Journal of King Saud University – Science 30 (2018) 352–358

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

New implementation of reproducing kernel Hilbert space method for solving a fuzzy integro-differential equation of integer and fractional orders

Asia Khalaf Albzeirat [a], Muhammad Zaini Ahmad [a], Shaher Momani [b,c], Israr Ahmad [d,*]

[a] Institute of Engineering Mathematics, Universiti Malaysia Perlis, Pauh Putra Main Campus, 02600 Arau, Perlis, Malaysia
[b] Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[c] Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan
[d] College of Applied Sciences Nizwa, Ministry of Higher Education, Oman

ARTICLE INFO

Article history:
Received 26 November 2016
Accepted 4 April 2017
Available online 12 April 2017

Keywords:
Fuzzy integro-differential
Fractional calculus
Caputo and Riemann-Liouville definitions
Reproducing Kernel theory

ABSTRACT

This paper presents a novel technique for solving two new form of equation with fuzzy and integro-differential equations. The proposed numerical iterative technique is based on the use of the reproducing Kernel theory. Two numerical examples are given to show the effectiveness and performance of the proposed technique. Simulation results are illustrated and comparative studies with past published works to the exact solution from Laplace transform of order integer have been performed to emphasize the simplicity and accuracy of the proposed technique. Moreover, future applications of the proposed technique are also discussed. Numerical experimental results fully support the findings of the proposed analytical approaches.

1. Introduction

The study of Integro-differential Equation has received increasing interest in various physical, biological and engineering sciences (Abu Arqub et al., 2013; Gushing, 2010; Cushing, 2013). For the last two decades, many researchers have paid their attention to the analytical and numerical methods for the solution of Integro-differential Equation. Certain iteration methods subject to periodic boundary conditions have been proposed to solve the Integro-differential Equation (Ben-Zvi et al., 2016; Oza and Callaway, 1987; Heydari et al., 2014; Maayah et al., 2014; Matache et al., 2005; Abbasbandy et al., 2013). The Reproducing Kernel Theory (RKT) has potential applications in integral equations, integro-differential equations, statistics, numerical analysis (Cattani, 2010; Abu Arqub et al., 2012; Jiang and Chen, 2013) among the other numerical and analytical methods. The RKT method has been successfully employed in the concerned literature to investigate certain scientific applications (Yang et al., 2012; Javadi et al., 2014; Abu Arqub, 2015).

Most recently, the authors proposed a reproducing kernel Hilbert space method for solving a system of integro-differential equations of integer order (Abu Arqub, 2015) and fractional order (Bushnaq et al., 2013). In this paper, the authors generalize the idea of RKT method to provide a numerical solution for solving both fuzzy and fractional orders as given in (1). The present work is the extension of the past published works (Yang et al., 2012; Javadi et al., 2014; Abu Arqub, 2015; Bushnaq et al., 2013). To the best of the author's knowledge, the said problem has not been discussed before.

Consider the following form of integro-differential equation of integer and fractional orders:

$$D^{\beta}_{c,0^+}y(x) = g(x, y(x)) + \int_{0^+}^{x} f(t, y(t))dt, y(x_0) = y_0^F, \tag{1}$$

where $\beta \in (0, 1]$ and $x \in [0, 1]$, $D^{\alpha}_{c,0^+}$, denotes the left fractional derivatives for Caputo of order β and y_0^F is a fuzzy value as a triangular number.

It can be observed that Eq. (1) is a general formulation of fuzzy integro-differential equation of fractional orders.

The rest of the paper is organized as follows: Section 2 provides the basic definitions about the integro-differential equations and for clarification of the general formula of the equation fuzzy integro-differential equation of integer and fractional orders,

* Corresponding author.
E-mail address: iak_2000plus@yahoo.com (I. Ahmad).
Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2017.04.002
1018-3647/

followed by some fuzzy and reproducing kernel definitions in Sections 3 and 4, respectively. Algorithm of the solution based on RKT is presented in Section 5. Two numerical examples are given in Section 6. Finally, the paper concludes in Section 7.

2. Caputo and Riemann-Liouville definitions

This section presents some important preliminaries and definitions.

2.1. Caputo definitions

The left and right fractional derivatives by Caputo sense (Manuel and Coito, 2012; Shaher and Odibat, 2007; Caputo, 1967) are defined, respectively as next form:

$$D^{\beta}_{c,a^+}y(x) = \frac{(1)}{\Gamma(\lceil\beta\rceil-\beta)}\int_a^x (x-\tau)^{\lceil\beta\rceil-\beta-1}y^{(\lceil\beta\rceil)}(\tau)d\tau. \quad (2)$$

$$D^{\beta}_{c,b^-}y(x) = \frac{(1)}{\Gamma(\lceil\beta\rceil-\beta)}\int_x^b (\tau-x)^{\lceil\beta\rceil-\alpha-1}y^{(\lceil\beta\rceil)}(\tau)d\tau, \quad (3)$$

where $\Gamma()$ represents the gamma-function, $y^{(\lceil\beta\rceil)}(\tau) = \frac{dy^{\beta}(\tau)}{d\tau^{\lceil\beta\rceil}}$ and $\lfloor\beta\rfloor \leqslant \beta < \lceil\beta\rceil, \beta \in Z^+$, $\lceil\rceil$ is used for the nearest integer number more than β and $\lfloor\rfloor$ is used for the nearest integer number less than β.

2.2. Riemann-Liouville definition

The left and right Riemann-Liouville fractional integral operators of order $\beta > 0$ (Oliveira and Machado, 2014) are defined respectively as next form:

$$I^{\beta}_{RL,a^+}y(x) = \frac{1}{\Gamma(\beta)}\int_a^x (x-\tau)^{\beta-1}y(\tau)d\tau \quad (4)$$

$$I^{\beta}_{RL,b^-}y(x) = \frac{1}{\Gamma(\beta)}\int_x^b (\tau-x)^{\beta-1}y(\tau)d\tau. \quad (5)$$

3. Fuzzy definition

In this section, the set of all real numbers is denoted by R and the space of n-dimensional fuzzy number by R^n_F, where $y^F(x) : R^n \to [0,1]$

Definition 3.1. Let $y^F(x) \in R^n_F$ and $r \in [0,1]$, and R^n_F denotes the space of n-dimensional fuzzy number. The r-cut off $y^F(x)$ is the crisp set $[y^F(x)]$ that contains all elements with degree in $y^F(x)$ either greater than or equal to r, that is; $[y^F(x)]^r = \{x \in R : y_F(x) \geqslant r\}$, for fuzzy number $u_F(x)$, its r-cut is closed and bounded interval in R and are denoted as follows:

$$[y^F(x)]^r = [y_{1,1r}(x), y_{1,2r}(x)],$$

where,

$$y_{1,1r} = \min\{x : x \in [y^F(x)]^r\}, \quad (6)$$

and

$$y_{1,2r} = \max\{x : x \in [y^F(x)]^r\}, \quad \text{for each} \quad r \in [0,1].$$

For more details, please refer to Buckley and Qu (1991), Arshad and Lupulescu (2011) and Yue et al. (1998).

Definition 3.2. The Triangular and trapezoidal fuzzy numbers respectively (Yue et al., 1998; Ahmad et al., 2013), are defined as follows:

$$y^{TRF}(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leqslant x \leqslant b \\ \frac{c-z}{c-b}, & b \leqslant x \leqslant c \\ 0, & x > c \end{cases} \quad (7)$$

where TRF represents the triangular fuzzy and $y^{TRF}(x) \in R_F$, and its r-cut as follows:

$$[y^{TRF}(x)]^r = [a + r(b-a), c - r(c-b)], \quad \text{for} \quad r \in [0,1] \quad (8)$$

$$y^{TLF}(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leqslant x \leqslant b \\ 1, & b \leqslant x \leqslant c \\ \frac{d-x}{d-c}, & c \leqslant x \leqslant d \\ 0, & x > c \end{cases} \quad (9)$$

where TLF represents the triangular fuzzy and $y^{TLF}(x) \in R_F$, and its r-cut as follows:

$$[y^{TLF}(x)]^r = [a + r(b-a), d - r(d-c)], \quad \text{for} \quad r \in [0,1]. \quad (10)$$

4. Reproducing kernel definitions

Definition 4.1. Let HS $\{y(x)|y(x)$ is a real value function or complex function, $x \in X$ is abstract set$\}$ is a Hilbert space, with inner product $\langle y(x), g(x)\rangle_{HS}$, $(y(x), g(x)) \in HS$

if $\exists U_y(x) \in HS$ $\forall$ fixed $y \in X$, then $U_y(x) \in HS$ and any $y(x) \in HS$ which satisfies the following:

$$\langle y(x), U_y(x)\rangle = y(y), \quad \text{for all} \quad y \in X$$

then,

(i) $U_y(x)$ is a reproducing kernel of HS.
(ii) HS is a reproducing kernel Hilbert space (RKHS).

Definition 4.2 ((Cui and Lin, 2009)). The function space $FS^m_2[a,b]$ is defined as follows:

$$FS^m_2[a,b] = \{y : y^{(i)} \text{ is absolutely continuous}, i = 0,\ldots,m-1, y^{(m)} \in L^2[a,b]\}, \quad (11)$$

where (i) denotes the order of derivative.

Definition 4.3 Cui and Lin, 2009. The inner product in the function space $FS^m_2[a,b]$ for any functions $y(x), v(x) \in FS^1_2[a,b]$ is given by:

$$\langle y, v\rangle_{FS^1_2[a,b]} = u(a)v(b) + \int_a^b y(x)v(x)dx. \quad (12)$$

Definition 4.4 Cui and Lin, 2009. The norm in the function space $FS^m_2[a,b]$ for any functions $y(x), v(x) \in FS^1_2[a,b]$ is defined as follows:

$$||y||_{FS^1_2[a,b]} = \sqrt{\langle y, y\rangle_{FS^1_2[a,b]}} \quad (13)$$

Definition 4.5 Cui and Lin, 2009. The inner product space of $FS^2_2[0,1]$ is defined as:

$$\langle y, v\rangle_{FS_2^2[a,b]} = \sum_{i=0}^{1} y^{(i)}(a)v^{(i)}(a) + \int_a^b y^{(2)}(x)v^{(2)}(x)dx = y(a)v(a) + y^{(1)}(a)v^{(1)}(a) + \int_a^b y^{(2)}(x)v^{(2)}(x)dx, \quad \forall y, v \in FS_2^2[a,b]. \tag{14}$$

Definition 4.6 Cui and Lin, 2009. The norm in the function space $FS_2^2[0,1]$ for any functions $y(x), v(x) \in FS_2^m[a,b]$ is defined as:

$$\|y\|_{FS_2^2[0,1]} = \sqrt{\langle y, y\rangle_{FS_2^2[0,1]}} \tag{15}$$

Theorem 4.1 Cui and Lin, 2009. *$FS_2^1[0,1]$ is RKHS with reproducing kernel and the first reproducing kernel (FRK) given by:*

$$FRK_x(y) = \begin{cases} 1+y, & y \leqslant x \\ 1+x, & y > x \end{cases} \tag{16}$$

Theorem 4.2 Cui and Lin, 2009. *$FS_2^2[0,1]$ is RKHS with reproducing kernel and the second reproducing kernel (SRK) given by:*

$$SRK_x(y) = \begin{cases} 1+xy+\frac{yx^2}{2}-\frac{x^3}{6}, & y \leqslant x \\ 1+xy+\frac{xy^2}{2}-\frac{y^3}{6}, & y > x \end{cases} \tag{17}$$

5. Proposed algorithm of solution

To solve Eq. (1) using *RKHS*, consider the following homogenizing the initial fuzzy condition $y(x_0) = y_0^F$ as:

$$y(x) = y^H(x) + y_0^F(x_0). \tag{18}$$

Therefore, the Eq. (1) becomes equivalent to the following:

$$D_{c,0^+}^{\beta}[y^H(x) + y(x_0)] = g(x, y^H(x) + y(x_0)) + \int_{0^+}^{x} f(t, y^H(t) + y(x_0))dt, \tag{19}$$

where $y(x_0) = y_0^F, y^H(x_0) = 0$, and $\beta \in (0,1], x \in [0,1]$, and Eq. (19) is equivalent to next form as given below:

$$D_{c,0^+}^{\beta}[y^H(x) + y_0^F] = g(x, y^H(x) + y_0^F) + \int_{0^+}^{x} f(t, y^H(t) + y_0^F)dt, \tag{20}$$

where $y^H(x_0) = 0$ and $\beta \in (0,1], x \in [0,1]$.

Under the effect of r-cut definition, Eq. (20) can be written as follows:

$$D_{c,0^+}^{\beta}[y^H(x) + [y_{1,1r}(x_0), y_{1,2r}(x_0)]] = g(x, y^H(x) + [y_{1,1r}(x_0), y_{1,2r}(x_0)]) + \int_{0^+}^{x} f(t, y^H(t) + [y_{1,1r}(x_0), y_{1,2r}(x_0)])dt, \tag{21}$$

where $y^H(x_0) = 0$ and $\beta \in (0,1], x \in [0,1]$.

Under the effect of fuzzy value over all equations, Eq. (21) can be written as the system of integro-differential equations in new next form as follows:

$$\begin{aligned} &D_{c,0^+}^{\beta}[y_{1,1r}^H(x) + y_{1,1r}(x_0)] = g(x, y^H(x) + y_{1,1r}(x_0)) + \textstyle\int_{0^+}^{x} f(t, y^H(t) + y_{1,1r}(x_0))dt, \\ &\text{where } y_{1,1r}^H(x_0) = 0 \text{ and } \beta \in (0,1], x \in [0,1]. \\ &D_{c,0^+}^{\beta}[y_{1,2r}^H(x) + y_{1,2r}(x_0)] = g(x, y^H(x) + y_{1,2r}(x_0)) + \textstyle\int_{0^+}^{x} f(t, y^H(t) + y_{1,2r}(x_0))dt, \\ &\text{where } y_{1,2r}^H(x_0) = 0 \text{ and } \beta \in (0,1], x \in [0,1]. \end{aligned} \tag{22}$$

Then, the following two cases arise.

Case 1 If $y(x_0) = y_0^F$ is a *TRF*, then Eq. (21) yields:

$$\begin{aligned} &D_{c,0^+}^{\beta}[y_{1,1r}^H(x) + a + r(b-a)] \\ &\quad = g(x, y^H(x) + a + r(b-a)) \\ &\qquad + \textstyle\int_{0^+}^{x} f(t, y^H(t) + a + r(b-a))dt, \\ &D_{c,0^+}^{\beta}[y_{1,2r}^H(x) + c - r(c-b)] \\ &\quad = g(x, y^H(x) + c - r(c-b)) \\ &\qquad + \textstyle\int_{0^+}^{x} f(t, y^H(t) + c - r(c-b))dt, \end{aligned} \tag{23}$$

where $r \in [0,1], y_{1,1r}^H(x_0) = 0, y_{1,2r}^H(x_0), \beta \in (0,1]$ and $x \in [0,1]$.

Case 2 If $y(x_0) = y_0^F$ is a *TLF*, then Eq. (21) gives:

$$\begin{aligned} &D_{c,0^+}^{\beta}[y_{1,1r}^H(x) + a + r(b-a)] = g(x, y^H(x) + a + r(b-a)) + \textstyle\int_{0^+}^{x} f(t, y^H(t) \\ &+ a + r(b-a))dt, \\ &D_{c,0^+}^{\beta}[y_{1,2r}^H(x) + d - r(d-c)] = g(x, y^H(x) + d - r(d-c)) + \textstyle\int_{0^+}^{x} f(t, y^H(t) \\ &+ d - r(d-c))dt, \end{aligned} \tag{24}$$

where $r \in [0,1], y_{1,1r}^H(x_0) = 0, y_{1,2r}^H(x_0), \beta \in (0,1]$ and $x \in [0,1]$.

Now solving Eq. (22) by using the RKHS tool, follows the next procedure:

(i) Need to construct a reproducing kernel defined in Eq. (11).

$$L_{1,ir} : FS_2^2[0,1] \to FS_2^1[0,1] \quad \text{such that} \quad L_{1,ir}y_{1,ir}^H(x) = y_{1,ir}^H(x), \tag{25}$$

then, $L_{1,ir}, i = 1,2$, are bounded linear operators.

(ii) Let

$$\{x_1, x_2, \ldots, x_\infty\} = \{x_k\}_{k=1}^{\infty} \text{ be a countable set in } [0,1]. \tag{26}$$

At this stage, there are two cases.

Case 1 Consider the order of derivative in integro-differential Eq. (1) is $\beta = 1$, then, the linear differential operator is used.

Case 2 Consider $\beta \in (0,1)$, then, the Caputo fractional operator is used.

(iii) Let

$$\varphi_k^{1,ir}(t) = SRK_x(x_k, x), \Upsilon_k^{1,ir}(x) = L_{1,ir}^{ad}\varphi_k^{1,ir}(x), \tag{27}$$

where $L_{1,ir}^{ad}$ represents the adjoint operator of $L_{1,ir}$.

Theorem 5.1. *Let $\{x_1, x_2, \ldots x_\infty\} = \{x_k\}_{k=1}^{\infty}$ is a dence set in $[0,1]$, Suppose that the inverse operator $L_{1,jr}^{-1}$ for Eq. (25) exists. Then,*

$$\{\Upsilon_1^{1,ir}(x), \Upsilon_2^{1,ir}(x), \ldots, \Upsilon_\infty^{1,ir}(x)\} = \{\Upsilon_k^{1,ir}(x)\}_{(k=1,i=1)}^{(\infty,2)} \tag{28}$$

is a complete fuzzy system of $FS_2^2[0,1]$.

Proof. $\forall y_{1,jr}^H(x) \in FS_2^2[0,1]$, let $\langle y_{1,jr}^H(x), \Upsilon_k^{1,jr}(x)\rangle = 0$, for $k = 1,2,\ldots$ Then

$$\begin{aligned} \langle y_{1,jr}^H(x), \Upsilon_{k,1,jr}(x)\rangle_{FS_2^2[0,1]} &= \langle y_{1,jr}^H(x), L_{1,jr}^{ad}\varphi_{k,1,jr}(x)\rangle_{FS_2^2[0,1]} \\ &= \langle L_{1,jr}y_{1,jr}^H(x), \varphi_{k,1,jr}(x)\rangle_{FS_2^1[0,1]} \\ &= L_{1,jr}y_{1,jr}^H(x) = 0, \end{aligned}$$

where $\{x_k\}_{k=1}^{\infty}$ is dense in $[0,1]$, then, $L_{1,jr}y^H_{1,jr}(x)=0$ from the existence of inverse and the continuity of $y^H_{1,jr}(x)$. □

Using Gram process to construct orthonormal system $\{\overline{\Upsilon}_k^{1,ir}(x)\}_{(k=1,i=1)}^{(\infty,2)}$ as the following:

$$\left\{\overline{\Upsilon}_1^{1,ir}(x),\overline{\Upsilon}_2^{1,ir}(x),\ldots,\overline{\Upsilon}_\infty^{1,ir}(x)\right\}=\left\{\overline{\Upsilon}_k^{1,ir}(x)\right\}_{(k=1,i=1)}^{(\infty,2)}\quad \text{of}\quad FS_2^1[0,1],$$

where

$$\overline{\Upsilon}_k^{1,ir}=\sum_{j=1}^{k}\beta_{kj}^{1,ir}\Upsilon_j^{1,ir}(x),\ \beta_{kk}^{1,1r}>0,\quad \forall k=1,2,\ldots,\infty,\ i=1,2,\tag{29}$$

where $\beta_{kj}^{1,ir}$ is orthogonalization coefficient (Abu Arqub et al., 2013).

Theorem 5.2. *Let* $\{x_1,x_2,\ldots,x_\infty\}=\{x_k\}_{k=1}^{\infty}$ *is a dence set in* $[0,1]$, *where the solution of* Eq. (22) *is unique on* $FS_2^2[0,1]$, *then, the solution of these* Eq. (22) *is given by*:

$$y^H_{1,ir}(x)=\sum_{k=1}^{\infty}Q_k^{1,ir}\overline{\Upsilon}_k^{1,ir},\tag{30}$$

where

$$Q_k^{1,1r}=\sum_{j=1}^{k}\beta_{kj}^{1,ir}\left(g(x_j,y^H(x_j)+y_{1,1r}(x_0))+\int_{0^+}^{x_j}f(t,y^H(t)+y_{1,1r}(x_0))dt\right),\tag{31}$$

where $i=1$.
And

$$Q_k^{1,2r}=\sum_{j=1}^{k}\beta_{kj}^{1,ir}\left(g(x_j,y^H(x_j)+y_{1,2r}(x_0))+\int_{0^+}^{x_j}f(t,y^H(t)+y_{1,2r}(x_0))dt\right),\tag{32}$$

where $i=2$.

Proof. Note that $\{\overline{\Upsilon}_k^{1,ir}(x)\}_{(k=1,i=1)}^{(\infty,2)}$ in Eq. (29) is a complete orthonormal basis of $FS_2^{m+1}[0,1]$. Thus, $y^H_{1,ir}(x)$ can be expanded in the Fourier series about the orthonormal system aswhere $i=1$:

$$y^H_{1,ir}(x)=\sum_{k}^{\infty}\langle y^H_{1,1r}(x),\overline{\Upsilon}_k^{1,ir}(x)\rangle\Upsilon_k^{1,jr}(x)$$

$$\begin{aligned}y^H_{1,ir}(x)&=\sum_{k=1}^{\infty}\langle y^H_{1,ir}(x),\overline{\Upsilon}_k^{1,ir}(x)\rangle\overline{\Upsilon}_k^{1,ir}(x)\\&=\sum_{k=1}^{\infty}\langle y^H_{1,ir}(x),\sum_{j=1}^{k}\beta_{kj}^{1,ir}\Upsilon_j^{1,ir}(x)\rangle\overline{\Upsilon}_k^{1,ir}(x)\\&=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\beta_{kj}^{1,1r}\langle y^H_{1,ir}(x),L^{ad}_{1,ir}\varphi_j^{1,ir}(x)\rangle\overline{\Upsilon}_k^{1,ir}(x)\\&=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\beta_{kj}^{1,1r}\langle L_{1,ir}y^H_{1,ir}(x),\varphi_j^{1,1r}(x)\rangle\overline{\Upsilon}_k^{1,ir}(x)\\&=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\beta_{kj}^{1,1r}\Big\langle g(x,y^H(x)+y_{1,1r}(x_0))+\textstyle\int_{0^+}^{x}f(t,y^H(t)\\&\quad+y_{1,1r}(x_0))dt,\varphi_j^{1,1r}(x)\Big\rangle\overline{\Upsilon}_k^{1,1r}(x)\\&=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\beta_{kj}^{1,1r}(g(x_j,y^H(x_j)+y_{1,1r}(x_0))+\textstyle\int_{0^+}^{x_j}f(t,y^H(t)\\&\quad+y_{1,1r}(x_0))dt)\overline{\Upsilon}_k^{1,1r}(x)\end{aligned}$$

Clearly, $y^H_{1,ir}(x)=\sum_{k=1}^{\infty}Q_k^{1,ir}\overline{\Upsilon}_k^{1,1r}, fori=1.$where $i=2$:

$$y^H_{1,ir}(x)=\sum_{k}^{\infty}\langle y^H_{1,ir}(x),\overline{\Upsilon}_k^{1,2r}(x)\rangle\Upsilon_k^{1,2r}(x)$$

$$\begin{aligned}y^H_{1,2r}(x)&=\sum_{k=1}^{\infty}\langle y^H_{1,ir}(x),\overline{\Upsilon}_k^{1,2r}(x)\rangle\overline{\Upsilon}_k^{1,2r}(x)\\&=\sum_{k=1}^{\infty}\langle y^H_{1,ir}(x),\sum_{j=1}^{k}\beta_{kj}^{1,ir}\Upsilon_j^{1,ir}(x)\rangle\overline{\Upsilon}_k^{1,2r}(x)\\&=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\beta_{kj}^{1,2r}\langle y^H_{1,2r}(x),L^{ad}_{1,ir}\varphi_j^{1,2r}(x)\rangle\overline{\Upsilon}_k^{1,2r}(x)\\&=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\beta_{kj}^{1,2r}\langle L_{1,ir}y^H_{1,2r}(x),\varphi_j^{1,2r}(x)\rangle\overline{\Upsilon}_k^{1,2r}(x)\\&=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\beta_{kj}^{1,ir}\langle g(x,y^H(x)+y_{1,2r}(x_0))+\textstyle\int_{0^+}^{x}f(t,y^H(t)\\&\quad+y_{1,2r}(x_0))dt,\varphi_j^{1,2r}(x)\rangle\overline{\Upsilon}_k^{1,2r}(x)\\&=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\beta_{kj}^{1,2r}\Big(g(x_j,y^H(x_j)+y_{1,2r}(x_0))+\textstyle\int_{0^+}^{x_j}f(t,y^H(t)\\&\quad+y_{1,2r}(x_0))dt\Big)\overline{\Upsilon}_k^{1,2r}(x)\end{aligned}$$

Clearly, $y^H_{1,2r}(x)=\sum_{k=1}^{\infty}Q_k^{1,2r}\overline{\Upsilon}_k^{1,2r}$, for $i=2$.

Through the application of the formula in Eqs. (23), (31) and (32) leads to the next forms, respectively.

$$\begin{aligned}Q_k^{1,1r}=\sum_{j=1}^{k}\beta_{kj}^{1,ir}(g(x,y^H(x_j)+a+r(b-a))+\int_a^{x_j}f(t,y^H(t)+a\\+r(b-a))dt),\end{aligned}\tag{33}$$

where $i=1$.
And

$$\begin{aligned}Q_k^{1,2r}=\sum_{j=1}^{k}\beta_{kj}^{1,ir}(g(x,y^H(x_j)+c-r(c-b))+\int_a^{x_j}f(t,y^H(t)+c\\-r(c-b))dt),\end{aligned}\tag{34}$$

where $i=2$.

Now, the approximate solution of $y^H_{1,ir}(t)$ can be obtained by taking finitely N-terms in Eq. (30), which is given as follows:

$$y^H_{1,ir}(x)=\sum_{k=1}^{N}Q_k^{1,ir}\overline{\Upsilon}_k^{1,ir}(x),\tag{35}$$

For the approximate solution of Eq. (22), let discuss the following two cases.

Case 1. If the initial value is a *TRF*, then:

$$\begin{aligned}y^H_{1,ir}(x)=\sum_{k=1}^{N}\sum_{j=1}^{k}\beta_{kj}^{1,ir}(g(x,y^H(x_j)+a+r(b-a))\\+\int_a^{x_j}f(t,y^H(t)+a+r(b-a))dt)\overline{\Upsilon}_k^{1,ir}(x),\end{aligned}\tag{36}$$

where $i=1$.
And

$$\begin{aligned}y^H_{1,ir}(x)=\sum_{k=1}^{N}\overline{\Upsilon}_k^{1,ir}(x)\sum_{j=1}^{k}\beta_{kj}^{1,ir}(g(x,y^H(x_j)+c-r(c-b))\\+\int_a^{x_j}f(t,y^H(x_j)+c-r(c-b))dt),\end{aligned}\tag{37}$$

where $i=2$.

Table 1
Results of $y_{1,1r}(x)$, when $r = 0$.

x	Exact Solution for $y_{1,1r}(x), r = 0$	Approximate Solution for $y_{1,1r}(x), r = 0$	Error
0.0	1.0	1.0	1.0
0.1	0.09139129761499401	0.09140732189	$1.602427395 \times 10^{-5}$
0.2	0.1703750454251937	0.1704008855	$2.584008915 \times 10^{-5}$
0.3	0.2427946105625625	0.2428272169	$3.260630034 \times 10^{-5}$
0.4	0.3131354255840278	0.3131738996	$3.847399588 \times 10^{-5}$
0.5	0.3849752557910373	0.3850201079	$4.485213601 \times 10^{-5}$
0.6	0.4613115423320764	0.4613642561	$5.271378091 \times 10^{-5}$
0.7	0.5448029996492808	0.5448658117	$6.281207332 \times 10^{-5}$
0.8	06379520045310086	0.6380278393	$7.583480056 \times 10^{-5}$
0.9	0.7432468092864707	0.7433393321	$9.252276769 \times 10^{-5}$
1.0	0.8632773332921827	0.8633911057	$1.137724473 \times 10^{-4}$

Table 2
Results of $y_{1,2r}(x)$, when $r = 0$.

x	Exact Solution for $y_{1,2r}(x), r = 0$	Approximate Solution for $y_{1,2r}(x), r = 0$	Error
0.0	2.0	2.0	0.0
0.1	1.7728457782490132	1.772756224	$8.955466741 \times 10^{-5}$
0.2	1.667543323444828	1.667406352	$1.369713923 \times 10^{-5}$
0.3	1.6572122730058399 4	1.657054154	$1.585757531 \times 10^{-6}$
0.4	1.724168412974534	1.724003197	$1.652161054 \times 10^{-6}$
0.5	1.8574476540096005	1.857284432	$1.632220096 \times 10^{-6}$
0.6	2.0510930798934095	2.050937136	$1.559440032 \times 10^{-6}$
0.7	2.30297424901518	2.302829493	$1.447555513 \times 10^{-6}$
0.8	2.613996998399984	2.613867349	$1.296488898 \times 10^{-6}$
0.9	2.987594278355136	2.987484746	$1.095328168 \times 10^{-6}$
1.0	3.429424290763692	3.429341985	$8.230529053 \times 10^{-6}$

Table 3
Results of $y_{1,1r}(x)$, when $r = 0.5$.

x	Exact Solution for $y_{1,1r}(x), r = 0.5$	Approximate Solution for $y_{1,1r}(x), r = 0.5$	Error
0.0	0.5	0.5	0.0
0.1	0.5117549177734989	0.5117445473	$1.037046139 \times 10^{-5}$
0.2	0.5446671149301022	0.5446522521	$1.486278121 \times 10^{-5}$
0.3	0.5963991404365216	0.5963839512	$1.518921303 \times 10^{-5}$
0.4	0.6658936724316544	0.6658812239	$1.244852945 \times 10^{-5}$
0.5	0.7530933553456781	0.7530861889	$7.166400387 \times 10^{-6}$
0.6	0.8587569267224099	0.8587574761	$5.493348798 \times 10^{-7}$
0.7	0.9843458119907555	0.9843567322	$1.092016716 \times 10^{-5}$
0.8	1.1319632528582564	1.131987717	$2.446387798 \times 10^{-5}$
0.9	1.304333676553637	1.304375685	$4.200887155 \times 10^{-5}$
1.0	1.5048140726600605	1.504878826	$6.475301284 \times 10^{-5}$

Table 4
Results of $y_{1,2r}(x)$, when $r = 0.5$.

x	Exact Solution for $y_{1,2r}(x), r = 0.5$	Approximate Solution for $y_{1,2r}(x), r = 0.5$	Error
0.0	1.5	1.5	0.0
0.1	1.3524821580905086	1.352418998	$1.037046139 \times 10^{-5}$
0.2	1.2932512539399192	1.293154985	$1.486278121 \times 10^{-5}$
0.3	1.3036082001844398	1.30349742	$1.518921303 \times 10^{-5}$
0.4	1.3714101661269074	1.371295873	$1.244852945 \times 10^{-5}$
0.5	1.4893295544549596	1.489218351	$7.166400387 \times 10^{-5}$
0.6	1.6536476955030763	1.653543916	$5.493348739 \times 10^{-5}$
0.7	1.863431436673705	1.863338573	$1.092016716 \times 10^{-5}$
0.8	2.119985749512751	2.119907472	$2.446387798 \times 10^{-5}$
0.9	2.42650741108797	2.426448392	$4.200887155 \times 10^{-5}$
1.0	2.787887551395815	2.787854266	$6.475301284 \times 10^{-5}$

Table 5
Results of $y_{1,1r}(x) and y_{1,2r}(x)$, when $r = 1.0$.

x	Exact Solution for $y_{1,1r}(x)$ and $y_{1,2r}(x)$, $r = 1.0$	Approximate Solution for $y_{1,1r}(x)$ and $y_{1,2r}(x)$, $r = 1.0$	Error
0.0	1.0	1.0	0.0
0.1	0.9321185379320037	0.9320817727	$3.676519673 \times 10^{-5}$
0.2	0.9189591844350107	0.9189036188	$5.556565157 \times 10^{-5}$
0.3	0.9500036703104807	0.9499406856	$6.298472639 \times 10^{-5}$
0.4	1.018651919279281	1.018588548	$6.337105478 \times 10^{-5}$
0.5	1.121211454900319	1.12115227	$5.918493679 \times 10^{-5}$
0.6	1.2562023111127427	1.256150696	$5.161511116 \times 10^{-5}$
0.7	1.4238886243322302	1.423847653	$4.097173901 \times 10^{-5}$
0.8	1.6259745011855036	1.625947594	$2.69070466 \times 10^{-5}$
0.9	1.8654205438208034	1.865412039	$8.505024577 \times 10^{-6}$
1.0	2.1463508120279373	2.146366546	$1.573357838 \times 10^{-5}$

Table 6
Results of $y_{1,1r}(x), y_{1,2r}(x)$, when $r = 0 and \beta = 0.9$.

x	Approximate Solution for $y_{1,1r}(x)$, $r = 0$ and $\beta = 0.9$	Approximate Solution for $y_{1,2r}(x)$, $r = 0$ and $\beta = 0.9$
0.0	0.0	2.0
0.1	0.1046233715	1.760498987
0.2	0.1901266044	1.659279709
0.3	0.2639141379	1.672267844
0.4	0.3345895685	1.766712006
0.5	0.407120093	1.927536637
0.6	0.4850255456	2.148099512
0.7	0.571182339	2.426921574
0.8	0.6681857686	2.765976621
0.9	0.7786341156	3.170065497
1.0	0.9053539844	3.646721337

Table 7
Results of $y_{1,1r}(x), y_{1,2r}(x)$, when $r = 0.5 and \beta = 0.9$.

x	Approximate Solution for $y_{1,1r}(x)$, $r = 0.5$ and $\beta = 0.9$	Approximate Solution for $y_{1,2r}(x)$, $r = 0.5$ and $\beta = 0.9$
0.0	0.5	1.5
0.1	0.5185922754	1.346530083
0.2	0.5574148807	1.291991433
0.3	0.6160025645	1.320179417
0.4	0.6926201778	1.408681396
0.5	0.787224229	1.547432501
0.6	0.9007940372	1.73233102
0.7	1.035117148	1.962986765
0.8	1.192633482	2.241528909
0.9	1.376491961	2.572207652
1.0	1.590695822	2.961379499

Table 8
Results of $y_{1,1r}(x), y_{1,2r}(x)$, when $r = 1.0 and \beta = 0.9$.

x	Approximate Solution for $y_{1,1r}(x)$, $r = 1.0$ and $\beta = 0.9$	Approximate Solution for $y_{1,2r}(x)$, $r = 1.0$ and $\beta = 0.9$
0.0	1.0	1.0
0.1	0.9325611793	0.9325611793
0.2	0.924703157	0.924703157
0.3	0.968090991	0.968090991
0.4	1.050650787	1.050650787
0.5	1.167328365	1.167328365
0.6	1.316562529	1.316562529
0.7	1.499051957	1.499051957
0.8	1.717081195	1.717081195
0.9	1.974349806	1.974349806
1.0	2.276037661	2.276037661

Case 2. If the initial value is a *TLF*, then:

$$y^H_{1,ir}(t) = \sum_{k=1}^{N} \overline{\Upsilon}_k^{1,ir}(x) \sum_{j=1}^{k} \beta_{kj}^{1,ir}(g(x, y^H(x_j) + a + r(b-a)) + \int_a^{x_j} f(t, y^H(t) + a + r(b-a))dt), \quad (38)$$

where $i = 1$.

And

$$y^H_{1,ir}(x) = \sum_{k=1}^{N} \overline{\Upsilon}_k^{1,ir}(x) \sum_{j=1}^{k} \beta_{kj}^{1,ir}(g(x, y^H(x_j) + d - r(d-c)) + \int_a^{x_j} f(t, y^H(x) + d - r(d-c))dt), \quad (39)$$

where $i = 2$. □

6. Numerical examples

Example 1. Consider the following fuzzy integro-differential equations with integer order

$D^{\beta}_{c,0^+} y(x) = -2y(x) + 5\int_0^x y(t)dt + 1$, when

$\beta = 1$,

$$y(x_0) = y(0) = y_0^F = (0, 1, 2), x \in [0, 1]. \quad (40)$$

Solution.

The exact solution for the last example by Laplace transform method give by:

(i)

$$y_{1,1r}(x) = -\frac{e^{(-1-\sqrt{6})x} - e^{(-1+\sqrt{6})x}}{2\sqrt{6}},$$

where $r = 0.0$.

(ii)

$$y_{1,2r}(x) = \frac{1}{12}(-24 + 12^{(-1-\sqrt{6})x} + \sqrt{6}^{(-1-\sqrt{6})x} + 12^{(-1+\sqrt{6})x} - \sqrt{6}^{(-1+\sqrt{6})x}) + 2,$$

where $r = 0.0$.

(iii)

$$y_{1,1r}(x) = 2.5(-0.2 + 0.0598e^{-3.4498x} + 0.1408e^{1.4495x}) + 0.5,$$

where $r = 0.5$.

(iv)

$$y_{1,2r}(x) = -2.0(-0.75 - 0.426e^{-3.4495x} - 0.324e^{1.4495x}) + 1.5,$$

where $r = 0.5$.

(v)

$$y_{1,1r}(x) = y_{1,2r}(x) = \frac{1}{2}(-2 + e^{(-1-\sqrt{6})x} + e^{(-1+\sqrt{6})x}) + 1,$$

where $r = 1.0$.

Approximate solution by *RKHS* at $N = 50$ is listed in Tables 1–5. From these tables, one can see the accuracy in comparing the approximate solution with the exact solution. These results show a convergence in results between the approximate solution and exact solution at different values of r.

Example 2. Consider the following fuzzy integro-differential equations with fractional order $\beta = 0.9$

$$D^{\beta}_{c,0^+} y(x) = -2y(x) + 5\int_0^x y(t)dt + 1, \quad (41)$$

where $y(x_0) = y_0^F = (0, 1, 2)$ and $x \in [0, 1]$.

Solution. In this example approximate solution by *RKHS* has been finding at $N = 15$. Results are listed in Tables 6–8.

As compared to the past published results, the proposed approach generalizes the idea of RKT method to provide a numerical solution for solving both fuzzy initial value and fractional orders. The proposed solution covers multi cases for the different initial conditions.

7. Conclusion

In this paper, new steps and procedures for solving fuzzy integro-differential equation of integer and fractional orders by the reproducing Kernel theory, where the initial value in problems given as a triangular fuzzy or trapezoidal fuzzy is focused. Numerical simulation results fully support analytical findings. The proposed approach appears well adaptable to fuzzy applications that need multiple and variable values. Moreover, the method and all the procedures are suitable to solve nonlinear problems in fractional calculus with uncertain situation.

The presented results can extended to the cases of other intervals with using another reproducing kernel suitable for these cases and can extended to the cases system fuzzy integro-differential equation.

References

Abu Arqub, O., Al-Smadi, M., Shawagfeh, N., 2013. Solving Fredholm integro-differential equations using reproducing kernel Hilbert space method. Appl. Math. Comput. 219 (17), 8938–8948.

Gushing, J.M., 2010. Volterra integro-differential equations in population dynamics. Mathematics of Biology. Springer, Berlin Heidelberg, pp. 81–148.

Cushing, J.M., 2013. Integro-differential equations & delay models in population dynamics. Springer Sci. Bus. Media 20.

Ben-Zvi, R., Scher, H., Jiang, S., Berkowitz, B., 2016. One-dimensional finite element method solution of a class of integro-differential equations: application to non-fickian transport in disordered media. Transp. Porous Media 115 (2), 239–263.

Oza, D.H., Callaway, J., 1987. Numerical solution of the Schroedinger equation with a non-local exchange kernel. J. Comp. Phys. 68 (1), 89–103.

Heydari, M.H., Hooshmandasl, M.R., Mohammadi, F., Cattani, C., 2014. Wavelets method for solving systems of nonlinear singular fractional Volterra integro-differential equations. Commun. Nonlinear Sci. Numer. Simul. 19 (1), 37–48.

Maayah, B., Bushnaq, S., Momani, S., Abu Arqub, O., 2014. Iterative multistep reproducing kernel hilbert space method for solving strongly nonlinear oscillators. Adv. Math. Phys., 7 Article ID: 758195

Matache, A.M., Schwab, C., Wihler, T.P., 2005. Fast numerical solution of parabolic integro-differential equations with applications in finance. SIAM J. Sci. Comp. 27 (2), 369–393.

Abbasbandy, S., Hashemi, M.S., Hashim, I., 2013. On convergence of homotopy analysis method and its application to fractional integro-differential equations. Questiones Mathematicae 36 (1), 93–105.

Cattani, C., 2010. Shannon wavelets for the solution of integro-differential equations. Math. Prob. Eng. 2010,, 22 Article ID: 408418.

Abu Arqub, O., Al-Smadi, M., Momani, S., 2012. Application of reproducing kernel method for solving nonlinear Fredholm-Volterra integro-differential equations. Abstr. Appl. Anal. 2012,, 16 Article ID: 839836.

Jiang, W., Chen, Z., 2013. Solving a system of linear Volterra integral equations using the new reproducing kernel method. Appl. Math. Comput. 219 (20), 10225–10230.

Yang, L.H., Shen, J.H., Wang, Y., 2012. The reproducing kernel method for solving the system of the linear Volterra integral equations with variable coefficients. J. Comput. Appl. Math. 236 (9), 2398–2405.

Javadi, S., Babolian, E., Moradi, E., 2014. New implementation of reproducing kernel Hilbert space method for solving a class of functional integral equations. Commun. Numer. Anal. 2014,, 7 Article ID: cna-00205.

Abu Arqub, O., 2015. An iterative method for solving fourth-order boundary value problems of mixed type integro-differential equations. J. Comput. Anal. Appl. 8, 857–874.
Bushnaq, S., Momani, S., Zhou, Y., 2013. A reproducing kernel Hilbert space method for solving integro-differential equations of fractional order. J. Optim. Theory Appl. 156 (1), 96–105.
Manuel, O., Coito, Fernando, C., 2012. On the Usefulness of Riemann-Liouville and Caputo Derivatives in Describing Fractional Shift-invariant Linear Systems. J. Applied Non. Dyns 1, 113–124.
Shaher, M., Odibat, Z., 2007. Numerical approach to differential equations of fractional order. J. Comput. Appl. Math. 207 (1), 96–110.
Caputo, M., 1967. Linear models of dissipation whose Q is almost frequency independent-II. Geophys. J. Int. 13 (5), 529–539.
Oliveira, E.C., Machado, J.T., 2014. A review of definitions for fractional derivatives and integral. Math. Prob. Eng., 6 Article ID: 238459
Buckley, J.J., Qu, Y., 1991. Solving systems of linear fuzzy equations. Fuzzy Sets Syst. 43 (1), 33–43.
Arshad, S., Lupulescu, V., 2011. On the fractional differential equations with uncertainty. Nonlinear Anal.: Theory, Methods Appl. 74 (11), 3685–3693.
Yue, Z., Guangyuan, W., Sufang, L., 1998. Frequency domain methods for the solutions of N-order fuzzy differential equations. Fuzzy Sets Syst. 94 (1), 45–59.
Ahmad, M.Z., Hasan, M.K., Abbasbandy, S., 2013. Solving fuzzy fractional differential equations using Zadeh's extension principle. Sci. World J. 2013,, 11 Article ID: 454969.
Cui, M., Lin, Y., 2009. Nonlinear Numerical Analysis in the Reproducing Kernel Hilbert Space. Nova Science Publishers, New York, NY, USA.