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Jacobi elliptic function solutions for a two-mode KdV equation

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ABSTRACT

In this paper we introduce a new type of KdV equations called Two-mode KdV (TMKdV). This equation represents the propagation of two-wave modes in the same direction simultaneously. We suggest a finite series of degree n in terms of a Jacobi elliptic functions as a possible solution for the TMKdV. We succeeded in obtaining new solutions of type solitons and periodics.

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1. Introduction

Most of nonlinear equations are defined by first-order partial differential equation (PDE) in time. They describe unidirectional waves where these equations model one right-moving for $x > o$. Such equations is the well-known KdV. Other equations model two left-right-moving as in Boussinesq which is defined by second-order PDE in time. Therefore, Korsunsky (1994) developed this phenomena and he was able to identify and derive two-mode KdV (TMKdV) as a nonlinear PDE of second order in time. This two-mode equation represents the propagation of two-wave modes in the same direction simultaneously. The TMKdV reads as

$$\begin{aligned} 0 = & u_{tt} + (c_1 + c_2)u_{xt} + c_1c_2u_{xx} \\ & + \left((\alpha_1 + \alpha_2)\frac{\partial}{\partial t} + (\alpha_1 c_2 + \alpha_2 c_1)\frac{\partial}{\partial x}\right)uu_x \\ & + \left((\beta_1 + \beta_2)\frac{\partial}{\partial t} + (\beta_1 c_2 + \beta_2 c_1)\frac{\partial}{\partial x}\right)u_{xxx}, \end{aligned} \tag{1.1}$$

where $u(x,t)$ is a field function, $-\infty < x, t < \infty, \alpha_i \geqslant -1$ and $\beta_i < 1$. The c_i are the phase velocities, the α_i are the parameters of nonlinearity, and the β_i are the dispersion parameters ore (modes). The field function $u(x,t)$ represents the height of the water's free surface above a flat bottom. The TMKdV equation describes the propagation of two different wave modes in the same direction simultaneously, with the same dispersion relation but different phase velocities, nonlinearity, and dispersion parameters (Korsunsky, 1994; Xiao et al., 2016; Lee and Liu, 2011; Lee et al., 2010; Lee and Lee, 2013; Zhu et al., 1997; Hong and Jung, 1999).

The TMKdV Eq. (1.1) was studied recently in the literature. Analytic methods for this equation have been used, such as the reductive perturbation (Lee and Liu, 2011), Hamiltonian system (Lee and Lee, 2013), Bell polynomials (Hong and Jung, 1999), and others. In Xiao et al. (2016), it was found that both the two solitons in the two modes separate without any change of their initial shapes and velocities except for the phase shifts after each collision. In Lee et al. (2010), three basic conserved quantities, namely, the mass, momentum, and Hamiltonian for the TMKdV equation were constructed by using the prolongation method.

Many models in mathematics and physics are described by nonlinear partial differential equations (NPDEs). The theory of solitary wave solution has contributed to understanding many experiments and complex phenomena in mathematical physics. Thus, the literature has been enriched by well imposed methods that produce exact solutions of such nonlinear equations. Such methods are: The sine-cosine method (Wazwaz, 2004; Alquran and Qawasmeh, 2013; Alquran, 2012), Rational sine-cosine (Qawasmeh and Alquran, 2014), tanh method (Alquran and Al-Khaled, 2011), extended tanh method (Shukri and Al-khaled, 2010), sech-tanh method (Alquran et al., 2012), Exp-function method (Raslan, 2009), the (G'/G)-

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expansion method (Alquran and Qawasmeh, 2014; Qawasmeh and Alquran, 2014) and so on. The aforementioned methods can be classified in two categories; special structure in terms of trigonometric or hyperbolic functions or a finite series of degree n in terms of a trigonometric-hyperbolic function. The order of this finite series is to be determined by a balance procedure. Finally, some new and important developments for searching for analytical solitary waves solutions for PDEs have been added recently to the literature (Seadawy, 2014, 2015, 2016, 2017; Helal and Seadawy, 2009; Seadawy and El-Rashidy, 2013).

The goal of this work is to explore more new exact solutions of the two-mode KdV equation given in (1.1) by using a finite series of degree n in terms of a Jacobi elliptic functions (Liu et al., 2001; Fu et al., 2001) which belongs to the well-known homogeneous balance principle and truncated function expansion method (Fan and Zhang, 1998, 2002; Dai and Zhang, 2006; Liu et al., 2001).

2. Jacobi elliptic sine and cosine functions

In this section we introduce in details the derivation of Jacobi elliptic functions. First, we consider the second order partial deferential equation (PDE)

$$\frac{\partial^2\phi}{\partial x\partial t} = \alpha\sin(\phi). \tag{2.1}$$

Applying the linear transformation $\xi = k(x-\lambda t)$ to the above PDE leads to the following ordinary differential equation (ODE)

$$\phi'' = \frac{-\alpha}{k^2\lambda}\sin(\phi), \tag{2.2}$$

which is equivalent to

$$\left[\frac{1}{2}\phi'\right]^2 = \frac{-\alpha}{k^2\lambda}\sin^2\frac{1}{2}(\phi) + c. \tag{2.3}$$

By forcing $c = 1, \frac{-\alpha}{k^2\lambda} = -m^2$ and $w = \frac{1}{2}(\phi)$, we write (2.3) as

$$(w')^2 = 1 - m^2\sin^2 w, \tag{2.4}$$

or

$$w' = \sqrt{1 - m^2\sin^2 w}. \tag{2.5}$$

Separating the variables in (2.5) leads to the following integral

$$\int\frac{1}{\sqrt{1-m^2\sin^2 w}}dw = \int d\xi, \tag{2.6}$$

which is known as the legendre elliptic integral of first kind and $m \in (0,1)$ is a parameter which is known as the module.

Now, we define

$$\begin{aligned} u =& u(t) = \int_0^\phi \frac{1}{\sqrt{1-m^2\sin^2 y}}dy \\ =& \int_0^{t\equiv\sin y}\frac{1}{\sqrt{(1-x^2)(1-m^2x^2)}}dx, \end{aligned} \tag{2.7}$$

and we propose that $u = f(t) \Rightarrow t = f^{-1}(u) = sn(u)$. Where $sn(u)$ is called the Jacobi elliptic sine function. To define the Jacobi elliptic cosine function, we let

$$u(t) = \int_0^\phi \frac{1}{\sqrt{1-m^2(1-\cos^2 y)}}dy, \tag{2.8}$$

and we set $\sqrt{1-t^2} = \cos y$ to get

$$u(t) = \int_0^{\sqrt{1-t^2}\equiv\cos y}\frac{1}{\sqrt{(1-x^2)(1-m^2x^2)}}dx. \tag{2.9}$$

Therefore, $u = f(\sqrt{1-t^2}) \Rightarrow \sqrt{1-t^2} = f^{-1}(u) = cn(u)$ and $cn(u)$ is called the Jacobi elliptic cosine function. Based on the above two definitions, we may write the following argument

$$\begin{aligned} t =& sn(u) \\ \sqrt{1-t^2} =& cn(u) \\ \sqrt{1-m^2t^2} =& dn(u). \end{aligned} \tag{2.10}$$

Thus, we reach to the identities, $cn^2(u) = 1 - sn^2(u)$ and $dn^2(u) = 1 - m^2sn^2(u)$. To know more about these functions, we study their derivatives. First we start with $sn(u)$,

$$\begin{aligned} u =& \int\frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}} \\ du =& \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}}. \end{aligned} \tag{2.11}$$

Thus,

$$\frac{dt}{du} = \sqrt{(1-t^2)(1-m^2t^2)}. \tag{2.12}$$

Now we are ready ti introduce the following formulas

$$\begin{aligned} \frac{d}{du}(sn(u)) =& \frac{d}{du}(t) = \sqrt{(1-t^2)(1-m^2t^2)} = cn(u)\ dn(u) \\ \frac{d}{du}(cn(u)) =& \frac{d}{du}(\sqrt{1-sn^2(u)}) = \frac{-2sn(u)}{2\sqrt{1-sn^2(u)}}\ cn(u)dn(u) \\ =& -sn(u)\ dn(u) \\ \frac{d}{du}(dn(u)) =& \frac{d}{du}(\sqrt{1-m^2sn^2(u)}) = \frac{-2m^2sn(u)}{2\sqrt{1-m^2sn^2(u)}}\ cn(u)dn(u) \\ =& -m^2sn(u)\ cn(u). \end{aligned} \tag{2.13}$$

Also, we can observe by the second equation of (2.13) that,

$$\begin{aligned} \frac{d}{du}(cn(u)) =& -sn(u)dn(u) \\ =& -\sqrt{1-cn^2(u)}\sqrt{1-m^2sn^2(u)} \\ =& -\sqrt{1-cn^2(u)}\sqrt{1-m^2+m^2cn^2(u)}. \end{aligned} \tag{2.14}$$

Therefore, $cn(u) = -\int_0^t \frac{dx}{\sqrt{(1-x^2)(1-m^2+m^2x^2)}}$. On the other hand, this can be obtained by substituting $t = \cos y$ in the elliptic integral

$$\int_0^\phi\frac{dy}{\sqrt{1-m^2\sin^2 y}} = \int_0^\phi\frac{dy}{\sqrt{1-m^2(1-\cos^2 y)}}. \tag{2.15}$$

Equivalently,

$$-\int_0^{t\equiv\cos y}\frac{dx}{\sqrt{(1-x^2)(1-m^2+m^2x^2)}} = f(t). \tag{2.16}$$

Thus, $u = f(t) \Rightarrow t = f^{-1}(u) = cn(u)$.

Also, $sn(u) = t = f^{-1}(u)$, where

$$u = u(t) = \int_0^t\frac{dx}{\sqrt{(1-x^2)(1-m^2x^2)}}. \tag{2.17}$$

Now, if $m \to 1$, then the integral become

$$\int_0^t\frac{dx}{(1-x^2)} = \frac{1}{2}(\ln(|t+1|) - \ln(|1-t|)) = tanh^{-1}(t), \tag{2.18}$$

and leads to:

$$\begin{aligned} sn(u) =& tanh(u) \\ cn(u) =& \sqrt{1-sn^2(u)} = \sqrt{1-tanh^2(u)} = sech(u) \\ dn(u) =& \sqrt{1-sn^2(u)} = sech(u). \end{aligned} \tag{2.19}$$

Figs. 1 and 2 represent the functions $sn(x, m)$ and $cn(x, m)$ for special values of the index m.

Now, we will highlight briefly the main steps of the Jacobi elliptic sine-cosine expansion method. We first unite the independent variables x and t into one wave variable $\zeta = x - ct$ to convert the partial differential equation (PDE)

$$P(u, u_t, u_x, u_{xt}, \ldots) \tag{2.20}$$

into an ordinary differential equation (ODE)

$$Q(u, -cu', u', -cu'', \ldots). \tag{2.21}$$

Eq. (2.21) is then integrated as long as all terms contain derivatives. The Jacobi elliptic sine-cosine technique is based on the assumption that the traveling wave solutions can be expressed in terms of the $sn(\xi, m)$ or $cn(\xi, m)$. We therefore introduce a new independent variable

$$Y = sn(\mu\zeta, m), \tag{2.22}$$

or

$$Y = cn(\mu\zeta, m). \tag{2.23}$$

Then, the solution can be proposed a finite power series in Y in the form:

$$u(\zeta) = S(Y) = \sum_{i=0}^{M} a_i Y^i. \tag{2.24}$$

The parameter M is a positive integer, in most cases, that will be determined by using a balance procedure, where by comparing the behavior of Y^i in the highest derivative against its counterpart within the nonlinear terms. With M determined, we collect all coefficients of powers of Y in the resulting equation where theses coefficients have to vanish, hence the coefficients a_i can be determined.

3. Solutions of the two-mode KdV equation

In this section, we apply the Jacobi elliptic sine-cosine expansion method to extract possible solutions of (1.1). By using the wave variable $\xi = k(x - \lambda t)$ and chain rule, the TMKdV equation is reduced to the following ordinary equation:

$$\begin{aligned} 0 &= (c_2\alpha_1 + c_1\alpha_2 - (\alpha_1 + \alpha_2)\lambda)u'^2 + (c_1 - \lambda)(c_2 - \lambda)u'' \\ &\quad + (c_2\alpha_1 + c_1\alpha_2 - (\alpha_1 + \alpha_2)\lambda)uu'' + k^2(c_2\beta_1 + c_1\beta_2 \\ &\quad - (\beta_1 + \beta_2)\lambda)u'''', \end{aligned} \tag{3.1}$$

where $u = u(\xi)$. Balancing the linear term u'''' with the nonlinear term u'^2 produces the algebraic equation $n + 4 = 2n + 2$ whose solution is $n = 2$. Thus, by using the Jacobi sine function, the solution of (3.1) has the form

$$u(\xi) = a_0 + a_1\ sn(\xi, m) + a_2\ sn^2(\xi, m). \tag{3.2}$$

To determine the coefficients a_0, a_1, a_2 and the wave's parameters λ and k, we substitute (3.2) into (3.1) and collect the coefficients for the same power of Jacobi sine function and set them to zero to obtain the following system:

$$\begin{aligned} 0 &= 10a_2m^2(a_2(c_2\alpha_1 + c_1\alpha_2 - (\alpha_1 + \alpha_2)\lambda) + 12k^2m^2(c_2\beta_1 + c_1\beta_2 - (\beta_1 + \beta_2)\lambda)) \\ 0 &= 2a_2(-4a_2(1 + m^2)(c_2\alpha_1 - (\alpha_1 + \alpha_2)\lambda) + c_1(3c_2m^2 - 4a_2(1 + m^2)\alpha_2 \\ &\quad - 3m^2(-a_0\alpha_2 + 20k^2(1 + m^2)\beta_2 + \lambda)) + 3m^2(-c_2(20k^2(1 + m^2)\beta_1 + \lambda) \\ &\quad + \lambda(20k^2(1 + m^2)(\beta_1 + \beta_2) + \lambda) + a_0(c_2\alpha_1 - (\alpha_1 + \alpha_2)\lambda))) \\ 0 &= -2a_2(-3a_2c_2\alpha_1 - 8c_2k^2\beta_1 - 52c_2k^2m^2\beta_1 - 8c_2k^2m^4\beta_1 - 2c_2\lambda - 2c_2m^2\lambda \\ &\quad + 3a_2\alpha_1\lambda + 3a_2\alpha_2\lambda + 8k^2\beta_1\lambda + 52k^2m^2\beta_1\lambda + 8k^2m^4\beta_1\lambda + 8k^2\beta_2\lambda + 52k^2m^2\beta_2\lambda \\ &\quad + 8k^2m^4\beta_2\lambda + 2\lambda^2 + 2m^2\lambda^2 + c_1(2c_2(1 + m^2) - 3a_2\alpha_2 + 2a_0(1 + m^2)\alpha_2 - 8k^2\beta_2 \\ &\quad - 52k^2m^2\beta_2 - 8k^2m^4\beta_2 - 2\lambda - 2m^2\lambda) + 2a_0(1 + m^2)(c_2\alpha_1 - (\alpha_1 + \alpha_2)\lambda)) \\ 0 &= 2a_2(-4c_2k^2\beta_1 - 4c_2k^2m^2\beta_1 + c_1(c_2 + a_0\alpha_2 - 4k^2\beta_2 - 4k^2m^2\beta_2 - \lambda) - c_2\lambda \\ &\quad + 4k^2\beta_1\lambda + 4k^2m^2\beta_1\lambda + 4k^2\beta_2\lambda + 4k^2m^2\beta_2\lambda + \lambda^2 + a_0(c_2\alpha_1 - (\alpha_1 + \alpha_2)\lambda)) \end{aligned} \tag{3.3}$$

Some of the equation in the above system have been deleted where $a_1 = 0$ is the solution of each. Now, solving this algebraic system yields

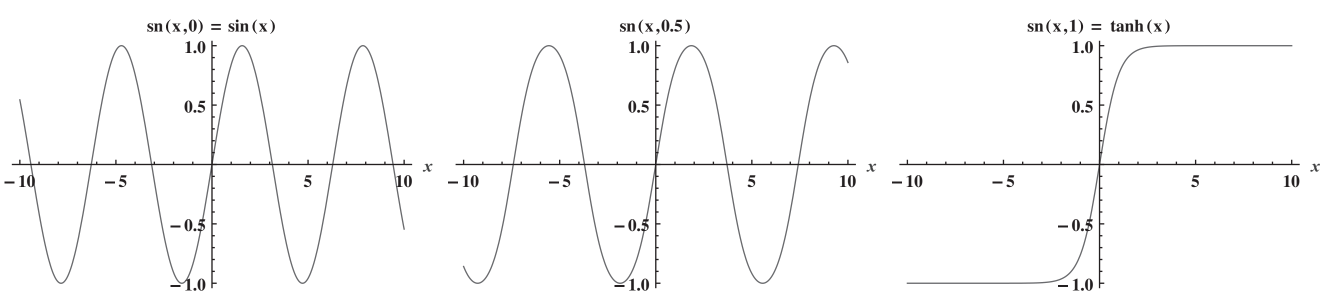

Fig. 1. Plots of the Jacobi $sn(x, m)$ function $sn(x, 0) = \sin(x)$, $sn(x, \frac{1}{2})$, $sn(x, 1) = \tanh(x)$ respectively and $-10 < x < 10$.

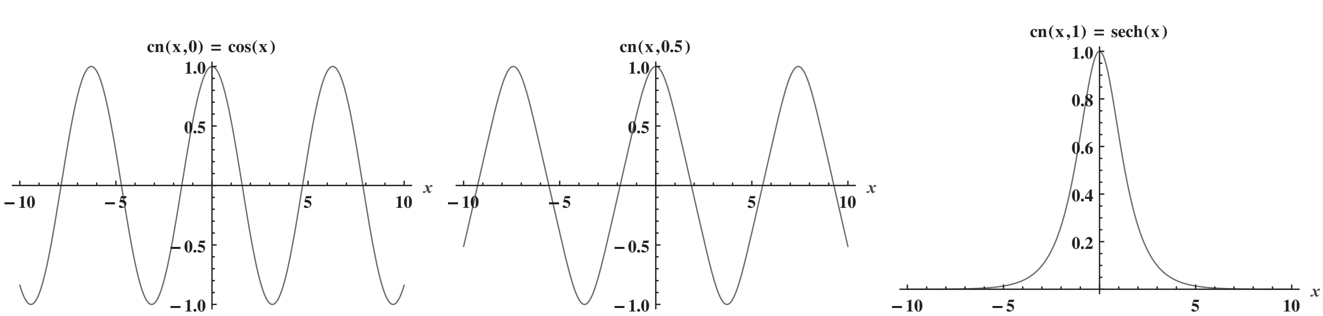

Fig. 2. Plots of the Jacobi $cn(x, m)$ function $cn(x, 0) = \cos(x)$, $cn(x, \frac{1}{2})$, $cn(x, 1) = \mathrm{sech}(x)$ respectively and $-10 < x < 10$.

$$a_0 = \frac{4k^2(1+m^2)(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda) + \lambda(c_1+c_2) - c_1c_2 - \lambda^2}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda}$$
$$a_2 = -\frac{12k^2m^2(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda)}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda}. \tag{3.4}$$

Therefore, the Jacobi sine solution of the two-mode KdV equation is

$$u(x,t) = \frac{4k^2(1+m^2)(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda) + \lambda(c_1+c_2) - c_1c_2 - \lambda^2}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda} - \frac{12k^2m^2(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda)}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda} sn^2(k(x-\lambda t), m). \tag{3.5}$$

If $m = 1$ in (3.5), then we obtain the hyperbolic tangent solution

$$u(x,t) = \frac{8k^2(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda) + \lambda(c_1+c_2) - c_1c_2 - \lambda^2}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda} - \frac{12k^2(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda)}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda} \tanh^2(k(x-\lambda t)), \tag{3.6}$$

where k and λ are free constants.

Now we proceed as above but replacing the Jacobi sine by Jacobi cosine function in (3.2), i.e.,

$$u(\xi) = b_0 + b_1\, cn(\xi, m) + b_2\, cn^2(\xi, m). \tag{3.7}$$

Following the procedures considered as in the Jacobi Sine case, one can reach to the following results

$$b_0 = \frac{-4k^2(2m^2-1)(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda) + \lambda(c_1+c_2) - c_1c_2 - \lambda^2}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda}$$
$$b_1 = 0$$
$$b_2 = \frac{12k^2m^2(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda)}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda}. \tag{3.8}$$

Therefore, the Jacobi cosine solution of the two-mode KdV equation is

$$u(x,t) = \frac{-4k^2(2m^2-1)(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda) + \lambda(c_1+c_2) - c_1c_2 - \lambda^2}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda} + \frac{12k^2m^2(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda)}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda} cn(k(x-\lambda t)). \tag{3.9}$$

If $m = 1$ in (3.9), then we obtain the hyperbolic secant solution

$$u(x,t) = \frac{-4k^2(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda) + \lambda(c_1+c_2) - c_1c_2 - \lambda^2}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda} + \frac{12k^2(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda)}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda} \operatorname{sech}^2(k(x-\lambda t)), \tag{3.10}$$

where k and λ are free constants.

Finally, if we use the Jacobi DN Solution to the two-mode KdV equation, we get the same obtained solution as given in (3.9) but with value of λ of 1.

4. Discussion and concluding remarks

In this work, we succeeded in obtaining Jacobi elliptic sine-cosine solutions for the proposed two-mode KdV Eq. (1.1). We may reduce the calculations performed in the previous section and obtain the Jacobi cosine solution given in (3.9) by using the identity $sn^2(\xi,m) + cn^2(\xi,m) = 1$. Indeed, if we replace $sn^2(\xi,m)$ in (3.5) by $1 - cn^2(\xi,m)$, then $b_0 = a_0 + a_2$ and $b_2 = -a_2$.

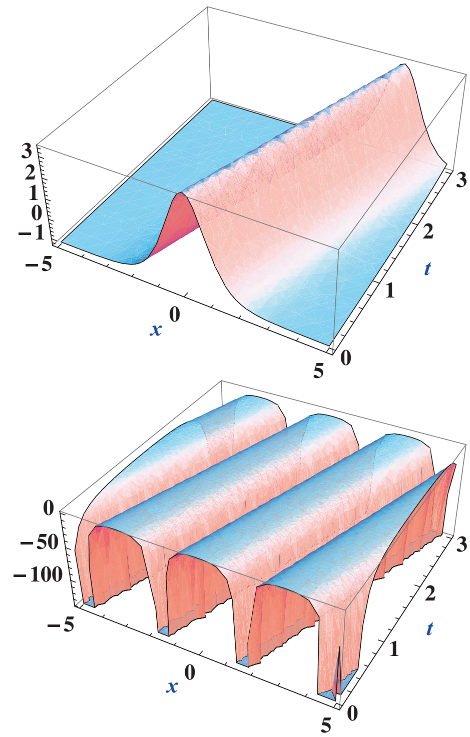

Fig. 3. The obtained solutions of the TMKdV given in (3.6) and (4.1) respectively when $c_1 = 0.5,\ c_2 = \frac{1}{3},\ \alpha_1 = 1,\ \alpha_2 = 0.8,\ \beta_1 = 0.25,\ \beta_2 = 0.5,\ k = \mu = 1$ and $\lambda = 1$.

The types of the obtained solutions given in (3.6) and (3.10) are solitons. Solitons are solutions in the form sech and $\operatorname{sech}^2(\equiv \tanh^2)$, the graph of a Soliton is a wave where in the limit $x \to \pm\infty$, the answer is 0.

By using the facts $\tanh(ix) = i\ \tan(x)$ and $\operatorname{sech}(ix) = \sec(x)$, if we let $k = i\ \mu$ in Eqs. (3.6) and (3.10), then two more periodic solutions will be obtained

$$u(x,t) = \frac{-8\mu^2(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda) + \lambda(c_1+c_2) - c_1c_2 - \lambda^2}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda} - \frac{12\mu^2(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda)}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda} \tan^2(\mu(x-\lambda t)), \tag{4.1}$$

and

$$u(x,t) = \frac{4\mu^2(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda) + \lambda(c_1+c_2) - c_1c_2 - \lambda^2}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda} - \frac{12\mu^2(c_2\beta_1 + c_1\beta_2 - (\beta_1+\beta_2)\lambda)}{c_2\alpha_1 + c_1\alpha_2 - (\alpha_1+\alpha_2)\lambda} \sec^2(\mu(x-\lambda t)), \tag{4.2}$$

where μ and λ are free constants. Finally, the last two periodic solutions have singularities at the characteristic lines $\mu(x - \lambda t) = \frac{\pi}{2} \pm n\pi$ (Fig. 3).

References

Alquran, M., 2012. Solitons and periodic solutions to nonlinear partial differential equations by the Sine-Cosine method. Appl. Math. Inf. Sci. 6 (1), 85–88.

Alquran, M., Al-Khaled, K., 2011. The tanh and sine-cosine methods for higher order equations of Korteweg-de Vries type. In: Phys. Scr. 84, 025010 (4pp).

Alquran, M., Qawasmeh, A., 2013. Classifications of solutions to some generalized nonlinear evolution equations and systems by the sine-cosine method. Nonlinear Stud. 20 (2), 263–272.

Alquran, M., Qawasmeh, A., 2014. Soliton solutions of shallow water wave equations by means of (G'/G)-expansion method. J. Appl. Anal. Comput. 4 (3), 221–229.

Alquran, M., Ali, M., Al-Khaled, K., 2012. Solitary wave solutions to shallow water waves arising in fluid dynamics. Nonlinear Stud. 19 (4), 555–562.

Dai, C.Q., Zhang, J.F., 2006. Jacobian elliptic function method for nonlinear differential-difference equations. Chaos, Solitons Fractals 27, 1042–1049.

Fan, E., Zhang, H., 1998. A note on the homogeneous balance method. Phys. Lett. A 246, 403–406.

Fan, E., Zhang, J., 2002. Applications of the Jacobi elliptic function method to special-type nonlinear equations. Phys. Lett A 305, 383–392.

Fu, Z.T., Liu, S.K., Liu, S.D., Zhao, Q., 2001. New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations. Phys. Lett. A 290, 72–76.

Helal, M.A., Seadawy, A.R., 2009. Variational method for the derivative nonlinear Schrodinger equation with computational applications. Phys. Scr. 80 (3), 350–360.

Hong, W.P, Jung, Y.D., 1999. New non-traveling solitary wave solutions for a second-order Korteweg-de Vries equation. Z. Naturforsch. 54a, 375–378.

Korsunsky, S.V., 1994. Soliton solutions for a second-order KdV equation. Phys. Lett. A 185, 174–176.

Lee, C.T., Lee, C.C., 2013. On wave solutions of a weakly nonlinear and weakly dispersive two-mode wave system. Waves Random Complex Medium 23, 56–76.

Lee, C.T., Liu, JL., 2011. A Hamiltonian model and soliton phenomenon for a two-mode KdV equation, Rocky Mountain. J. Math. 41 (4), 1273–1289.

Lee, C.C., Lee, C.T., Liu, J.L., Huang, W.Y., 2010. Quasi-solitons of the two-mode Korteweg-de Vries equation. Eur. Phys. J. Appl. Phys. 52, 11–301.

Liu, S.K., Fu, Z.T., Liu, S.D., Zhao, Q., 2001. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Phys. Lett. A 289, 69–74.

Liu, S., Fu, Z., Liu, S., Zhao, Q., 2001. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Phys. Lett. A 289 (1–2), 69–74.

Qawasmeh, A., Alquran, M., 2014. Reliable study of some new fifth-order nonlinear equations by means of (G'/G)-expansion method and rational sine-cosine method. Appl. Math. Sci. 8 (120), 5985–5994.

Qawasmeh, A., Alquran, M., 2014. Soliton and periodic solutions for $(2+1)$-dimensional dispersive long water-wave system. Appl. Math. Sci. 8 (50), 2455–2463.

Raslan, K.R., 2009. The application of Hes Exp-function method for MKdV and Burgers equations with variable coefficients. Int. J. Nonlinear Sci. 7 (2), 174–181.

Seadawy, A.R., 2014. Stability analysis for Zakharov-Kuznetsov equation of weakly nonlinear ion-acoustic waves in a plasma. Comp. Math. Appl. 67, 172–180.

Seadawy, A.R., 2015. Fractional solitary wave solutions of the nonlinear higher-order extended KdV equation in a stratified shear flow: part I. Comp. Math. Appl. 70, 345–352.

Seadawy, A.R., 2016. Stability analysis solutions for nonlinear three-dimensional modified Korteweg-de Vries-Zakharov-Kuznetsov equation in a magnetized electron-positron plasma. Phys. A 455, 44–51.

Seadawy, A.R., 2017. Travelling-wave solutions of a weakly nonlinear two-dimensional higher-order Kadomtsev-Petviashvili dynamical equation for dispersive shallow-water waves. Eur. Phys. J. Plus 132, 29.

Seadawy, A.R., El-Rashidy, K., 2013. Traveling wave solutions for some coupled nonlinear evolution equations. Math. Comput. Modell. 57 (5), 1371–1379.

Shukri, S., Al-khaled, K., 2010. The extended tanh method for solving systems of nonlinear wave equations. Appl. Math. Comput. 217 (5), 1997–2006.

Wazwaz, A.M., 2004. A sine-cosine method for handling nonlinear wave equations. Math. Comput. Modell. 40 (5–6), 499–508.

Xiao, Z.J., Tian, B., Zhen, H.L., Chai, J., Wu, X.Y., 2016. Multi-soliton solutions and Bäcklund transformation for a two-mode KdV equation in a fluid. Waves Random Complex Medium 31 (6), 1–4.

Zhu, Z., Huang, H.C., Xue, W.M., 1997. Solitary wave solutions having two wave modes of KdV-type and KdV-burgers-type. Chin. J. Phys. 35 (6), 633–639.