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Original article

Some exact solutions of the Yu–Toda–Sasa–Fukuyama equation with time-dependent coefficients via two different methods

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A R T I C L E I N F O

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A B S T R A C T

In the current study, the generalized (G'/G)-expansion method and a new modified simple equation method (NMSEM) have been utilized to secure the exact solutions of the Yu-Toda-Sasa-Fukuyama (YTSF) equation with time-dependent coefficients, which describes two layers of liquid (or in an elastic) media. By implementing these methods, we have secured the three different families of travelling wave solutions (such as hyperbolic, rational and trigonometric functions) of the independent variables and the arbitrary parameters of the equation. Further, the graphical representation of the solitons are done with the help of the software Maple.

1. Introduction

Several complex phenomena in applied sciences and engineerings, such as plasma physics, quantum field theory, fluid dynamics, optics, optical fibres communication, and elastic media, are represented mathematically by nonlinear evolution equations (NLEEs). Such equations are the nonlinear Schrödinger equation, the Korteweg–de Vries equation, the Zakharov-Kuznetsov equation, the Boussinesq equation, and the Kadomtsev–Petviashvili equation (Ak et al., 2020; Biswas et al., 2020; Jyoti and Kumar, 2020; Kumar and Malik, 2022; Kumar et al., 2012; Malik et al., 2021; Ma, 2002; Ma, 2020; Nisar et al., 2022; Shumaila et al., 2021; Wang et al., 2021; Wazwaz, 2008). The exact solutions of these NLEEs perform a significant role in investigating different real-world phenomena and dynamical processes in several scientific disciplines. In the current scenario, various methods have been developed to secure the analytical solutions of these NLEEs, like the Hirota bilinear approach (Rizvi et al., 2020; Wazwaz, 2016), Bäcklund transform method (Rogers and Shadwick, 1982), inverse scattering transform (Ablowitz et al., 1991), Kudryashov scheme (Malik et al., 2021), Riccati equation method (Esen et al., 2022; Yıldırım et al., 2020), homogeneous balance method (Fan and Zhang, 1998), semi-inverse variational principle (Biswas et al., 2017; Kohl et al., 2020), modified extended tanh-method (El-Wakil and Abdou, 2007; Nuruddeen et al., 2020), Lie symmetry analysis (Kumar et al., 2020; Malik et al., 2021) and many more. Along with these methods, generalized (G'/G)-expansion scheme (Wang et al., 2008; Zhang et al., 2008) and NMSEM (Irshad et al., 2017) gain attention of the many researcher. In the generalized (G'/G)-expansion scheme, we utilized the basic knowledge of solutions of a second-order linear ordinary differential equation $G''(\theta)+\lambda G'(\theta)+\mu G(\theta)=0$, which produces the solutions in the form of hyperbolic, rational, and trigonometric functions. We didn't use any widely accepted fundamental equations in the NMSEM. As a result, it is impossible to predict, based on prior experience, what kinds of solutions can be obtained using this strategy. This is the uniqueness and attractiveness of the NMSEM approach. The current study explores the YTSF equation with time-dependent coefficients, which describes the elastic quasi plane wave in a lattice or interfacial wave in a two-layer liquid

$$\left(v_\varrho+\varphi(\varrho)v_{xxz}+\chi(\varrho)vv_z+\psi(\varrho)v_x\partial_x^{-1}v_z\right)_x+\omega(\varrho)v_{yy}=0, \tag{1}$$

where the integral operator is defined as $\partial_x^{-1}f=\int f dx$ and $\varphi(\varrho),\chi(\varrho),\psi(\varrho)$,and $\omega(\varrho)$ are the arbitrary functions of time-dependent variable ϱ. The constant coefficients version of Eq. (1)

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has been extensively exploited in literature. Nowadays, the variable version of this equation has been gaining significant attention. In Abdel-Gawad and Tantawy (2017), the authors find the single and double soliton rational solutions of Eq. (1) by using the two different versions of the unified method. Along with this, the authors in Abdel-Gawad and Tantawy (2017) show the propagation of semi-self similar wave solutions, which are a mixture of other wave solutions. In Wazwaz and Osman (2018), Wazwaz and Osman extended the work of Abdel-Gawad and Tantawy (2017) and derived different kinds of combined multi-wave polynomial solutions for Eq. (1) by the generalized unified approach. In this study, we use the generalized (G'/G)-expansion scheme and NMSEM to derive the exact solution of the (3 + 1)-dimensional YTSF Eq. (1) by inspiring the work of Abdel-Gawad and Tantawy (2017) and Wazwaz and Osman (2018). To the best of our knowledge, the problem under consideration has never been studied using these methods before.

The remainder of the study is organized as follows: The generalized (G'/G)-expansion method and NMSEM are described in Sec. (2). In Sec. (3), the suggested approaches are used to find exact solutions to the (3 + 1)-dimensional YTSF equation with time-dependent coefficients. Sec. (4) shows some graphic representations of some solutions that have been found. Finally, the conclusion is presented.

2. Methods and their ideas

This section describes the generalized $(\frac{G'}{G})$-expansion method and the NMSEM, both of which are used to generate exact solutions to NLEEs. For this, consider the following NLEE

$$F(u, u_\varrho, u_x, u_y, u_z, \dots u_{x\varrho}, u_{y\varrho}, u_{z\varrho}, u_{\varrho\varrho}, \dots) = 0, \tag{2}$$

where $u = u(x, y, z, \dots, \varrho)$ is an unknown function, $F$is a polynomial in $u = u(x, y, z, \dots, \varrho)$ and its partial derivatives and nonlinear terms.

2.1. The Generalized $(\frac{G'}{G})$-Expansion Method:

The following are the major steps in the generalized $(\frac{G'}{G})$-expansion scheme:

Step 1: Assume the solution of Eq. (2) can be stated as a polynomial in $(\frac{G'}{G})$ as follows:

$$u = \varphi_0(X) + \sum_{i=1}^{m} \varphi_i(X)\left(\frac{G'}{G}\right)^i, \varphi_m(X) \neq 0, \tag{3}$$

where $\varphi_0, \varphi_i, (i = 1, 2, \dots, m)$ are functions of $(x, y, z, \dots, \varrho)$, which will be revealed later and $G = G(\theta)$ satisfies following equation

$$G''(\theta) + \lambda G'(\theta) + \mu G(\theta) = 0, \tag{4}$$

where $\theta = \kappa(\varrho)x + \rho(\varrho)y + \sigma(\varrho)z + \tau(\varrho), \kappa(\varrho), \rho(\varrho), \sigma(\varrho)$ and $\tau(\varrho)$ are functions to be determined.

Step 2: Now the positive integer $m$is evaluated by applying the homogeneous balance between the highest order derivative and nonlinear terms of $u$in Eq. (2), which helps us to determine $u$explicitly.

Step 3: Putting Eq. (3) and Eq. (4) into Eq. (2), and gathering up all terms of the same order of (G'/G) together, the Eq. (2) is transferred into a polynomial in (G'/G). Then, for $\varphi_0(X), \varphi_i(X)$ and θ, we construct a set of over-determined partial differential equations by setting each coefficient of this polynomial to zero.

Step 4: By solving the system of partial differential equations obtained in the previous step, the expressions for $\varphi_0(X), \varphi_i(X)$ and θ can be found. Thus the solutions of Eq. (2) can be achieved based on the general solution of (4) and demonstrated as in the form of (G'/G), which is further dependent on the sign of the determinant $\lambda^2 - 4\mu$ as follows:

$$\left(\frac{G'}{G}\right) = \begin{cases} \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{a_1 \cosh((\sqrt{\lambda^2-4\mu}/2)\theta)+a_2 \sinh((\sqrt{\lambda^2-4\mu}/2)\theta)}{a_1 \sinh((\sqrt{\lambda^2-4\mu}/2)\theta)+a_2 \cosh((\sqrt{\lambda^2-4\mu}/2)\theta)}\right), & \lambda^2 - 4\mu > 0, \\ \frac{-\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{a_1 \cos((\sqrt{4\mu-\lambda^2}/2)\theta)-a_2 \sin((\sqrt{4\mu-\lambda^2}/2)\theta)}{a_1 \sin((\sqrt{4\mu-\lambda^2}/2)\theta)+a_2 \cos((\sqrt{4\mu-\lambda^2}/2)\theta)}\right), & \lambda^2 - 4\mu < 0, \\ -\frac{\lambda}{2} + \frac{a_2}{a_1+a_2\xi}, & \lambda^2 - 4\mu = 0, \end{cases} \tag{5}$$

where a_1 and a_2 are arbitrary constants.

2.2. The new modified simple equation method

The next steps will outline the essence of the NMSEM.

Step 1: Suppose that Eq. (2) has solution in the form of

$$v(x, y, z, \varrho) = A_0 + \sum_{r=1}^{N}\sum_{i+j=r} A_r \left[\frac{\psi_1'(\xi_1)}{\psi_1(\xi_1)}\right]^i \left[\frac{\psi_2'(\xi_2)}{\psi_2(\xi_2)}\right]^j, \tag{6}$$

where $\xi_i = \kappa_i(\varrho)x + q_i(\varrho)y + r_i(\varrho)z + s_i(\varrho), i = 1, 2.$ $A_0, A_r, (r = 1, 2, \dots, N)$ are unknown functions of ϱ, while $\psi_i(\xi_i), i = 1, 2$ are unknown functions to be revealed later.

Step 2: Now balance the highest-order derivatives and nonlinear terms in Eq. (2), to find the positive integer N.

Step 3: Calculate the necessary derivatives $v_x, v_{x\varrho}, v_{xy}, \dots$of the unknown function $v(x, y, z, \varrho)$ and replace them in Eq. (2). As a result, a polynomial of $\psi_1^{-i}\psi_2^{-j}(i, j = 0, 1, 2, \dots)$ can be obtained.

Step 4: From the obtained polynomial, now we gather all the terms of same power $\psi_1^{-i}\psi_2^{-j}, (i, j = 0, 1, 2, \dots)$ and by equating all these coefficients equal to zero, we yields a system of algebraic equations which can be solved to find A_k and $\psi_i(\xi_i), i = 1, 2$. As an outcome, we have the exact solutions of Eq. (2).

3. Application of proposed methods to the YTSF equation with time-dependent coefficients

In this section, we have derived the several exact solutions of the YTSF Eq. (1) by utilized the methods described in the above section. To overcome with the integral part in Eq. (1), we use the transformation $v = u_x$ and then by integrating Eq. (1) w.r.t. $x$with integral constant zero, we have

$$u_{x\varrho} + \varphi(\varrho)u_{xxxz} + \chi(\varrho)u_x u_{xz} + \psi(\varrho)u_{xx}u_z + \omega(\varrho)u_{yy} = 0. \tag{7}$$

Now the aim to find the exact solutions of the (3 + 1)-dimensional nonlinear evolutionary Eq. (1) is convert to find the exact solutions of Eq. (7). For this, first we find the solutions of Eq. (7) using the proposed methods, then we presents the solutions of governing Eq. (1).

3.1. Application of the $(\frac{G'}{G})$-expansion method

According to the method described in the previous section, by applying the balance principle, we have $m = 1$. Thus the solution of Eq. (7) can be written as:

$$u = \varphi_0(\varrho) + \varphi_1(\varrho)\left(\frac{G'}{G}\right). \tag{8}$$

Plugging Eq. (8) into Eq. (7) and using Eq. (4), collect all terms with the same order of $(\frac{G'}{G})$ together, these coefficients of different powers $(\frac{G'}{G})$ are polynomials in x, y and z, equating these polynomials equal to zero, we get the equations as follows:

$$
\begin{aligned}
0 = \; & -2\psi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho) - 2\chi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho) + 24\varphi(\varrho)\sigma(\varrho)\varphi_1(\varrho)\kappa(\varrho)^3,\\
0 = \; & 60\varphi(\varrho)\sigma(\varrho)\varphi_1(\varrho)\kappa(\varrho)^3\lambda - 5\psi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\lambda - 5\chi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\lambda,\\
0 = \; & 2\omega(\varrho)\varphi_1(\varrho)\rho(\varrho)^2 - 4\chi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\lambda^2 + 2\kappa(\varrho)\varphi_1(\varrho)\kappa'(\varrho)x + 2\kappa(\varrho)\varphi_1(\varrho)\rho'(\varrho)y + 2\kappa(\varrho)\varphi_1(\varrho)\\
& \sigma'(\varrho)z + 2\kappa(\varrho)\varphi_1(\varrho)\tau'(\varrho) + 40\varphi(\varrho)\sigma(\varrho)\varphi_1(\varrho)\kappa(\varrho)^3\mu - 4\chi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\mu + 50\varphi(\varrho)\sigma(\varrho)\varphi_1(\varrho)\\
& \kappa(\varrho)^3\lambda^2 - 4\psi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\mu - 4\psi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\lambda^2,\\
0 = \; & -6\chi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\lambda\mu + 3\kappa(\varrho)\varphi_1(\varrho)\lambda\rho'(\varrho)y + 3\kappa(\varrho)\varphi_1(\varrho)\lambda\kappa'(\varrho)x - \kappa'(\varrho)\varphi_1(\varrho) + 3\kappa(\varrho)\varphi_1(\varrho)\\
& \lambda\tau'(\varrho) - \psi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\lambda^3 + 60\varphi(\varrho)\sigma(\varrho)\varphi_1(\varrho)\kappa(\varrho)^3\lambda\mu + 3\kappa(\varrho)\varphi_1(\varrho)\lambda\sigma'(\varrho)z - \chi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\\
& \sigma(\varrho)\lambda^3 - \kappa(\varrho)\varphi'(\varrho) - 6\psi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\lambda\mu + 15\varphi(\varrho)\sigma(\varrho)\varphi_1(\varrho)\kappa(\varrho)^3\lambda^3 + 3\omega(\varrho)\varphi_1(\varrho)\rho(\varrho)^2\lambda,\\
0 = \; & \kappa(\varrho)\varphi_1(\varrho)\lambda^2\kappa'(\varrho)x - 2\chi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\lambda^2\mu + 22\varphi(\varrho)\sigma(\varrho)\varphi_1(\varrho)\kappa(\varrho)^3\lambda^2\mu - 2\psi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\\
& \sigma(\varrho)\lambda^2\mu + \varphi(\varrho)\sigma(\varrho)\varphi_1(\varrho)\kappa(\varrho)^3\lambda^4 + 2\kappa(\varrho)\varphi_1(\varrho)\mu\tau'(\varrho) - 2\chi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\mu^2 + 2\kappa(\varrho)\varphi_1(\varrho)\mu\rho'(\varrho)y\\
& -\kappa(\varrho)\varphi'(\varrho)\lambda - 2\psi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\mu^2 + \kappa(\varrho)\varphi_1(\varrho)\lambda^2\rho'(\varrho)y - \kappa'(\varrho)\varphi_1(\varrho)\lambda + 2\kappa(\varrho)\varphi_1(\varrho)\mu\kappa'(\varrho)x\\
& +\kappa(\varrho)\varphi_1(\varrho)\lambda^2\tau'(\varrho) + 2\omega(\varrho)\varphi_1(\varrho)\rho(\varrho)^2\mu + 16\varphi(\varrho)\sigma(\varrho)\varphi_1(\varrho)\kappa(\varrho)^3\mu^2 + \omega(\varrho)\varphi_1(\varrho)\rho(\varrho)^2\lambda^2 + \kappa(\varrho)\varphi_1(\varrho)\\
& \lambda^2\sigma'(\varrho)z + 2\kappa(\varrho)\varphi_1(\varrho)\mu\sigma'(\varrho)z,\\
0 = \; & -\psi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\lambda\mu^2 + \kappa(\varrho)\varphi_1(\varrho)\lambda\mu\tau'(\varrho) + \kappa(\varrho)\varphi_1(\varrho)\lambda\mu\kappa'(\varrho)x + \kappa(\varrho)\varphi_1(\varrho)\lambda\mu\sigma'(\varrho)z + \kappa(\varrho)\\
& \varphi_1(\varrho)\lambda\mu\rho'(\varrho)y + \varphi(\varrho)\sigma(\varrho)\varphi_1(\varrho)\kappa(\varrho)^3\lambda^3\mu - \kappa'(\varrho)\varphi_1(\varrho)\mu - \chi(\varrho)\varphi_1(\varrho)^2\kappa(\varrho)^2\sigma(\varrho)\mu^2\lambda + 8\varphi(\varrho)\sigma(\varrho)\\
& \varphi_1(\varrho)\kappa(\varrho)^3\lambda\mu^2 - \kappa(\varrho)\varphi'(\varrho)\mu + \omega(\varrho)\varphi_1(\varrho)\rho(\varrho)^2\lambda\mu.
\end{aligned} \tag{9}
$$

Solving system (9) with the help of Maple software, gives

$$
\begin{aligned}
&\kappa(\varrho) = c_1, \quad \rho(\varrho) = c_2, \quad \sigma(\varrho) = c_3,\\
&\tau(\varrho) = \int -\left(\frac{\omega(\varrho)c_2^2 - 4\varphi(\varrho)c_3c_1^3\mu + \varphi(\varrho)c_3c_1^3\lambda^2}{c_1}\right)d\varrho + c_4,\\
&\chi(\varrho) = \frac{12\varphi(\varrho)c_1^2 - \psi(\varrho)c_5}{c_5}, \quad \varphi_1(\varrho) = \frac{c_5}{c_1},
\end{aligned} \tag{10}
$$

where c_i, $(i = 1, 2, \ldots, 5)$ are arbitrary constants. Substituting the general solution (5) of Eq. (4) into Eq. (8), we have three types of exact solution of Eq. (7) as follows:

Case 1: When $\lambda^2 - 4\mu > 0$, the solution of Eq. (7) is given as

$$
u(x,y,z,\varrho) = \varphi_0(\varrho) + \frac{c_5}{c_1}\left[\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}\left(\frac{a_1\cosh\phi + a_2\sinh\phi}{a_1\sinh\phi + a_2\cosh\phi}\right)\right]. \tag{11}
$$

Hence, derivative of Eq. (11) w.r.t. x, yields the hyperbolic solution of Eq. (1) as:

$$
v(x,y,z,\varrho) = \frac{1}{4}\,\frac{c_5(-\lambda^2 + 4\mu)(a_1^2 - a_2^2)}{(a_2\cosh\phi - a_1\sinh\phi)^2}, \tag{12}
$$

where a_1, a_2 are free parameters and ϕ is represent by

$$
\phi = \frac{-\sqrt{\lambda^2 - 4\mu}}{2}\left[c_1x + c_2y + c_3z - \int\left(\frac{\omega(\varrho)c_2^2 - 4\varphi(\varrho)c_3c_1^3\mu + \varphi(\varrho)c_3c_1^3\lambda^2}{c_1}\right)d\varrho + c_4\right].
$$

Case 2: When $\lambda^2 - 4\mu = 0$, the solution of Eq. (7) is given as:

$$
u(x,y,z,\varrho) = \varphi_0(\varrho) + \frac{c_5}{c_1}\left(\frac{-\lambda}{2} + \frac{a_2}{a_1 + a_2\theta}\right). \tag{13}
$$

Hence, derivative of Eq. (13) w.r.t. x, yields the following rational solution of Eq. (1)

$$
v(x,y,z,\varrho) = -\frac{c_5a_2^2}{(a_1 + a_2\theta)^2}, \tag{14}
$$

where a_1, a_2 are free parameters and θ is represent by

$$
\theta = c_1x + c_2y + c_3z - \int\left(\frac{\omega(\varrho)c_2^2 - 4\varphi(\varrho)c_3c_1^3\mu + \varphi(\varrho)c_3c_1^3\lambda^2}{c_1}\right)d\varrho + c_4.
$$

Case 3: When $\lambda^2 - 4\mu < 0$, the solution of Eq. (7) is given as

$$
u(x,y,z,\varrho) = \varphi_0(\varrho) + \frac{c_5}{c_1}\left[\frac{-\lambda}{2} + \frac{\sqrt{-\lambda^2 + 4\mu}}{2}\left(\frac{a_1\cos\zeta - a_2\sin\zeta}{a_1\sin\zeta + a_2\cos\zeta}\right)\right]. \tag{15}
$$

Finally, the trigonometric solution of Eq. (1) is expressed in the following manner

$$
v(x,y,z,\varrho) = \frac{c_5(a_1^2 + a_2^2)(\lambda^2 - 4\mu)}{4(a_1\sin\zeta + a_2\cos\zeta)^2}, \tag{16}
$$

where a_1, a_2 are arbitrary constants and ζ is given by

$$
\zeta = \frac{\sqrt{-\lambda^2 + 4\mu}}{2}\left[c_1x + c_2y + c_3z - \int\left(\frac{\omega(\varrho)c_2^2 - 4\varphi(\varrho)c_3c_1^3\mu + \varphi(\varrho)c_3c_1^3\lambda^2}{c_1}\right)d\varrho + c_4\right].
$$

3.2. Application of NMSEM

According to the method described in previous section, from Eq. (7), the positive integer $N$is determined by classical balance procedure as $N = 1$. Thus the solution of Eq. (7) according to Eq. (6) can be written as:

$$
u(x,y,z,\varrho) = \varphi_0(\varrho) + \varphi_1(\varrho)\frac{\psi_1'(\xi_1)}{\psi_1(\xi_1)} + \varphi_2(\varrho)\frac{\psi_2'(\xi_2)}{\psi_2(\xi_2)}, \tag{17}
$$

where $\xi_1 = \kappa_1(\varrho)x + \rho_1(\varrho)y + \sigma_1(\varrho)z + \tau_1(\varrho)$, $\xi_2 = \kappa_2(\varrho)x + \rho_2(\varrho)y + \sigma_2(\varrho)z + \tau_2(\varrho)$. Now substituting Eq. (17) into Eq. (7) we may get a polynomial of $\psi_1^{-i}\psi_2^{-j}$, gather all the coefficients of same power of $\psi_1^{-i}\psi_2^{-j}$ we get the following system of equations

$$
\begin{aligned}
0 = {}& \kappa_2(\varrho)\varphi_2(\varrho)\psi_2'''(\xi_2)\kappa_2'(\varrho)\kappa_2'(\varrho)x + \varphi(\varrho)\kappa_2(\varrho)^3\sigma_2(\varrho)\varphi_2(\varrho)\psi_2'''''(\xi_2) + \kappa_2'(\varrho)\varphi_2(\varrho)\tau_2(\varrho)\psi_2''(\xi_2) + \kappa_2'(\varrho)\\
& \kappa_2(\varrho)\psi_2''(\xi_2)\varphi_2(\varrho)x + \kappa_2'(\varrho)\rho_2(\varrho)\psi_2''(\xi_2)\varphi_2(\varrho)y + \kappa_2(\varrho)\varphi_2(\varrho)\rho_2'(\varrho)\psi_2'''(\xi_2)y + \kappa_2(\varrho)\varphi_2(\varrho)\psi_2'''(\xi_2)\\
& \sigma_2'(\varrho)z + \kappa_2(\varrho)\varphi_2(\varrho)\tau_2'(\varrho)\psi_2'''(\xi_2) + \kappa_2(\varrho)\varphi_2'(\varrho)\psi_2''(\xi_2) + \kappa_2'(\varrho)\sigma_2(\varrho)\varphi_2(\varrho)\psi_2''(\xi_2)z + \omega(\varrho)\rho_2(\varrho)^2\\
& \varphi_2(\varrho)\psi_2'''(\xi_2),\\
0 = {}& \kappa_1(\varrho)\varphi_1(\varrho)\psi_1'''(\xi_1)\tau_1'(\varrho) + \varphi(\varrho)\kappa_1(\varrho)^3\sigma_1(\varrho)\varphi_1(\varrho)\psi_1'''''(\xi_1) + \varphi_1'(\varrho)\kappa_1(\varrho)\psi_1''(\xi_1) + \kappa_1(\varrho)\varphi_1(\varrho)\psi_1'''(\xi_1)\\
& \kappa_1'(\varrho)x + \kappa_1(\varrho)\varphi_1(\varrho)\psi_1'''(\xi_1)\rho_1'(\varrho)y + \omega(\varrho)\rho_1(\varrho)^2\varphi_1(\varrho)\psi_1'''(\xi_1) + \varphi_1(\varrho)\psi_1''(\xi_1)\kappa_1'(\varrho) + \kappa_1(\varrho)\varphi_1(\varrho)\\
& \sigma_1'(\varrho)\psi_1'''(\xi_1)z,\\
0 = {}& \chi(\varrho)\kappa_1(\varrho)\varphi_1(\varrho)\kappa_2(\varrho)\sigma_2(\varrho)\varphi_2(\varrho)\psi_1''(\xi_1)\psi_2'''(\xi_2) + \chi(\varrho)\kappa_2(\varrho)\varphi_2(\varrho)\kappa_1(\varrho)\sigma_1(\varrho)\varphi_1(\varrho)\psi_1'''(\xi_1)\psi_2''(\xi_2)\\
& +\psi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)\sigma_1(\varrho)\varphi_1(\varrho)\psi_1'''(\xi_1)\psi_2''(\xi_2) + \psi(\varrho)\kappa_1(\varrho)^2\varphi_1(\varrho)\varphi_2(\varrho)\sigma_2(\varrho)\psi_1'''(\xi_1)\psi_2''(\xi_2),\\
0 = {}& -2\chi(\varrho)\kappa_2(\varrho)\varphi_2(\varrho)\kappa_1(\varrho)\sigma_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)^3\psi_2'(\xi_2)^2 - 2\psi(\varrho)\kappa_1(\varrho)^2\varphi_1(\varrho)\sigma_2(\varrho)\varphi_2(\varrho)\psi_1'(\xi_1)^3\psi_2'(\xi_2)^2,\\
0 = {}& -3\psi(\varrho)\kappa_1(\varrho)^2\varphi_1(\varrho)\sigma_2(\varrho)\varphi_2(\varrho)\psi_1'(\xi_1)\psi_1''(\xi_1)\psi_2''(\xi_2) - \psi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)\sigma_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)^2\psi_2'''(\xi_2)\\
& -3\chi(\varrho)\sigma_1(\varrho)\kappa_2(\varrho)\varphi_2(\varrho)\varphi_1(\varrho)\kappa_1(\varrho)\psi_1'(\xi_1)\psi_1''(\xi_1)\psi_2''(\xi_2) - \chi(\varrho)\sigma_2(\varrho)\kappa_1(\varrho)\varphi_1(\varrho)\kappa_2(\varrho)\varphi_2(\varrho)\psi_1(\xi_1)^2\\
& \psi_2'''(\xi_2),\\
0 = {}& -\chi(\varrho)\kappa_2(\varrho)\varphi_2(\varrho)\kappa_1(\varrho)\sigma_1(\varrho)\varphi_1(\varrho)\psi_1'''(\xi_1)\psi\prime_2(\xi_2)^2 - 3\psi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)\varphi_1(\varrho)\sigma_1(\varrho)\psi_2'(\xi_2)\psi_2''(\xi_2)\psi_1''(\xi_1)\\
& -3\chi(\varrho)\kappa_1(\varrho)\varphi_1(\varrho)\kappa_2(\varrho)\sigma_2(\varrho)\varphi_2(\varrho)\psi_2'(\xi_2)\psi_2''(\xi_2)\psi_1''(\xi_1) - \psi(\varrho)\kappa_1(\varrho)^2\varphi_1(\varrho)\sigma_2(\varrho)\varphi_2(\varrho)\psi_1'''(\xi_1)\psi_2'(\xi_2)^2,\\
0 = {}& 3\chi(\varrho)\kappa_1(\varrho)\varphi_1(\varrho)\kappa_2(\varrho)\sigma_2(\varrho)\varphi_2(\varrho)\psi_1'(\xi_1)^2\psi_2''(\xi_2)\psi_2'(\xi_2) + 3\psi(\varrho)\kappa_1(\varrho)^2\varphi_1(\varrho)\sigma_2(\varrho)\varphi_2(\varrho)\psi_1'(\xi_1)\psi_2'(\xi_2)^2\\
& \psi_1''(\xi_1) + 3\psi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)\sigma_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)^2\psi_2'(\xi_2)\psi_2''(\xi_2) + 3\chi(\varrho)\kappa_2(\varrho)\varphi_2(\varrho)\kappa_1(\varrho)\sigma_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)\\
& \psi_2'(\xi_2)^2\psi_1''(\xi_1),\\
0 = {}& 2\chi(\varrho)\kappa_1(\varrho)\varphi_1(\varrho)\kappa_2(\varrho)\sigma_2(\varrho)\varphi_2(\varrho)\psi_1''(\xi_1)\psi_2'(\xi_2)^3 + 2\psi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)\sigma_1(\varrho)\varphi_1(\varrho)\psi_1''(\xi_1)\psi_2'(\xi_2)^3,\\
0 = {}& -2\chi(\varrho)\kappa_1(\varrho)\varphi_1(\varrho)\kappa_2(\varrho)\sigma_2(\varrho)\varphi_2(\varrho)\psi_1'(\xi_1)^2\psi_2'(\xi_2)^3 - 2\psi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)\sigma_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)^2\psi_2'(\xi_2)^3,\\
0 = {}& 2\psi(\varrho)\kappa_1(\varrho)^2\varphi_1(\varrho)\sigma_2(\varrho)\varphi_2(\varrho)\psi_1\prime^3(\xi_1)\psi_2''(\xi_2) + 2\chi(\varrho)\kappa_2(\varrho)\varphi_2(\varrho)\kappa_1(\varrho)\sigma_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)^3\psi_2''(\xi_2),\\
0 = {}& \psi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)^2\psi_2''(\xi_2)\sigma_2(\varrho)\psi_2'''(\xi_2) - \varphi_2'(\varrho)\kappa_2(\varrho)\psi_2'(\xi_2)^2 - 3\kappa_2(\varrho)\varphi_2(\varrho)\psi_2''(\xi_2)\psi_2'(\xi_2)\tau_2'(\varrho)\\
& -\kappa_2'(\varrho)\varphi_2(\varrho)\psi_2'(\xi_2)^2\kappa_2(\varrho)x - \kappa_2'(\varrho)\varphi_2(\varrho)\rho_2(\varrho)\psi_2'(\xi_2)^2y - \kappa_2'(\varrho)\varphi_2(\varrho)\psi_2'(\xi_2)^2\sigma_2(\varrho)z - 3\kappa_2(\varrho)\\
& \varphi_2(\varrho)\kappa_2'(\varrho)\psi_2''(\xi_2)\psi_2'(\xi_2)x + \chi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)^2\sigma_2(\varrho)\psi_2''(\xi_2)\psi_2'''(\xi_2) - 3\omega(\varrho)\rho_2(\varrho)^2\varphi_2(\varrho)\psi_2''(\xi_2)\\
& \psi_2'(\xi_2) - 10\varphi(\varrho)\kappa_2(\varrho)^3\sigma_2(\varrho)\varphi_2(\varrho)\psi_2'''(\xi_2)\psi_2''(\xi_2) - 3\kappa_2(\varrho)\varphi_2(\varrho)\psi_2'(\xi_2)\psi_2''(\xi_2)\sigma_2'(\varrho)z - 5\varphi(\varrho)\\
& \kappa_2(\varrho)^3\sigma_2(\varrho)\varphi_2(\varrho)\psi_2'''\prime(\xi_2)\psi_2'(\xi_2) - \kappa_2'(\varrho)\varphi_2(\varrho)\psi_2'(\xi_2)^2\tau_2(\varrho) - 3\kappa_2(\varrho)\varphi_2(\varrho)\psi_2''(\xi_2)\psi_2'(\xi_2)\rho_2'(\varrho)y,\\
0 = {}& 5\psi(\varrho)\kappa_1(\varrho)^2\varphi_1(\varrho)^2\sigma_1(\varrho)\psi_1'(\xi_1)^3\psi_1''(\xi_1) - 60\varphi(\varrho)\kappa_1(\varrho)^3\sigma_1(\varrho)\varphi_1(\varrho)\psi_1''(\xi_1)\psi_1'(\xi_1)^3 + 5\chi(\varrho)\kappa_1(\varrho)^2\\
& \varphi_1(\varrho)^2\sigma_1(\varrho)\psi_1'(\xi_1)^3\psi_1''(\xi_1),\\
0 = {}& -60\varphi(\varrho)\kappa_2(\varrho)^3\sigma_2(\varrho)\varphi_2(\varrho)\psi_2''(\xi_2)\psi_2'(\xi_2)^3 + 5\psi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)^2\sigma_2(\varrho)\psi_2''(\xi_2)\psi_2'(\xi_2)^3 + 5\kappa_2(\varrho)^2\\
& \chi(\varrho)\varphi_2(\varrho)^2\sigma_2(\varrho)\psi_2''(\xi_2)\psi_1'(\xi_2)^3,\\
0 = {}& 2\kappa_2(\varrho)\varphi_2(\varrho)\sigma_2'(\varrho)\psi_2'(\xi_2)^3z + 2\kappa_2(\varrho)\varphi_2(\varrho)\psi_2'(\xi_2)^3\kappa_2'(\varrho)x + 2\kappa_2(\varrho)\varphi_2(\varrho)\psi_2'(\xi_2)^3\rho_2'(\varrho)y + 2\omega(\varrho)\\
& \rho_2(\varrho)^2\varphi_2(\varrho)\psi_2'(\xi_2)^3 + 30\varphi(\varrho)\kappa_2(\varrho)^3\sigma_2(\varrho)\varphi_2(\varrho)\psi_2''(\xi_2)^2\psi_2'(\xi_2) - 3\chi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)^2\sigma_2(\varrho)\psi_2''(\xi_2)^2\\
& \psi_2'(\xi_2) - 3\psi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)^2\sigma_2(\varrho)\psi_2''(\xi_2)^2\psi_2'(\xi_2) - \psi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)^2\sigma_2(\varrho)\psi_2'''(\xi_2)\psi_2'(\xi_2)^2 - \chi(\varrho)\\
& \kappa_2(\varrho)^2\varphi_2(\varrho)^2\sigma_2(\varrho)\psi_2'(\xi_2)^2\psi_2'''(\xi_2) + 20\varphi(\varrho)\kappa_2(\varrho)^3\sigma_2(\varrho)\varphi_2(\varrho)\psi_2'''(\xi_2)\psi_2'(\xi_2)^2 + 2p_2\varphi_2(\varrho)\tau_2'(\varrho)\psi_2'(\xi_2)^3,\\
0 = {}& 2\kappa_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)^3\sigma_1'(\varrho)z - \psi(\varrho)\kappa_1(\varrho)^2\varphi_1(\varrho)^2\psi_1'(\xi_1)^2\sigma_1(\varrho)\psi_1'''(\xi_1) - 3\chi(\varrho)\kappa_1(\varrho)^2\varphi_1(\varrho)^2\psi_1''(\xi_1)^2\\
& \sigma_1(\varrho)\psi_1'(\xi_1) + 2\kappa_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)^3\rho_1'(\varrho)y + 2\omega(\varrho)\rho_1(\varrho)^2\varphi_1(\varrho)\psi_1'(\xi_1)^3 + 20\varphi(\varrho)\kappa_1(\varrho)^3\sigma_1(\varrho)\varphi_1(\varrho)\\
& \psi_1'''(\xi_1)\psi_1'(\xi_1)^2 + 2\kappa_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)^3\kappa_1'(\varrho)x + 30\varphi(\varrho)\kappa_1(\varrho)^3\sigma_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)\psi_1''(\xi_1)^2 - \chi(\varrho)\kappa_1(\varrho)^2\\
& \varphi_1(\varrho)^2\sigma_1(\varrho)\psi_1'(\xi_1)^2\psi_1'''(\xi_1) + 2\kappa_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)^3\tau_1'(\varrho) - 3\psi(\varrho)\kappa_1(\varrho)^2\varphi_1(\varrho)^2\sigma_1(\varrho)\psi_1''(\xi_1)^2\psi_1'(\xi_1),\\
0 = {}& -\kappa_1'(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)^2 - \varphi_1'(\varrho)\kappa_1(\varrho)\psi_1'(\xi_1)^2 - 5\varphi(\varrho)\kappa_1(\varrho)^3\sigma_1(\varrho)\varphi_1(\varrho)\psi_1'''\prime(\xi_1)\psi_1'(\xi_1) - 3\omega(\varrho)\rho_1(\varrho)^2\\
& \varphi_1(\varrho)\psi_1'(\xi_1)\psi_1''(\xi_1) - 3\kappa_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)\psi_1''(\xi_1)\sigma_1'(\varrho)z + \chi(\varrho)\kappa_1(\varrho)^2\sigma_1(\varrho)\varphi_1(\varrho)^2\psi_1'''(\xi_1)\psi_1''(\xi_1) - 3\kappa_1(\varrho)\\
& \varphi_1(\varrho)\psi_1''(\xi_1)\psi_1'(\xi_1)\kappa_1'(\varrho)x + \psi(\varrho)\kappa_1(\varrho)^2\varphi_1(\varrho)^2\sigma_1(\varrho)\psi_1''(\xi_1)\psi_1'''(\xi_1) - 10\varphi(\varrho)\kappa_1(\varrho)^3\sigma_1(\varrho)\varphi_1(\varrho)\psi_1''(\xi_1)\\
& \psi_1'''(\xi_1) - 3\kappa_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)\psi_1''(\xi_1)\rho_1'(\varrho)y - 3\kappa_1(\varrho)\varphi_1(\varrho)\psi_1''(\xi_1)\psi_1'(\xi_1)\tau_1'(\varrho),\\
0 = {}& -2\chi(\varrho)\kappa_1(\varrho)^2\varphi_1(\varrho)^2\sigma_1(\varrho)\psi_1'(\xi_1)^5 + 24\varphi(\varrho)\kappa_1(\varrho)^3\sigma_1(\varrho)\varphi_1(\varrho)\psi_1'(\xi_1)^5 - 2\psi(\varrho)\kappa_1(\varrho)^2\varphi_1(\varrho)^2\sigma_1(\varrho)\psi_1'(\xi_1)^5,\\
0 = {}& -2\chi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)^2\sigma_2(\varrho)\psi_2'(\xi_2)^5 - 2\psi(\varrho)\kappa_2(\varrho)^2\varphi_2(\varrho)^2\sigma_2(\varrho)\psi_2'(\xi_2)^5 + 24\varphi(\varrho)\kappa_2(\varrho)^3\sigma_2(\varrho)\varphi_2(\varrho)\psi_2'(\xi_2)^5.
\end{aligned}
\tag{18}
$$

Solving system (18) with the help of maple software, gives

Case 1:

$$\begin{aligned} &\kappa_1(\varrho) = c_1, \quad \rho_1(\varrho) = c_2, \quad \sigma_1(\varrho) = c_3, \quad \kappa_2(\varrho) = c_4, \\ &\rho_2(\varrho) = c_5, \quad \sigma_2(\varrho) = c_6, \quad \psi(\varrho) = -\chi(\varrho) + \varphi(\varrho)k_5, \\ &\tau_2(\varrho) = -\int \tfrac{\omega(\varrho)c_5^2}{c_4} d\varrho + k_1, \quad \varphi_2(\varrho) = 12\tfrac{c_4}{k_5}, \\ &\psi_1(\xi_1) = k_5, \quad \psi_2(\xi_2) = k_2\xi_2 + k_3. \end{aligned} \tag{19}$$

So, from Eqs. (17) and (19), the rational solution corresponding to Eq. (7) can be expressed as

$$u(x,y,z,\varrho) = \varphi_0(\varrho) + 12\frac{c_4}{k_5}\left(\frac{k_2}{k_2\xi_2 + k_3}\right). \tag{20}$$

Hence, derivative of Eq. (20) w.r.t. x, gives the solution of our main Eq. (1) as

$$v(x,y,z,\varrho) = \frac{-12k_2^2c_4^2}{(k_2\xi_2 + k_3)^2}k_5, \tag{21}$$

where $\xi_2 = c_4x + c_5y + c_6z - \int \frac{\omega(\varrho)c_5^2}{c_4} d\varrho + k_1$.

Case 2:

$$\begin{aligned} &\kappa_1(\varrho) = c_1, \quad \rho_1(\varrho) = c_2, \quad \sigma_1(\varrho) = c_3, \\ &\kappa_2(\varrho) = c_4, \quad \rho_2(\varrho) = c_5, \quad \sigma_2(\varrho) = c_6, \quad \psi(\varrho) = -\chi(\varrho) + \varphi(\varrho)k_6, \\ &\tau_2(\varrho) = -\int \left(\tfrac{\varphi(\varrho)c_4^3c_6k_2^2 + \omega(\varrho)c_5^2}{c_4}\right) d\varrho + k_1, \quad \varphi_2(\varrho) = 12\tfrac{c_4}{k_6}, \\ &\psi_1(\xi_1) = k_5, \quad \psi_2(\xi_2) = \tfrac{e^{k_2(\xi_2+k_3)}}{k_2} + k_4. \end{aligned} \tag{22}$$

So, the solution of Eq. (7) can be represented as

$$u(x,y,z,\varrho) = \varphi_0(\varrho) + 12\frac{c_4k_2}{k_6}\left(\frac{e^{k_2(\xi_2+k_3)}}{e^{k_2(\xi_2+k_3)} + k_2k_4}\right). \tag{23}$$

Hence, derivative of Eq. (23) w.r.t. x, gives the solution of our main Eq. (1) as

$$v(x,y,z,\varrho) = 12\frac{c_4^2k_2^2}{k_6}\left(\frac{e^{k_2(\xi_2+k_3)}}{e^{k_2(\xi_2+k_3)} + k_2k_4} - \frac{e^{2k_2(\xi_2+k_3)}}{(e^{k_2(\xi_2+k_3)} + k_2k_4)^2}\right), \tag{24}$$

where $\xi_2 = c_4x + c_5y + c_6z - \int \left(\frac{\varphi(\varrho)c_4^3c_6k_2^2 + \omega(\varrho)c_5^2}{c_4}\right) d\varrho + k_1$.

Case 3:

$$\begin{aligned} &\kappa_1(\varrho) = c_1, \quad \rho_1(\varrho) = c_2, \quad \sigma_1(\varrho) = c_3, \quad \kappa_2(\varrho) = c_4, \\ &\rho_2(\varrho) = c_5, \quad \sigma_2(\varrho) = c_6, \quad \chi(\varrho) = \varphi(\varrho)k_4, \\ &\tau_1(\varrho) = \int \left(\tfrac{-\omega(\varrho)c_2^2}{c_1}\right) d\varrho + k_1, \quad \psi(\varrho) = -\tfrac{\varphi(\varrho)k_4c_4c_3}{c_1c_6}, \\ &\varphi_1(\varrho) = \tfrac{12c_1^2c_6}{k_4c_1c_6 - k_4c_4c_3}, \quad \varphi_2(\varrho) = 0, \\ &\psi_1(\xi_1) = k_2\xi_1 + k_3, \quad \psi_2(\xi_2) = \psi_2(\xi_2). \end{aligned} \tag{25}$$

So, the solution of Eq. (7) can be represented as

$$u(x,y,z,\varrho) = \varphi_0(\varrho) + \frac{12c_1^2c_6}{k_4c_1c_6 - k_4c_4c_3}\left(\frac{k_2}{k_2\xi_1 + k_3}\right). \tag{26}$$

Hence, derivative of Eq. (26) w.r.t. x, gives the following rational solution of our main Eq. (1) as

$$v(x,y,z,\varrho) = \frac{-12c_1^3c_6k_2^2}{(k_4c_1c_6 - k_4c_4c_3)(k_2\xi_1 + k_3)^2}, \tag{27}$$

where

$$\xi_1 = c_1x + c_2y + c_3z + \int \left(\frac{-\omega(\varrho)c_2^2}{c_1}\right) d\varrho + k_1.$$

Case 4:

$$\begin{aligned} &\kappa_1(\varrho) = c_1, \quad \rho_1(\varrho) = c_2, \quad \sigma_1(\varrho) = c_3, \quad \kappa_2(\varrho) = c_4, \\ &\rho_2(\varrho) = c_5, \quad \sigma_2(\varrho) = c_6, \quad \chi(\varrho) = k_7\varphi(\varrho), \\ &\psi(\varrho) = \tfrac{-\varphi(\varrho)k_7c_4c_3}{c_1c_6}, \quad \tau_1(\varrho) = -\int \left(\tfrac{\varphi(\varrho)c_1^3c_3k_4^2 + \omega(\varrho)c_2^2}{c_1}\right) d\varrho + k_1, \\ &\varphi_1(\varrho) = \tfrac{12c_1^2c_6}{k_7c_1c_6 - k_7c_4c_3}, \quad \varphi_2(\varrho) = 0, \\ &\psi_1(\xi_1) = \tfrac{e^{k_4(\xi_1+k_5)}}{k_4} + k_6, \quad \psi_2(\xi_2) = k_2\xi_2 + k_3. \end{aligned} \tag{28}$$

So, the solution of Eq. (7) becomes

$$u(x,y,z,\varrho) = \varphi_0(\varrho) + \frac{12c_1^2c_6k_4}{k_7c_1c_6 - k_7c_4c_3}\left(\frac{e^{k_4(\xi_1+k_5)}}{e^{k_4(\xi_1+k_5)} + k_6k_4}\right). \tag{29}$$

Hence, we arrive to the solution of our main Eq. (1) as

$$v(x,y,z,\varrho) = 12\frac{c_1^2c_6k_4^2}{k_7c_1c_6 - k_7c_4c_3}\left(\frac{e^{k_4(\xi_1+k_5)}}{e^{k_4(\xi_1+k_5)} + k_6k_4} - \frac{e^{2k_4(\xi_1+k_5)}}{(e^{k_4(\xi_1+k_5)} + k_6k_4)^2}\right), \tag{30}$$

where

$$\xi_1 = c_1x + c_2y + c_3z - \int \left(\frac{\varphi(\varrho)c_1^3c_3k_4^2 + \omega(\varrho)c_2^2}{c_1}\right) d\varrho + k_1.$$

4. Graphical representation

The physical interpretation of the solution of nonlinear differential equation can be done by its graphical representation, hence is important to be given. Fig. 1 demonstrate the solution Eq. (12) against $x$and ϱ axis, at the points $\lambda = 3, \mu = c_1 = c_2 = c_3 = c_5 = 1, c_4 = 0, a_1 = 2, a_2 = 1, \omega(\varrho) = \varphi(\varrho) = 1, y = -z = k$, which shows the periodic solution. Fig. 2 represents the solution Eq. (14) against $x$and ϱ axis, at the points $\lambda = a_1 = 2, \mu = c_1 = c_2 = c_3 = c_5 = a_2 = 1, \omega(\varrho) = \varphi(\varrho) = 1$. Fig. 3 demonstrate the solution Eq. (16) against $x$and ϱ axis, at the points $\lambda = \mu = c_1 = c_2 = c_3 = c_5 = 1, c_4 = 0, a_1 = 2, a_2 = 1, \omega(\varrho) = 1, \varphi(\varrho) = 1, y = -z = k$. Similarly, we have shown the graphical representation of solutions Eq. (21), Eq. (24), Eq. (27) and Eq. (30) as Fig. 4-Fig. 7, respectively. The Fig. 5 and Fig. 7 represent the bright wave solutions, while the other figures represents the periodical solutions. (See Fig. 6).

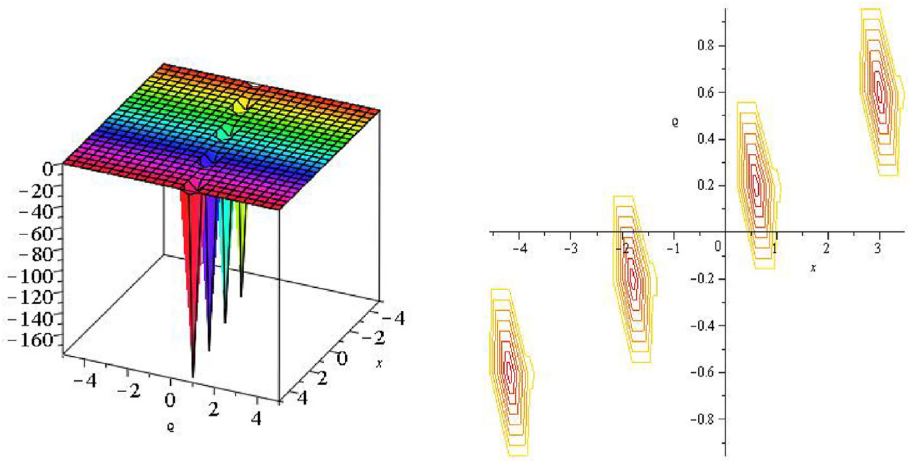

Fig. 1. 3D and contour plot of Eq. (12).

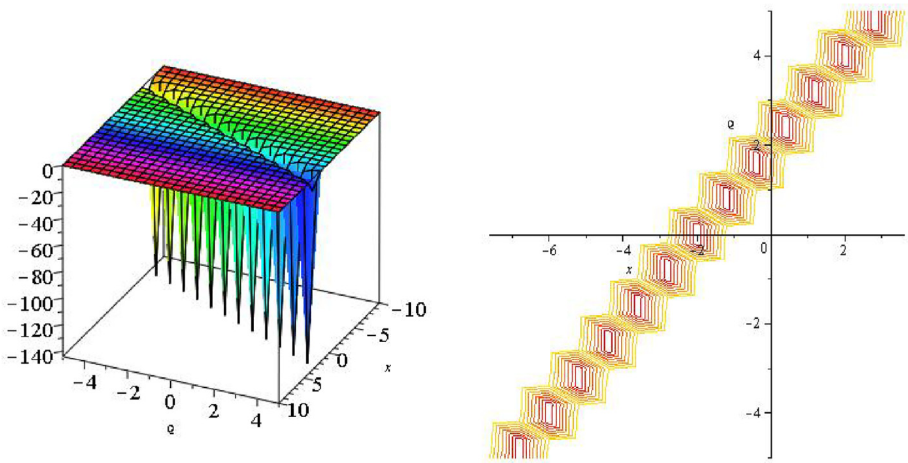

Fig. 2. 3D and contour plot of Eq. (14).

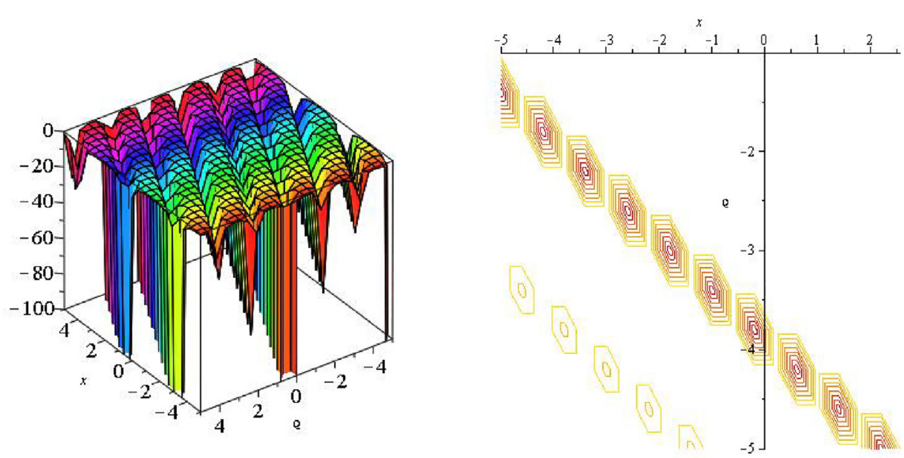

Fig. 3. 3D and contour plot of Eq. (16).

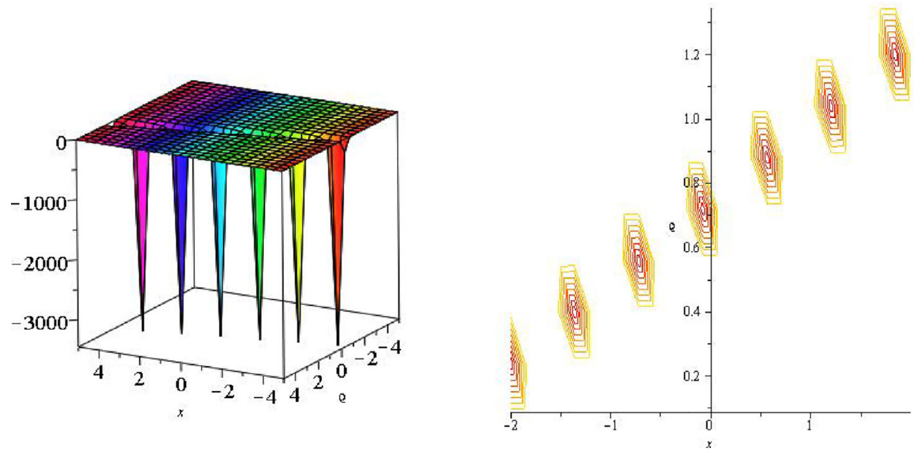

Fig. 4. 3D and contour plot of Eq. (21).

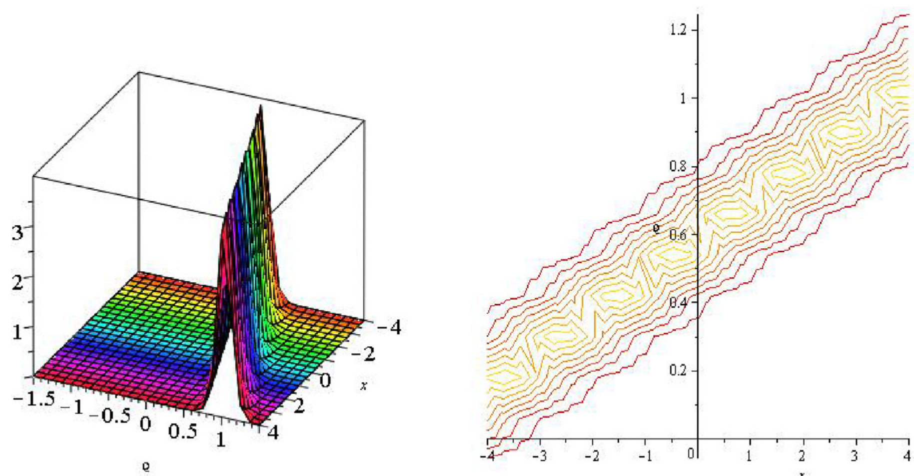

Fig. 5. 3D and contour plot of Eq. (24).

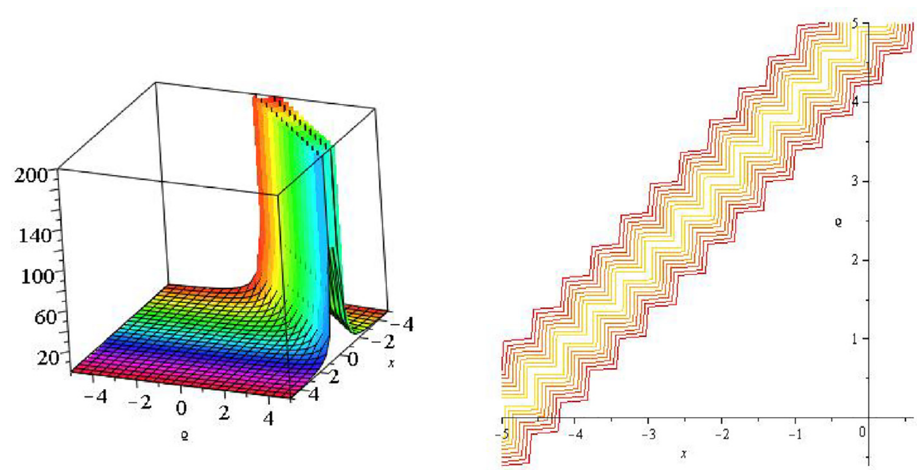

Fig. 6. 3D and contour plot of Eq. (27).

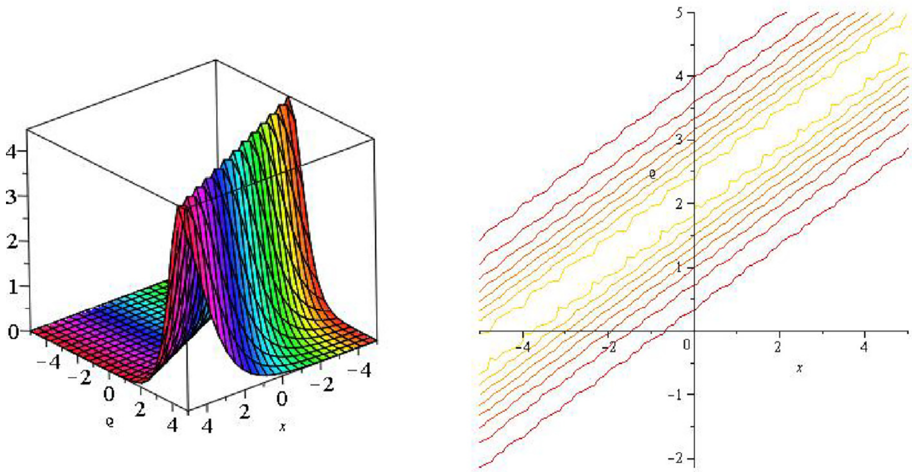

Fig. 7. 3D and contour plot of Eq. (30).

5. Conclusion

In this paper, we observed that the generalized (G'/G)-expansion method and NMSEM are convenient tools to obtain the solutions of the nonlinear (3 + 1)-dimensional YTSF equation. We study this equation with time-dependent coefficients, as in the real world phenomena, coefficients of nonlinear evolution equations vary with time and space. As a result, the exact solution of the variable coefficient nonlinear evolution equations has a broader range of applications. The obtained solutions can be divided into three families of the hyperbolic, rational and trigonometric functions, including independent variables and arbitrary parameters. Some solutions are represented graphically by giving particular values to the arbitrary constants and arbitrary parameters. Using these elementary and concise methods, namely the generalized (G'/G)-expansion approach and NMSEM, many nonlinear evolution equations of mathematical physics and other applied sciences can be solved efficiently. In the future, we will try to study the (3 + 1)-dimensional YTSF equation with time-dependent coefficients with the help of Lie symmetry analysis and derive some invariant and soliton solutions.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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