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Non-polynomial quadratic spline method for solving fourth order singularly perturbed boundary value problems

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ABSTRACT

In this paper, a non-polynomial quadratic spline method is described for solving fourth-order boundary value problems whose highest-order derivative is multiplied by a small perturbation parameter. This method is applied directly to the solution of the problem without reducing the order of the problem. Convergence analysis of the fourth order method is discussed. To illustrate the efficiency of the method, a boundary value problem is considered with different type of boundary conditions and obtained numerical results are compared with the existing methods.

1. Introduction

Fourth order singularly perturbed boundary value problems occur frequently in many areas of applied sciences such as solid mechanics, Newtonian fluid mechanics, chemical reactor theory, aerodynamics, hydrodynamics, optimal control, convection diffusion processes, quantum mechanics, etc. These problems have various important applications in fluid dynamics. Ghasemi et al. (2014) gave the analysis of electrohydrodynamic flow in a circular cylindrical conduit using least square method and Hatami and Domairry (2014) investigated the transient vertically motion of a soluble particle in a Newtonian fluid media. Hatami and Ganji (2014) studied the motion of a spherical particle in a fluid forced vortex by DQM and DTM. Hatami et al. (2016) gave the optimization of a circular-wavy cavity filled by nanofluid under the natural convection heat transfer condition. Nadeem and Haq (2014) studied the effect of thermal radiation for megnetohydrodynamic boundary layer flow of a nanofluid past a stretching sheet with convective boundary conditions. Sheikholeslami and Ganji investigated the nanofluid flow and heat transfer between parallel plates considering Brownian motion using DTM in Sheikholeslami and Ganji (2015). Sheikholeslami et al. (2013) gave investigation of squeezing unsteady nanofluid flow using ADM. Sheikholeslami et al. (2012) discussed analytical investigation of Jeffery-Hamel flow with high magnetic field and nanoparticle by Adomian Decomposition Method. Sheikholeslami et al. (2012) investigated the laminar viscous flow in a semi-porous channel in the presence of uniform magnetic field using Optimal Homotopy Asymptotic Method. Zhou et al. (2016) designed the microchannel heat sink with wavy channel and its time-efficient optimization with combined RSM and FVM methods.

Singularly perturbed problems are classified on the fact that how the order of the differential equation is affected if $\epsilon \to 0$, here ϵ is a small positive parameter multiplying the highest order derivative of the differential equation. The solution of singularly perturbed boundary value problem has a multiscale character; that is, there are thin transition layers where the solution varies rapidly, while away from the layers the solution varies very slowly.

In this paper, we develop a computational method to solve fourth order singularly perturbed boundary value problems of the form:

$$-\epsilon y^{(4)}(x) + p(x)y(x) = q(x), \quad a < x < b \tag{1}$$

subject to the boundary conditions

$$\textbf{Case (i)} \quad y(a) = \alpha_1,\ y(b) = \beta_1,\ y^{(2)}(a) = \alpha_2,\ y^{(2)}(b) = \beta_2 \tag{2}$$

OR

$$\textbf{Case (ii)} \quad y(a) = \alpha_1,\ y(b) = \beta_1,\ y^{(1)}(a) = \alpha_2,\ y^{(1)}(b) = \beta_2 \tag{3}$$

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where p and q are sufficiently continuously differentiable functions in the interval $[a, b]$, $\alpha_i's$ and $\beta_i's$ are real constants and ϵ is a small positive parameter.

In literature, we found many numerical methods which were developed for solving second order singularly perturbed BVPs. These methods are exponentially fitted finite difference scheme (Kadalbajoo and Kumar, 2009), non-polynomial spline method (Tirmizi, 2008), cubic spline method (Kumar et al., 2007) and quintic spline method (Rashidinia et al., 2010) etc. There are few methods available for solving higher order singularly perturbed BVPs such as asymptotic finite element method (Babu and Ramanujam, 2007), reproducing kernel method (Akram and Rehman, 2012). Shanthi and Ramanujam (Shanthi and Ramanujam, 2002) solved singularly perturbed fourth-order ordinary differential equations of convection–diffusion type using asymptotic numerical methods. The authors in Akram and Amin (2012, 2013) used quintic spline and septic spline respectively for solving fourth order singularly perturbed BVPs.

However, most of these methods were used to solve fourth-order singularly perturbed boundary value problem by using a higher degree spline. Here, we use a non-polynomial quadratic spline method for solving the problem (1). In this paper, we discuss two types of boundary value problems with boundary conditions (2) and (3). Firstly a numerical system is obtained by using non-polynomial quadratic spline. Then finite difference formula of $O(h^2)$ is used for making the system consistent with the given boundary value problem. Finally the obtained scheme is used to solve fourth order singularly perturbed boundary value problems. After implementation of the problem over the method we get a system of pentadiagonal matrix which is solved by using LU decomposition method. The paper describing a non-polynomial quadratic spline method is organized into five sections. Section 2 gives a brief derivation of the method, along with boundary conditions. In Section 3 truncation error has been obtained for fourth order method. Application of the method for solving fourth order singularly perturbed BVPs is discussed in Section 4. Convergence analysis of the method is discussed in Section 5. In Section 6, numerical examples and their comparison with the existing methods are presented which demonstrate the efficiency of our method. Conclusion and the figures are presented in Section 7 also proves the accuracy.

2. Derivation of the scheme

Let $a = x_0 < x_1 < x_2 < \cdots < x_{n+1} = b$, we first divide the interval [a,b] into $n+1$ equal parts by introducing

$$x_i = a + ih, \ i = 0, 1, \ldots, n+1 \quad and \quad h = (b-a)/(n+1)$$

Let

$$Q_i(x) = a_i \sin \tau(x - x_i) + b_i e^{\tau(x-x_i)} + c_i \tag{4}$$

be a non-polynomial quadratic spline Q_i is defined on [a, b] which interpolates the uniform mesh points x_i depends on a parameter τ, reduces to an ordinary quadratic spline in [a,b] as $\tau \longrightarrow 0$ and $\tau > 0$.

To determine the coefficients a_i, b_i *and* c_i, we define the following interpolatory conditions as

$$Q_i(x_i) = y_i, \ Q_i(x_{i+1}) = y_{i+1}, \quad Q_i^{(2)}(x_i) = \frac{1}{2}(D_i + D_{i+1}), \quad i = 0, 1, \ldots, n \tag{5}$$

By using above conditions we calculated the coefficients as

$$\begin{aligned} a_i &= \frac{y_{i+1} - y_i}{\sin\theta} + \frac{(1-e^\theta)}{2\tau^2 \sin\theta}(D_i + D_{i+1}) \\ b_i &= \frac{1}{2\tau^2}(D_i + D_{i+1}) \\ c_i &= y_i - \frac{1}{2\tau^2}(D_i + D_{i+1}) \end{aligned}$$

where, $\theta = \tau h$

Using the continuity of first derivative, $Q_{i-1}^m(x_i) = Q_i^m(x_i), \ m = 1$ the following consistency relation is derived

$$\alpha_1 D_{i-1} + \beta D_i + \alpha_2 D_{i+1} = \frac{1}{h^2}(\gamma y_{i-1} + \delta y_i + y_{i+1}), \quad i = 1, 2, \ldots, n \tag{6}$$

where,

$$\begin{aligned} \alpha_1 &= \frac{(1-e^\theta)\cos\theta + e^\theta \sin\theta}{2\theta^2} \\ \beta &= \frac{(-1+e^\theta)(1-\cos\theta) + (-1+e^\theta)\sin\theta}{2\theta^2} \\ \alpha_2 &= \frac{-1+e^\theta - \sin\theta}{2\theta^2} \\ \gamma &= \cos\theta \\ \delta &= -1 - \cos\theta \end{aligned}$$

Remark:. Our method reduces to Al-Said (2008) based on quadratic spline when

$$(\alpha_1, \beta, \alpha_2) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right). \tag{7}$$

For making the system consistent with the given boundary conditions, we use finite difference formula of $O(h^2)$

$$D_{i-1} - 2D_i + D_{i+1} = h^2 y_i^{(4)} + O(h^4), \quad i = 1, 2, \ldots, n \tag{8}$$

Using (6) and (8), we obtained the following relation

$$\begin{aligned} &\gamma y_{i-2} + (\delta - 2\gamma) y_{i-1} + (1 - 2\delta + \gamma) y_i + (-2 + \delta) y_{i+1} + y_{i+2} \\ &\quad = h^4(\alpha_1 y_{i-1}^{(4)} + \beta y_i^{(4)} + \alpha_2 y_{i+1}^{(4)}) \end{aligned}$$

$$i = 2, 3, \ldots, n-1 \tag{9}$$

Eq. (9) form a system of $n-2$ linear equations in n unknowns $y_i, \ i = 1, 2, \ldots, n$. Thus, we need two more equations, one at each end of the range of integration.

For case (i), the equations are obtained as

$$\begin{aligned} 5y_1 - 4y_2 + y_3 = 2y_0 - h^2 y_0^{(2)} \\ + h^4\left(\frac{7}{90}u_0^{(4)} + \frac{49}{72}u_1^{(4)} + \frac{7}{45}u_2^{(4)} + \frac{1}{360}u_3^{(4)}\right) + t_1 \end{aligned} \tag{10}$$

$$\begin{aligned} y_{n-2} - 4y_{n-1} + 5y_n = 2y_{n+1} - h^2 y_{n+1}^{(2)} \\ + h^4\left(\frac{1}{360}u_{n-2}^{(4)} + \frac{7}{45}u_{n-1}^{(4)} + \frac{49}{72}u_n^{(4)} + \frac{7}{90}u_{n+1}^{(4)}\right) + t_n \end{aligned} \tag{11}$$

For case (ii), the equations are obtained as

$$\begin{aligned} 9y_1 - \frac{9}{2}y_2 + y_3 = \frac{11}{2}y_0 - 3hy_0^{(1)} \\ + h^4\left(\frac{8}{280}u_0^{(4)} + \frac{151}{280}u_1^{(4)} + \frac{52}{280}u_2^{(4)} - \frac{1}{280}u_3^{(4)}\right) + t_1 \end{aligned} \tag{12}$$

$$y_{n-2}-\frac{9}{2}y_{n-1}+9y_n=\frac{11}{2}y_{n+1}-3hy^{(1)}_{n+1}$$
$$+h^4\left(-\frac{1}{280}u^{(4)}_{n-2}+\frac{52}{280}u^{(4)}_{n-1}+\frac{151}{280}u^{(4)}_n+\frac{8}{280}u^{(4)}_{n+1}\right)+t_n \qquad (13)$$

where, t_1 and t_n are of $O(h^4)$.

3. Truncation error

Expanding (9) by using Taylor series, we obtained the following truncation error

$$t_i=h^2(1+\gamma+\delta)y^{(2)}_i+h^3(1-\gamma)y^{(3)}_i+h^4\left(\frac{14+14\gamma+2\delta}{4!}-(\alpha_1+\alpha_2+\beta)\right)y^{(4)}_i$$
$$+h^5\left(\frac{1-\gamma}{4}-(-\alpha_1+\alpha_2)\right)y^{(5)}_i+h^6\left(\frac{62+62\gamma+2\delta}{6!}-\frac{\alpha_1+\alpha_2}{2!}\right)y^{(6)}_i$$
$$+h^7\left(\frac{1-\gamma}{40}-\frac{\alpha_2-\alpha_1}{3!}\right)y^{(7)}_i+h^8\left(\frac{254+254+\gamma+2\delta}{8!}-\frac{\alpha_1+\alpha_2}{4!}\right)y^{(8)}_i$$
$$+h^9\left(\frac{17-17\gamma}{12096}-\frac{\alpha_2-\alpha_1}{5!}\right)y^{(9)}_i+O(h^{10}),\quad i=2,3\ldots,n-1 \qquad (14)$$

For different values of parameters, we get the method of second order as well as fourth order. Here we discuss only fourth order method. The local truncation error for (10) and (11) is

$$t_i=\begin{cases}-\frac{20162/3}{8!}h^8y^{(8)}_i+O(h^9), & \mathrm{i}=1\\ \frac{5904/133}{8!}h^8y^{(8)}_i+O(h^9), & \mathrm{i}=\mathrm{n}\end{cases}$$

and the local truncation error for (12) and (13) is

$$t_i=\begin{cases}-\frac{6554}{8!}h^8y^{(8)}_i+O(h^9), & \mathrm{i}=1\\ \frac{211}{8!}h^8y^{(8)}_i+O(h^9), & \mathrm{i}=\mathrm{n}\end{cases}$$

For $(\alpha_1,\beta,\alpha_2,\gamma,\delta)=\left(\frac{1}{6},\frac{1}{3},\frac{1}{6},1,-2\right)$ the truncation error is

$$t_i=\left(-\frac{1}{720}\right)h^8y^{(8)}_i+O(h^9),\quad i=2,3,\ldots,n-1$$

4. Application to the fourth order singularly perturbed boundary value problems

We consider a fourth order singularly perturbed boundary value problem of the form

$$-\epsilon y^{(4)}(x)+p(x)y(x)=q(x),\quad a<x<b$$

subject to the boundary conditions

$y(a)=\alpha_1,\ y(b)=\beta_1$
$y^{(2)}(a)=\alpha_2,\ y^{(2)}(b)=\beta_2$

OR

$y(a)=\alpha_1,\ y(b)=\beta_1$
$y^{(1)}(a)=\alpha_2,\ y^{(1)}(b)=\beta_2$

where p, q are sufficiently continuously differentiable functions in the interval $[a,b]$, α_i and β_i are real constants.

After applying the scheme (9) to the problem (1) with boundary conditions (2) and (3), we get the following relation

$$Ay_{i-2}+By_{i-1}+Cy_i+Dy_{i+1}+Ey_{i+2}$$
$$=-h^4(\alpha_1q_{i-1}+\beta q_i+\alpha_2q_{i+1}),\quad i=2,3,\ldots,n-1 \qquad (15)$$

where,

$$A=\gamma\epsilon$$
$$B=(\delta-2\gamma)\epsilon-h^4\alpha_1p_{i-1}$$
$$C=(1-2\delta+\gamma)\epsilon-h^4\beta p_i$$
$$D=(-2+\delta)\epsilon-h^4\alpha_2p_{i+1}$$
$$E=\epsilon$$

Eqs. (10) and (11) takes the form

$$A_1y_1+A_2y_2+A_3y_3=\epsilon A_0y_0-h^2\epsilon y^{(2)}_0$$
$$-h^4\left(\frac{7}{90}q_0+\frac{49}{72}q_1+\frac{7}{45}q_2+\frac{1}{360}q_3\right)$$

$$C_{n-2}y_{n-2}+C_{n-1}y_{n-1}+C_ny_n=\epsilon C_{n+1}y_{n+1}-h^2\epsilon y^{(2)}_{n+1}$$
$$-h^4\left(\frac{7}{90}q_{n+1}+\frac{49}{72}q_n+\frac{7}{45}q_{n-1}+\frac{1}{360}q_{n-2}\right)$$

where,

$$A_0=2\epsilon-\frac{7}{9}h^4p_0,\ A_1=5\epsilon-\frac{49}{72}h^4p_1,$$
$$A_2=-4\epsilon-\frac{7}{45}h^4p_2,\ A_3=\epsilon-\frac{1}{360}h^4p_3$$
$$C_{n-2}=5\epsilon-\frac{49}{72}h^4p_{n-2},\ C_{n-1}=-4\epsilon-\frac{7}{45}h^4p_{n-1},$$
$$C_n=\epsilon-\frac{1}{360}h^4p_n,\ C_{n+1}=2\epsilon-\frac{7}{90}h^4p_{n+1}$$

Eqs. (12) and (13) takes the form

$$A_1y_1+A_2y_2+A_3y_3=\epsilon A_0y_0-3\epsilon hy^{(1)}_0$$
$$-h^4\left(\frac{8}{280}q_0+\frac{151}{280}q_1+\frac{52}{280}q_2-\frac{1}{280}q_3\right)$$

$$C_{n-2}y_{n-2}+C_{n-1}y_{n-1}+C_ny_n$$
$$=\epsilon C_{n+1}y_{n+1}-3\epsilon hy^{(1)}_{n+1}-h^4\left(\frac{8}{280}q_{n+1}+\frac{151}{280}q_n+\frac{52}{280}q_{n-1}-\frac{1}{280}q_{n-2}\right)$$

where,

$$A_0=\frac{11}{2}\epsilon-\frac{8}{280}h^4p_0,\ A_1=9\epsilon-\frac{151}{280}h^4p_1,$$
$$A_2=-\frac{9}{2}\epsilon-\frac{52}{280}h^4p_2,\ A_3=\epsilon+\frac{1}{280}h^4p_3$$
$$C_{n-2}=9\epsilon-\frac{151}{280}h^4p_{n-2},\ C_{n-1}=-\frac{9}{2}\epsilon-\frac{52}{280}h^4p_{n-1},$$
$$C_n=\epsilon+\frac{1}{280}h^4p_n,C_{n+1}=\frac{11}{2}\epsilon-\frac{8}{280}h^4p_{n+1}$$

5. Convergence analysis

The developed method leads to the following matrix form

$$PY=C \qquad (16)$$

where,

$$P=\begin{bmatrix}A_1 & A_2 & A_3 & & & & \\ B & C & D & E & & & \\ A & B & C & D & E & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & A & B & C & D & E & \\ & & A & B & C & D & \\ & & & C_{n-2} & C_{n-1} & C_n & \end{bmatrix}$$

$Y=[y_1,y_2,\ldots,y_{n-1}]^T$ and the right hand side vector is $C=[c_1,c_2,\ldots,c_{n-1}]^T$.

For Case (i), we have

$$c_1 = A_0\epsilon y_0 - h^2\epsilon y_0^{(2)} - h^4\left(\frac{7}{90}q_0 + \frac{49}{72}q_1 + \frac{7}{45}q_2 + \frac{1}{360}q_3\right)$$
$$c_2 = -A\epsilon y_0 - h^4(\alpha_1 q_1 + \beta q_2 + \alpha_2 q_3)$$
$$c_i = -h^4(\alpha_1 q_{i-1} + \beta q_i + \alpha_2 q_{i+1}), i = 3, 4, \ldots, n-1$$
$$c_n = \epsilon C_{n+1} y_{n+1} - h^2\epsilon y_{n+1}^{(2)} - h^4\left(\frac{7}{90}q_{n+1} + \frac{49}{72}q_n + \frac{7}{45}q_{n-1} + \frac{1}{360}q_{n-2}\right)$$

For Case (ii),we have

$$c_1 = A_0\epsilon y_0 - 3h\epsilon y_0^{(1)} - h^4\left(\frac{8}{280}q_0 + \frac{151}{280}q_1 + \frac{52}{280}q_2 - \frac{1}{280}q_3\right)$$
$$c_2 = -A\epsilon y_0 - h^4(\alpha_1 q_1 + \beta q_2 + \alpha_2 q_3)$$
$$c_i = -h^4(\alpha_1 q_{i-1} + \beta q_i + \alpha_2 q_{i+1}), \quad i = 3, 4, \ldots, n-1$$
$$c_n = \epsilon C_{n+1} y_{n+1} - 3h\epsilon y_{n+1}^{(1)} - h^4\left(\frac{8}{280}q_{n+1} + \frac{151}{280}q_n + \frac{52}{280}q_{n-1} - \frac{1}{280}q_{n-2}\right)$$

Also we have,

$$P\tilde{Y} = C + T \tag{17}$$

where $\tilde{Y} = [\tilde{y}(x_1), \tilde{y}(x_2), \ldots, \tilde{y}(x_{n-1})]^T$ be the exact solution and $T = [T_1, T_2, \ldots, T_{n-1}]^T$ be the local truncation error.

From (16) and (17) we have

$$P(\tilde{Y} - Y) = T$$
$$PE = T$$
$$E = \tilde{Y} - Y = [e_1, e_2, \ldots, e_{n-1}]^T$$

Definition:. A pentadiagonal matrix $P = (p_{i,j})$, where $p_{i,j} = 0$ for $|i-j| > 2$, is irreducible iff

$p_{i,i-1} \neq 0, \quad i = 2, 3, \ldots, n$
$p_{i,i-2} \neq 0, \quad i = 3, 4, \ldots, n p_{i,i+1} \neq 0,$
$i = 1, 2, \ldots, n-1$ *and*
$p_{i,i+2} \neq 0, \quad i = 1, 2, \ldots, n-1$

Now we have to calculate sum of each row of matrix P:

For case (i),

$$S_1 = 2\epsilon - \frac{49}{72}h^4 p_1 - \frac{7}{45}h^4 p_2 - \frac{1}{360}h^4 p_3$$
$$S_2 = -\gamma\epsilon - h^4\alpha_1 p_1 - h^4\beta p_2 - h^4\alpha_2 p_3$$
$$S_i = -h^4\alpha_1 p_{i-1} - h^4\beta p_i - h^4\alpha_2 p_{i+1}, \quad i = 3, \ldots, n-1$$
$$S_n = 2\epsilon - \frac{49}{72}h^4 p_{n-2} - \frac{7}{45}h^4 p_{n-1} - \frac{1}{360}h^4 p_n$$

For case (ii),

$$S_1 = \frac{11}{2}\epsilon - \frac{151}{280}h^4 p_1 - \frac{52}{280}h^4 p_2 + \frac{1}{280}h^4 p_3$$
$$S_2 = -\gamma\epsilon - h^4\alpha_1 p_1 - h^4\beta p_2 - h^4\alpha_2 p_3$$
$$S_i = -h^4\alpha_1 p_{i-1} - h^4\beta p_i - h^4\alpha_2 p_{i+1}, \quad i = 3, \ldots, n-1$$
$$S_n = \frac{11}{2}\epsilon - \frac{151}{280}h^4 p_{n-2} - \frac{52}{280}h^4 p_{n-1} + \frac{1}{280}h^4 p_n$$

Let $0 < M \in Z^+$ is the minimum of $|p_{i-1}|, |p_i|, |p_{i+1}|$. For sufficiently small h we can say that:

For case (i),

$$S_i \geqslant \begin{cases} \frac{151}{180}h^4 M, & i = 1 \\ (\alpha_1 + \beta + \alpha_2)h^4 M, & i = 2 \\ (\alpha_1 + \beta + \alpha_2)h^4 M, & i = 3, \ldots, n-1 \\ \frac{151}{180}h^4 M, & i = n \end{cases}$$

For case (ii),

$$S_i \geqslant \begin{cases} \frac{51}{70}h^4 M, & i = 1 \\ (\alpha_1 + \beta + \alpha_2)h^4 M, & i = 2 \\ (\alpha_1 + \beta + \alpha_2)h^4 M, & i = 3, \ldots, n-1 \\ \frac{51}{70}h^4 M, & i = n \end{cases}$$

Further,we get for case (i)

$$\frac{1}{S_i} \leqslant \begin{cases} \frac{180}{151 h^4 M}, & i = 1, n \\ \frac{1}{(\alpha_1+\beta+\alpha_2)h^4 M}, & i = 2, 3, \ldots, n-1 \end{cases}$$

For sufficiently small h, we can easily show that the matrix P is irreducible and monotone. Therefore, P^{-1} exist and $P^{-1} \geqslant 0$. Hence,

$$\|E\| = \|P^{-1}\|\|T\| \tag{18}$$

and let $P^{-1} = (p^*_{i,j})$, then by theory of matrices Varga (1962), we get

$$\sum_{i=1}^{n-1} p^*_{i,j} S_i = 1, \quad j = 1, \ldots, n \tag{19}$$

Therefore,

$$p^*_{i,j} \leqslant \frac{1}{S_i}$$
$$\|P^{-1}\| = \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}|p^*_{i,j}| \leqslant \sum_{i=1}^{n}\frac{1}{S_i} = \frac{1}{h^4 M}\left(\frac{360}{151} + \frac{2}{\alpha_1+\beta+\alpha_2}\right), \quad i = 1, \ldots, n \text{ and}$$
$$\|T_i\| = \max_{1\leqslant i\leqslant n}\sum_{i=1}^{n}|T_i|$$

and error is given by:

For case (i),

$$\|E\| = \|P^{-1}\|\|T\| \leqslant \frac{1}{h^4 M}\left(\frac{360}{151} + \frac{2}{\alpha_1+\beta+\alpha_2}\right)\|T\|$$

Similarly for Case (ii), the error is given by

$$\|E\| = \|P^{-1}\|\|T\| \leqslant \frac{1}{h^4 M}\left(\frac{140}{51} + \frac{2}{\alpha_1+\beta+\alpha_2}\right)\|T\|$$

By using (14), we have truncation error $\|T\| = O(h^8)$, then we get

$$\|E\| \leqslant \frac{1}{h^4 M}\left(\frac{360}{151} + \frac{2}{\alpha_1+\beta+\alpha_2}\right)O(h^8) = O(h^4)$$

Hence, the scheme is fourth order convergent.

$$\|E\| = O(h^4)$$

Theorem.. *The method given by Eq. (9) for solving the given singularly perturbed boundary value problem for sufficiently small h has a fourth order convergence.*

6. Numerical illustrations

In the present paper, we consider two linear fourth order singularly perturbed boundary value problems whose exact solutions are known. The maximum absolute errors for h = 1/16, 1/32, 1/64 and 1/128 are tabulated in Tables 1 and 2 and comparison are also shown in graphs 1–2. The obtained results are compared with the results of quintic spline method (Akram and Amin, 2012) and septic spline method (Akram and Naheed, 2013).

Table 1
Maximum absolute error for Example 1.

	ϵ	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
Our method	1/16	2.6309×10^{-9}	1.5786×10^{-10}	1.8007×10^{-11}	1.2579×10^{-10}
Akram and Amin (2012)		2.3722×10^{-6}	5.9529×10^{-7}	1.4896×10^{-7}	3.7214×10^{-8}
Our method	1/32	5.0874×10^{-10}	3.0216×10^{-11}	3.8255×10^{-13}	2.4353×10^{-11}
Akram and Amin (2012)		4.5647×10^{-7}	1.1462×10^{-7}	2.8684×10^{-8}	7.1730×10^{-9}
Our method	1/64	1.1660×10^{-10}	6.926×10^{-12}	7.7907×10^{-13}	6.1730×10^{-12}
Akram and Amin (2012)		1.0356×10^{-7}	2.6027×10^{-8}	6.514×10^{-9}	1.6304×10^{-9}

Table 2
Maximum absolute error for Example 2.

	ϵ	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
Our method	1/16	1.0499×10^{-7}	9.8529×10^{-9}	7.0265×10^{-10}	4.6045×10^{-11}
Akram and Amin (2012)	–	1.315×10^{-6}	1.617×10^{-7}	2.853×10^{-8}	$6.682 \times$ 10(-9)
Akram and Naheed (2013)		1.666×10^{-6}	1.310×10^{-7}	2.614×10^{-9}	6.716×10^{-11}
Our method	1/32	5.3745×10^{-8}	5.0610×10^{-9}	3.6108×10^{-10}	2.3663×10^{-11}
Akram and Amin (2012)		6.703×10^{-7}	8.170×10^{-8}	1.434×10^{-8}	3.355×10^{-9}
Akram and Naheed (2013)		8.537×10^{-7}	6.736×10^{-8}	1.344×10^{-9}	3.452×10^{-11}
Our method	1/64	2.8376×10^{-8}	2.6766×10^{-9}	1.9132×10^{-10}	1.2538×10^{-11}
Akram and Amin (2012)		3.489×10^{-7}	4.177×10^{-8}	7.249×10^{-9}	1.692×10^{-9}
Akram and Naheed (2013)		4.520×10^{-7}	3.569×10^{-8}	7.128×10^{-10}	1.829×10^{-11}
Our method	1/128	1.6215×10^{-8}	1.5345×10^{-9}	1.0968×10^{-10}	7.1910×10^{-12}
Akram and Amin (2012)		1.195×10^{-7}	2.201×10^{-8}	3.717×10^{-9}	8.619×10^{-10}
Akram and Naheed (2013)		2.60×10^{-7}	2.049×10^{-8}	4.092×10^{-10}	1.05×10^{-11}

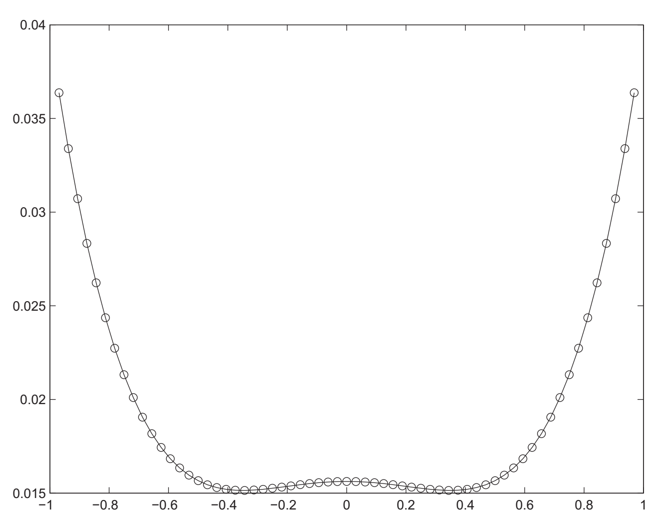

Fig. 1. Graph of the exact solution versus the approximate solution for N = 64 for Example 1.

Example 1. For $-1 \leqslant x \leqslant 1$, consider the following differential equation:

$$-\epsilon y^{(4)}(x) + y(x) = \epsilon(2x^4 + \cos x - \epsilon(48 + \cos x)), \tag{20}$$

$$y(-1) = \epsilon(2 + \cos 1), \quad y(1) = \epsilon(24 - \cos 1)$$
$$y^{(2)}(-1) = \epsilon(2 + \cos 1), \quad y^{(2)}(1) = \epsilon(24 - \cos 1)$$

The analytical solution of the problem (20) is

$$y(x) = \epsilon(2x^4 + \cos x).$$

Maximum absolute errors for Example 1 are given in Table 1. For the sake of comparison we are reported results of Akram and Amin (2012) in Table 1.

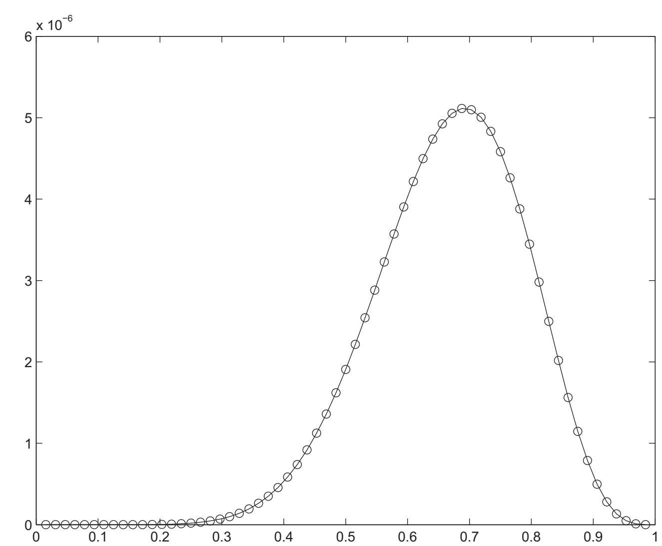

Fig. 2. Graph of the exact solution versus the approximate solution for N = 64 for Example 2.

Example 2. For $x \in [0, 1]$, consider the following differential equation:

$$-\epsilon y^{(4)}(x) + y(x) = f(x) \tag{21}$$

where

$$\begin{aligned} f(x) = & (x-1)^4 x^8 \sin(\epsilon x) - \epsilon x^4(-16\epsilon^3(x-1)^3 x^3(3x-2)\cos(\epsilon x) \\ & + 96\epsilon x(14 - 84x + 180x^2 - 165x^3 + 55x^4)\cos(\epsilon x) \\ & + \epsilon^4(x-1)^4 x^4 \sin(\epsilon x) - 24\epsilon^2(x-1)^2 x^2(14 - 44x + 33x^2)\sin(\epsilon x) \\ & + 24(70 - 504x + 1260x^2 - 1320x^3 + 495x^4)\sin(\epsilon x)) \end{aligned}$$

and

$$y(0) = 0, y(1) = 0, y^{(1)}(0) = 0, y^{(1)}(1) = 0,$$

The analytical solution of (21) is

$$y(x) = x^8(1-x)^4 \sin(\epsilon x).$$

Maximum absolute errors for Example 2 are given in Table 2. For the sake of comparison we are also reported results of Akram and Amin (2012) and Akram and Naheed (2013) in the Table 2.

7. Conclusion

Non-polynomial quadratic spline method is developed for the approximate solution of fourth order singularly perturbed boundary value problems with two types of boundary conditions. Convergence analysis of the method proved that our scheme (9) is fourth order convergent. A lower degree non-polynomial quadratic spline is used in this paper. However, in previous papers higher degree quintic and septic splines were used. This method is also applicable to solve linear boundary value problems as $\epsilon \to 1$. Maximum absolute errors in Tables 1 and 2 shows that our method is better than existing methods. Graphs between exact and approximate solutions for the Examples 1 and 2 are shown in Fig. 1 and 2 respectively which also shows the superiority of our method.

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