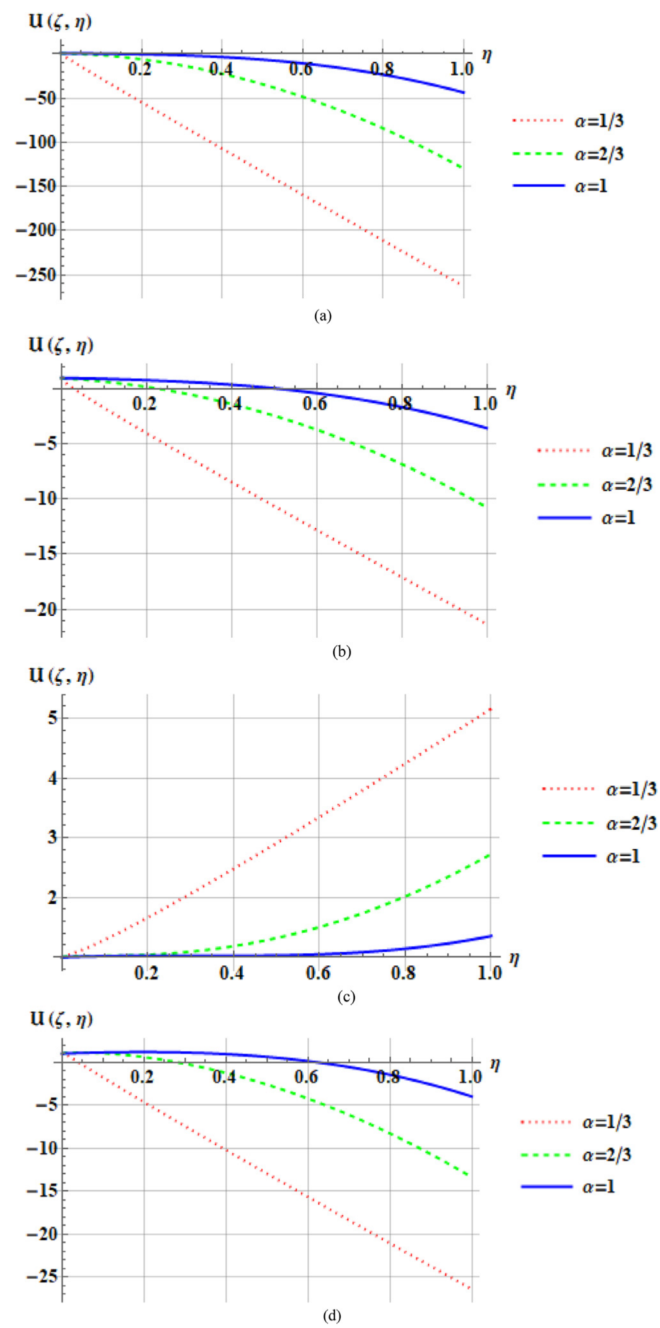

Fig. 2. Plots of $\mathfrak{U}(\xi,\eta)$ vs. ηonce (a) $\lambda=-2$ (b) $\lambda=-1$ (c) $\lambda=1$(d)$\lambda=2$at $\xi=1$, $a=1$for different values of $\alpha=\frac{1}{3},\frac{2}{3},1$.

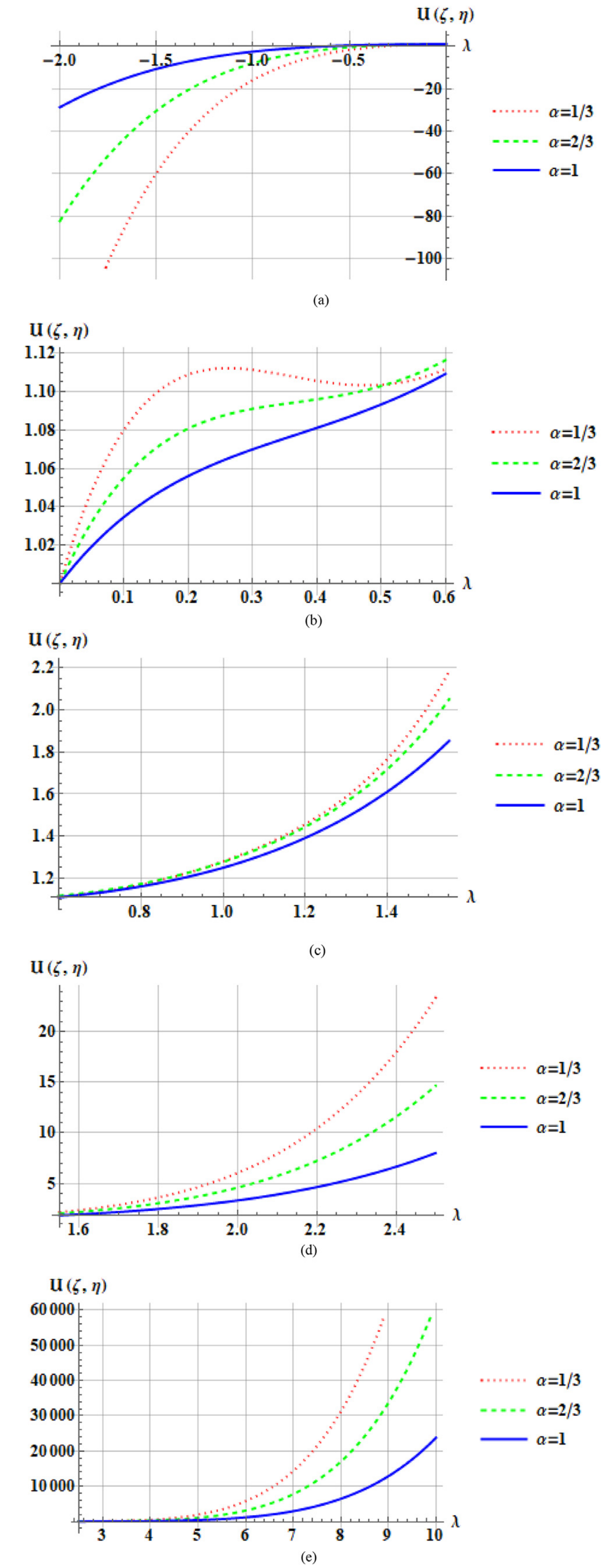

Fig. 3. Plots of $\mathfrak{U}(\xi,\eta)$ vs. λ once (a) $-2\le\lambda\le0$ (b) $0\le\lambda\le0.6$ (c) $0.6\le\lambda\le1.55$ (d) $1.55\le\lambda\le2.5$ (e) $2.5\le\lambda\le10$ at $\xi=1$, $\eta=1$,$a=0$for various values of $\alpha=\frac{1}{3},\frac{2}{3},1$.

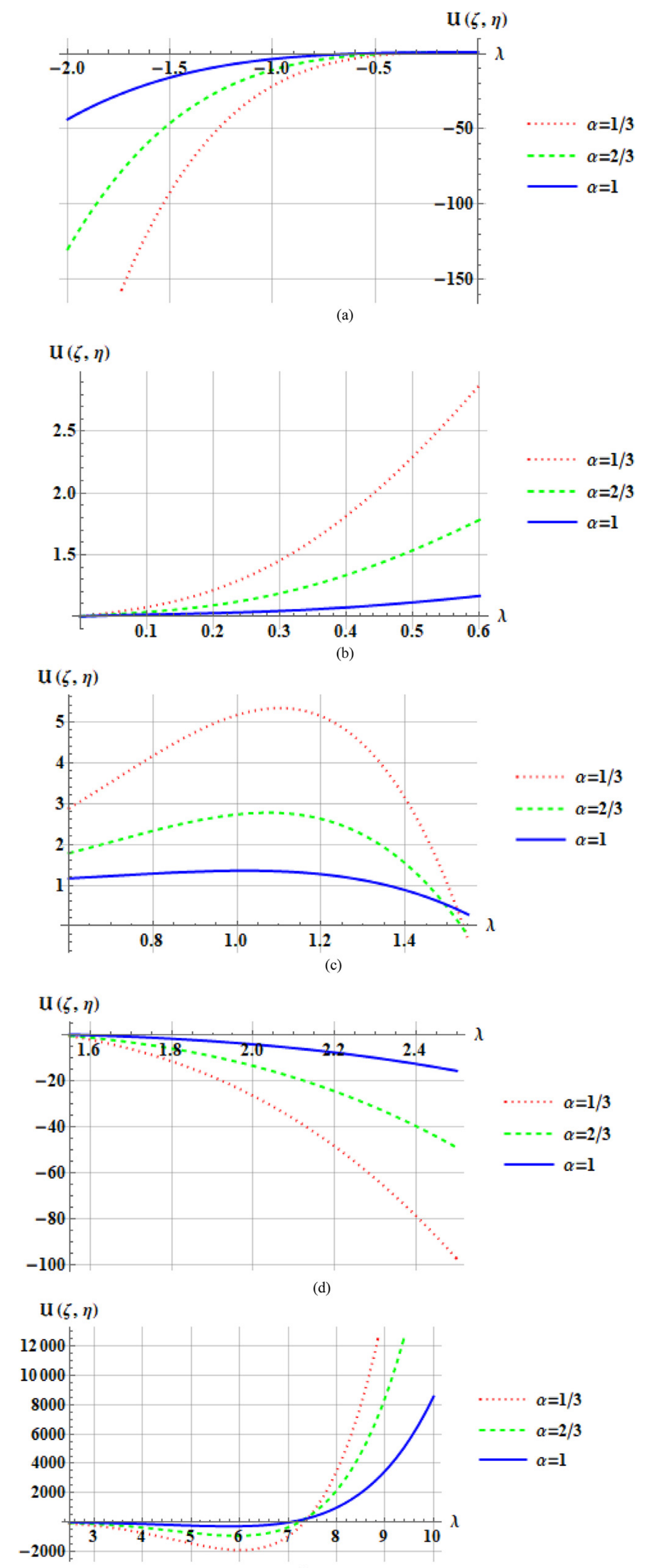

Fig. 4. Plots of $\mathfrak{U}(\xi,\eta)$vs. λ when (a) $-2 \le \lambda \le 0$ (b) $0 \le \lambda \le 0.6$ (c) $0.6 \le \lambda \le 1.55$ (d) $1.55 \le \lambda \le 2.5$ (e) $2.5 \le \lambda \le 10$at $\xi = 1,\ \eta = 1, a = 1$for various values of $\alpha = \frac{1}{3}, \frac{2}{3}, 1$.

$$(g(\xi))^4 g''(\xi) - \lambda (g(\xi))^{2\lambda+2} (g'(\xi))^2 (a\lambda(7 + 3\lambda(4\lambda - 7)(g'(\xi))^2 - 6(\lambda - 1)(7\lambda - 3)$$

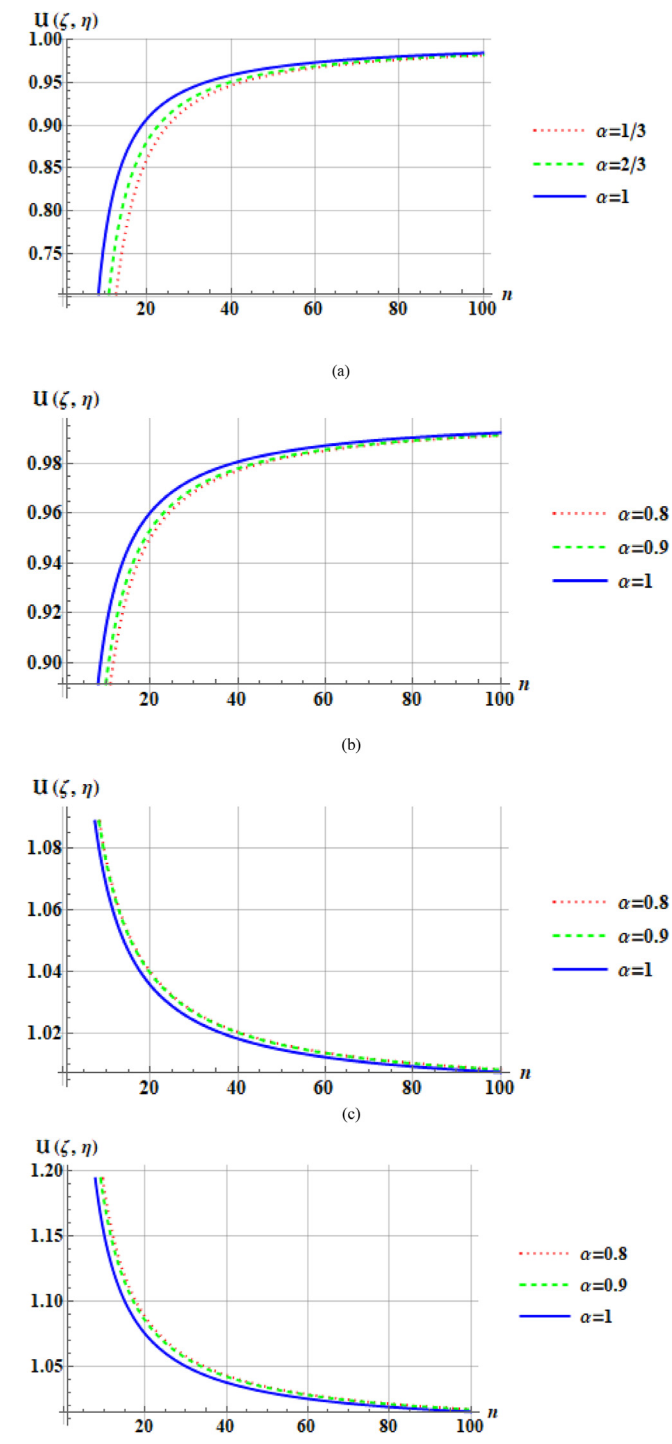

Fig. 5. Plots of n -curves when (a) $\lambda = -2$ (b) $\lambda = -1$ (c) $\lambda = 1$ (d) $\lambda = 2$at$\xi = 1, \eta = 1,\ a = 0$for different values of.$\alpha$

$$(g''(\xi))^2 - 2(\lambda - 1)(7\lambda - 3) g'(\xi) g''(\xi) - 2a(g(\xi))^{2\lambda+5}(a(\lambda - 2)(\lambda + 1)g''(\xi) - (\lambda - 1)g''(\xi)) - 2a(g(\xi))^{\lambda+5}(a(\lambda^2 - 3)g''(\xi) - (\lambda - 1)g'^{v}(\xi)) + (g(\xi))^{2\lambda+4}(2a^2\lambda(\lambda + 1)(3\lambda - 4)(g'(\xi))^2 + a(3 + (7 - 8\lambda)\lambda)(g''(\xi))^2$$

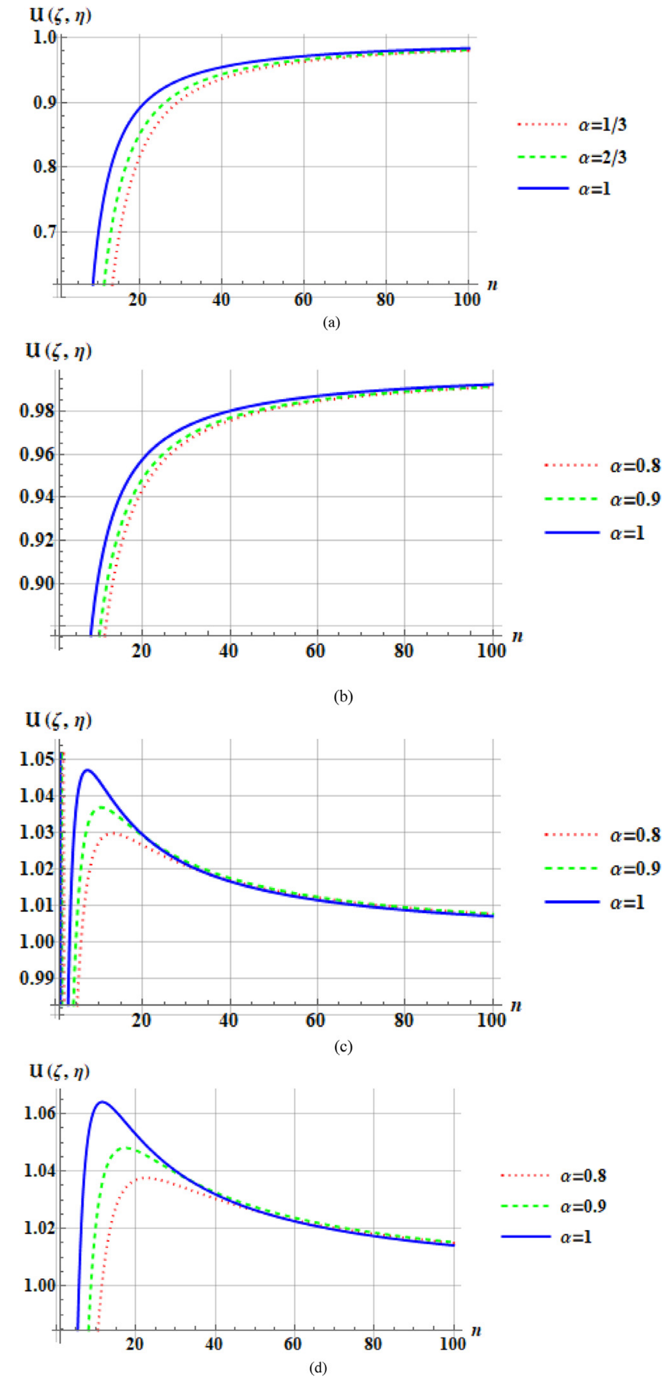

Fig. 6. Plots of n -curves when (a) $\lambda = -2$ (b) $\lambda = -1$ (c) $\lambda = 1$ (d) $\lambda = 2$at$\xi = 1, \eta = 1$, $a = 1$for different values of.α

$$-2a(\lambda-1)(7\lambda+1)g'(\xi)g''(\xi)+2(\lambda-1)\left(g''(\xi)\right)^2+2(\lambda -1)g''(\xi)g'^{v}(\xi)+2\left(g(\xi)\right)^{2\lambda+3}$$

$$(3(\lambda-1)\lambda\left(g''(\xi)\right)^3+(\lambda-1)(11\lambda-1)g'(\xi)g''(\xi)g'''(\xi) + \left(g'(\xi)\right)^2\left(a(-1+\lambda+20\lambda^2-18\lambda^3)g''(\xi)+\lambda(\lambda -1)g'^{v}(\xi)\right))]\frac{\Gamma(2\alpha+1)}{2(\Gamma(\alpha+1))^2}\frac{\eta^{3\alpha}}{\Gamma(3\alpha+1)} \quad (22)$$

Proceeding, other iterations of $\mathfrak{U}_m(\xi,\eta)$for $m \geq 4$ can be attained entirely, obtained the complete solution of series.

Conclusively, we get the solution of $\mathfrak{U}(\xi,\eta)$approximately by the series in truncated form

$$\mathfrak{U}(\xi,\eta) = \lim_{N\to\infty} \Phi_N(x,t) \quad (23)$$

$$\Phi_N(x,t) = \sum_{m=0}^{N-1} \mathfrak{U}_m(\xi,\eta)$$

The preceding series solution usually converges very quickly.

4. Numerical results and discussion

Numerical consequences of the chemical concentration $\mathfrak{U}(\xi,\eta)$ for $\alpha = \frac{1}{3},\frac{2}{3}$ and $\alpha = 1$ for $g(\xi) = \frac{\xi+1}{2}$ are showed through Figs. 1-6. From Fig. 1 noticed that the $\mathfrak{U}(\xi,\eta)$ decreases by decrease increases in time but then again increase by decrease in $\alpha$for $\lambda = -2,-1,2$. However it is found that $\mathfrak{U}(\xi,\eta)$ decreases by decrease in time t and increase with decrease in$\alpha$for $\lambda = 1$ i.e. as λbecoming more negative, diffusion $\mathfrak{U}(\xi,\eta)$ increases more rapidly in reverse direction with increase inη.

From Fig. 2 when reaction term is absent i.e. a, $\mathfrak{U}(\xi,\eta)$ is gradually shifting in upward direction for $\lambda = -2,-1$ but shows opposite nature for $\lambda = 1,2$.

Figs. 3 and 4, demonstrate the variation of $\mathfrak{U}(\xi,\eta)$ for various value of λ at $\xi = 1, \eta = 1$ without reaction term and with reaction term respectively.

It is detected that in Fig. 3, $\mathfrak{U}(\xi,\eta)$ increases in w. r. t.λ, where $-2 \leq \lambda \leq 10$, for various values of $\alpha = \frac{1}{3},\frac{2}{3},1$. It is seen that in Fig. 4 the nature of $\mathfrak{U}(\xi,\eta)$ changes several times w. r. t. λ. The influences of $\hbar$ on the convergence of the solution.

Figs. 5 and 6 plots of n-curves for $\lambda = -2,-1,1,2$ at $\xi = 1, \eta = 1$ without reaction term and with reaction term respectively for different values of α.

5. Conclusion

The present study exhibits the prosperous purpose of q-HATM to achieve solutions of a reaction-diffusion equation with fractional time derivatives approximately. The q-HATM gives quantitatively reliable outcomes with fewer computational effort and adequately selecting the control parameters and gives a more accurate approximate solution. The study's significant part is the presentation graphically of the consequence of the reaction part on the nonlinear fractional diffusion equation solution. The prominent characteristic of the projected work is the demonstration of the considered fractional-order equation's stochastic nature. The article effectively demonstrates the consequence of damping with the occurrence of reaction term.

The effect of reaction term in the range $-2 \leq \lambda \leq 10$, it is seen that anomalous diffusion present with sub-diffusion in $0 < \lambda < 4$and super-diffusion in $4 \leq \lambda < 10$ but we can see that no sub-diffusions or super-diffusion present in $-2 < \lambda < 0$.

CRediT authorship contribution statement

Supriya Yadav: Conceptualization, Formal analysis, Methodology, Writing - review & editing. **Devendra Kumar:** Conceptualization, Formal analysis, Investigation, Writing - original draft, Writing - review & editing. **Kottakkaran Sooppy Nisar:** Conceptualization, Formal analysis, Investigation, Methodology, Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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