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ORIGINAL ARTICLE

Numerical solution of linear Fredholm integral equations via modified Simpson's quadrature rule

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Abstract The main purpose of this paper is to demonstrate that using modified Simpson's quadrature rule for solving linear Fredholm integral equations of the second kind. We convert the integral equations to a system of linear equations, and by using numerical examples we show our estimation have a good degree of accuracy.

1. Introduction and notations

In recent years, many different basic functions have used to estimate the solution of integral equations, such as orthonormal bases and wavelets (Jung and Schanfelberger, 1992; Maleknejad and Hadizadeh, 1999; Razzaghi and Arabshahi, 1989).

In this paper, we apply modified Simpson's quadrature rule to solve the linear Fredholm integral equations of the second kind.

Modified Simpson's quadrature rule formula for solving definite integral $\int_{x_i}^{x_{i+2}} f(x)dx$ is as follows:

$$\int_{x_i}^{x_{i+2}} f(x)dx = \frac{h}{3}[f_i + 4f_{i+1} + f_{i+2}] + \frac{h^4}{180}[f_i''' - f_{i+2}'''] - \frac{h^7}{1260}f^{(6)}(\zeta_i); \quad \zeta_i \in (x_i, x_{i+2}). \tag{1}$$

In general for integral $[a, b]$ we have:

$$\int_a^b f(x)dx = \sum_{i=0}^{\frac{N}{2}-1} \int_{x_{2i}}^{x_{2i+2}} f(x)dx \simeq \frac{h}{3}f(a) + \frac{4h}{3}\sum_{i=0}^{\frac{N}{2}-1} f_{2i+1} + \frac{2h}{3}\sum_{i=1}^{\frac{N}{2}-1} f_{2i} + \frac{h}{3}f(b) + \frac{h^4}{180}[f'''(a) - f'''(b)], \tag{2}$$

where N is even.

2. Development of modified Simpson's quadrature rule for solving linear Fredholm integral equations

Consider linear Fredholm integral equations of the second kind:

$$y(t) = x(t) + \int_a^b k(t,s)y(s)ds; \quad a \leqslant t \leqslant b, \tag{3}$$

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where $k(t,s)$ and $x(t)$ are known functions, but $y(t)$ is an unknown function (Atkinson, 1997; Baker and Miller, 1982; Delves and Mohamed, 1985; Delves and Walsh, 1974; Jerri, 1999; Kenwal, 1971; Kondo, 1991; Kress, 1989; Kyte and Puri, 2002).

If for solving (3), we approximate the right-hand integral (3) with the repeated modified Simpson's quadrature rule, we have:

$$\begin{aligned} y(t) &= x(t) + \frac{h}{3}k(t,s_0)y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} k(t,s_{2j+1})y_{2j+1} \\ &\quad + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} k(t,s_{2j})y_{2j} + \frac{h}{3}k(t,s_N)y_N \\ &\quad + \frac{h^4}{180}\Big[J''(t,s_0)y_0 + k(t,s_0)y_0''' + 3J'(t,s_0)y_0' \\ &\quad + 3J(t,s_0)y_0'' - J''(t,s_N)y_N - k(t,s_N)y_N''' \\ &\quad - 3J'(t,s_N)y_N' - 3J(t,s_N)y_N''\Big], \end{aligned} \tag{4}$$

where

$$J(t,s) = \frac{\partial k(t,s)}{\partial s}, \quad J'(t,s) = \frac{\partial^2 k(t,s)}{\partial s^2}, \quad J''(t,s) = \frac{\partial^3 k(t,s)}{\partial s^3},$$

must exist.

Hence for $t = t_0, t_1, \cdots, t_N$, we get the following system of equations:

$$\begin{aligned} y_i &= x_i + \frac{h}{3}k_{i,0}y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} k_{i,2j+1}y_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} k_{i,2j}y_{2j} + \frac{h}{3}k_{i,N}y_N \\ &\quad + \frac{h^4}{180}\Big[J''_{i,0}y_0 + k_{i,0}y_0''' + 3J'_{i,0}y_0' + 3J_{i,0}y_0'' - J''_{i,N}y_N \\ &\quad - k_{i,N}y_N''' - 3J'_{i,N}y_N' - 3J_{i,N}y_N''\Big]; \quad i = 0(1)(N). \end{aligned} \tag{5}$$

By taking three derivative from Eq. (3) we obtain:

$$y'(t) = x'(t) + \int_a^b H(t,s)y(s)ds; \qquad a \leqslant t \leqslant b, \tag{6}$$

$$y''(t) = x''(t) + \int_a^b H'(t,s)y(s)ds; \quad a \leqslant t \leqslant b, \tag{7}$$

$$y'''(t) = x'''(t) + \int_a^b H''(t,s)y(s)ds; \quad a \leqslant t \leqslant b, \tag{8}$$

where $H(t,s) = \frac{\partial k(t,s)}{\partial t}$, $H'(t,s) = \frac{\partial^2 k(t,s)}{\partial t^2}$, $H''(t,s) = \frac{\partial^3 k(t,s)}{\partial t^3}$, $x'(t)$, $x''(t)$, $x'''(t)$ must exist.

Now, for solving the afore-mentioned equation, we must consider two position.

Position 1. The partial derivatives $\frac{\partial J(t,s)}{\partial t}$, $\frac{\partial^2 J(t,s)}{\partial t^2}$, $\frac{\partial^3 J(t,s)}{\partial t^3}$, $\frac{\partial J'(t,s)}{\partial t}$, $\frac{\partial^2 J'(t,s)}{\partial t^2}$, $\frac{\partial^3 J'(t,s)}{\partial t^3}$, $\frac{\partial J''(t,s)}{\partial t}$, $\frac{\partial^2 J''(t,s)}{\partial t^2}$, $\frac{\partial^3 J''(t,s)}{\partial t^3}$ does not exist:

In this position, we solve Eqs. (6)–(8) with repeated Simpson's quadrature rule (Stoer and Bulirsch, 1993), and for $t = t_0, t_1, \cdots, t_N$, we obtain:

$$\begin{aligned} y_i' &= x_i' + \frac{h}{3}H_{i,0}y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} H_{i,2j+1}y_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} H_{i,2j}y_{2j} \\ &\quad + \frac{h}{3}H_{i,N}y_N; \quad i = 0(1)(N), \end{aligned} \tag{9}$$

$$\begin{aligned} y_i'' &= x_i'' + \frac{h}{3}H'_{i,0}y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} H'_{i,2j+1}y_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} H'_{i,2j}y_{2j} \\ &\quad + \frac{h}{3}H'_{i,N}y_N; \quad i = 0(1)(N), \end{aligned} \tag{10}$$

$$\begin{aligned} y_i''' &= x_i''' + \frac{h}{3}H''_{i,0}y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} H''_{i,2j+1}y_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} H''_{i,2j}y_{2j} \\ &\quad + \frac{h}{3}H''_{i,N}y_N; \quad i = 0(1)(N), \end{aligned} \tag{11}$$

For $i = 0$, N from systems (9)–(11), and system (5) for $i = 0(1)(N)$, we have the following system:

$$\begin{aligned} y_i &= x_i + \left(\frac{h}{3}k_{i,0} + \frac{h^4}{180}J''_{i,0}\right)y_0 + \left(\frac{h}{3}k_{i,N} - \frac{h^4}{180}J''_{i,N}\right)y_N \\ &\quad + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} k_{i,2j+1}y_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} k_{i,2j}y_{2j} \\ &\quad + \frac{h^4}{180}\Big[k_{i,0}y_0''' + 3J'_{i,0}y_0' + 3J_{i,0}y_0'' - k_{i,N}y_N''' \\ &\quad - 3J'_{i,N}y_N' - 3J_{i,N}y_N''\Big]; \quad i = 0(1)(N), \end{aligned} \tag{12}$$

$$\begin{aligned} y_0' &= x_0' + \frac{h}{3}H_{0,0}y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} H_{0,2j+1}y_{2j+1} \\ &\quad + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} H_{0,2j}y_{2j} + \frac{h}{3}H_{0,N}y_N, \end{aligned} \tag{13}$$

$$\begin{aligned} y_0'' &= x_0'' + \frac{h}{3}H'_{0,0}y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} H'_{0,2j+1}y_{2j+1} \\ &\quad + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} H'_{0,2j}y_{2j} + \frac{h}{3}H'_{0,N}y_N, \end{aligned} \tag{14}$$

$$\begin{aligned} y_0''' &= x_0''' + \frac{h}{3}H''_{0,0}y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} H''_{0,2j+1}y_{2j+1} \\ &\quad + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} H''_{0,2j}y_{2j} + \frac{h}{3}H''_{0,N}y_N, \end{aligned} \tag{15}$$

$$\begin{aligned} y_N' &= x_N' + \frac{h}{3}H_{N,0}y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} H_{N,2j+1}y_{2j+1} \\ &\quad + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} H_{N,2j}y_{2j} + \frac{h}{3}H_{N,N}y_N, \end{aligned} \tag{16}$$

$$\begin{aligned} y_N'' &= x_N' + \frac{h}{3}H'_{N,0}y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} H'_{N,2j+1}y_{2j+1} \\ &\quad + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} H'_{N,2j}y_{2j} + \frac{h}{3}H'_{N,N}y_N, \end{aligned} \tag{17}$$

$$\begin{aligned} y_N''' &= x_N''' + \frac{h}{3}H''_{N,0}y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} H''_{N,2j+1}y_{2j+1} \\ &\quad + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} H''_{N,2j}y_{2j} + \frac{h}{3}H''_{N,N}y_N. \end{aligned} \tag{18}$$

By solving above system with $(N + 7)$ equations and $(N + 7)$ unknowns, the approximate solution of Eq. (3), is obtained.

Position 2. The partial derivatives $\frac{\partial J(t,s)}{\partial t}$, $\frac{\partial^2 J(t,s)}{\partial t^2}$, $\frac{\partial^3 J(t,s)}{\partial t^3}$, $\frac{\partial J'(t,s)}{\partial t}$, $\frac{\partial^2 J'(t,s)}{\partial t^2}$, $\frac{\partial^3 J'(t,s)}{\partial t^3}$, $\frac{\partial J''(t,s)}{\partial t}$, $\frac{\partial^2 J''(t,s)}{\partial t^2}$, $\frac{\partial^3 J''(t,s)}{\partial t^3}$ exist:

In this position, we solve Eqs. (6)–(8) with repeated modified Simpson's quadrature rule, and for $t = t_0, t_1, \cdots, t_N$ we obtain:

$$y'_i = x'_i + \frac{h}{3}H_{i,0}y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} H_{i,2j+1}y_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} H_{i,2j}y_{2j} + \frac{h}{3}H_{i,N}y_N + \frac{h^4}{180}[D_{i,0}y_0 + H_{i,0}y'''_0 + 3M_{i,0}y'_0 + 3L_{i,0}y''_0 - D_{i,N}y_N - H_{i,N}y'''_N - 3M_{i,N}y'_N - 3L_{i,N}y''_N]; \quad i = 0(1)(N), \tag{19}$$

$$y''_i = x''_i + \frac{h}{3}H'_{i,0}y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} H'_{i,2j+1}y_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} H'_{i,2j}y_{2j} + \frac{h}{3}H'_{i,N}y_N + \frac{h^4}{180}[D'_{i,0}y_0 + H'_{i,0}y'''_0 + 3M'_{i,0}y'_0 + 3L'_{i,0}y''_0 - D'_{i,N}y_N - H'_{i,N}y'''_N - 3M'_{i,N}y'_N - 3L'_{i,N}y''_N]; \quad i = 0(1)(N), \tag{20}$$

$$y'''_i = x'''_i + \frac{h}{3}H''_{i,0}y_0 + \frac{4h}{3}\sum_{j=0}^{\frac{N}{2}-1} H''_{i,2j+1}y_{2j+1} + \frac{2h}{3}\sum_{j=1}^{\frac{N}{2}-1} H''_{i,2j}y_{2j} + \frac{h}{3}H''_{i,N}y_N + \frac{h^4}{180}[D''_{i,0}y_0 + H''_{i,0}y'''_0 + 3M''_{i,0}y'_0 + 3L''_{i,0}y''_0 - D''_{i,N}y_N - H''_{i,N}y'''_N - 3M''_{i,N}y'_N - 3L''_{i,N}y''_N]; \quad i = 0(1)(N), \tag{21}$$

where

$L(t,s) = \frac{\partial J(t,s)}{\partial t}$,
$L'(t,s) = \frac{\partial^2 J(t,s)}{\partial t^2}$, $L''(t,s) = \frac{\partial^3 J(t,s)}{\partial t^3}$,
$M(t,s) = \frac{\partial J'(t,s)}{\partial t}$, $M'(t,s) = \frac{\partial^2 J'(t,s)}{\partial t^2}$, $M''(t,s) = \frac{\partial^3 J'(t,s)}{\partial t^3}$,
$D(t,s) = \frac{\partial J''(t,s)}{\partial t}$, $D'(t,s) = \frac{\partial^2 J''(t,s)}{\partial t^2}$, $D''(t,s) = \frac{\partial^3 J''(t,s)}{\partial t^3}$.

For $i = 0, N$ from systems (19)–(21), and system (5) for $i = 0(1)(N)$ we obtain a system with $(N + 7)$ equations and $(N + 7)$ unknowns.

By solving system, the approximate solution of Eq. (3), is obtained.

Table 1 Solution of Love's equation with repeated trapezoid quadrature rule from Saberi-Nadjafi and Heidari (2007).

Nodes t	$N = 8$	$N = 16$	$N = 32$	$N = 64$
$t = \pm 1$	1.63639	1.63887	1.63949	1.63964
$t = \pm 0.75$	1.74695	1.75070	1.75164	1.75187
$t = \pm 0.5$	1.83641	1.84089	1.84201	1.84229
$t = \pm 0.25$	1.89332	1.89804	1.89922	1.89952
$t = 0$	1.91268	1.91744	1.91863	1.91839

Table 2 Solution of Love's equation with repeated Simpson's quadrature rule.

Nodes t	$N = 8$	$N = 16$	$N = 32$	$N = 64$
$t = \pm 1$	1.63638	1.63639	1.63970	1.63970
$t = \pm 0.75$	1.75193	1.75195	1.75195	1.75195
$t = \pm 0.5$	1.84237	1.84238	1.84238	1.84238
$t = \pm 0.25$	1.89960	1.89961	1.89962	1.89962
$t = 0$	1.91902	1.91903	1.91903	1.91903

Table 3 Solution of Love's equation with modified trapezoid quadrature rule from Saberi-Nadjafi and Heidari (2007).

Nodes t	$N = 8$	$N = 16$	$N = 32$	$N = 64$
$t = \pm 1$	1.63638	1.63639	1.63970	1.63970
$t = \pm 0.75$	1.75193	1.75195	1.75195	1.75195
$t = \pm 0.5$	1.84237	1.84238	1.84238	1.84238
$t = \pm 0.25$	1.89960	1.89961	1.89962	1.89962
$t = 0$	1.91902	1.91903	1.91903	1.91903

3. Illustrative examples

In this section, we intend to compare modified Simpson's quadrature rule with other methods such as repeated trapezoid quadrature rule (Table 1), repeated Simpson's quadrature rule (Table 2), and modified trapezoid quadrature rule (Table 3). We solve these example by using MATLAB v7.6.

Example 1. Love's integral equation, is generally defined as follows (Maleknejad and Mirzaee, 2003; Saberi-Nadjafi and Heidari, 2007);

$$y(t) = x(t) + \frac{1}{\pi}\int_{-1}^{1} \frac{d}{d^2 + (t-s)^2} y(s)ds; \quad -1 \leqslant t \leqslant 1. \tag{22}$$

Consider this equation in particular case when $d = 1$ and $x(t) = 1$. Numerical results by using modified Simpson's quadrature rule are show in Table 4.

Table 4 Solution of Love's equation with modified Simpson's quadrature rule.

Nodes t	$N = 8$	$N = 16$	$N = 32$	$N = 64$
$t = \pm 1$	1.639711609574	1.639695664013	1.639695241498	1.639695222888
$t = \pm 0.75$	1.751984714220	1.751955291249	1.751954724517	1.751954702172
$t = \pm 0.5$	1.842392432181	1.842385252689	1.842384817993	1.842384796330
$t = \pm 0.25$	1.899630047187	1.899615594985	1.899615191928	1.899615169014
$t = 0$	1.919038494096	1.919032539776	1.919032025165	1.919031995048

4. Conclusion

In this work, we applied an application of modified Simpson's quadrature rule for solving the linear Fredholm integral equations. According to the numerical results which obtaining from the illustrative examples, we conclude that for sufficiently small **h** we get a good accuracy, since by reducing step size length the least square error will be reduced. In *i*th equation of quadrature system in (5) (for using Position 2) the error of approximation of integral given in linear integral equation with repeated modified Simpson's quadrature rule is $\frac{i}{1260}h^7f^{(6)}(\zeta)$, but this, for instance, by using repeated Simpson's quadrature rule is $\frac{i}{180}h^5f^{(4)}(\zeta)$, and by using repeated modified trapezoid quadrature rule is $\frac{i}{720}h^5f^{(4)}(\zeta)$.

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