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New best proximity point results on orthogonal F-proximal contractions with applications

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ABSTRACT

In this work, we propose the ideas of orthogonal F-proximal contraction mappings, generalized orthogonal F-proximal contraction mappings and establish several best proximity point theorems in an orthogonally complete metric space ($\mathcal{OCMS}$) through a non-self mapping. Hence, utilizing these recently discovered results, numerous of the existing findings in the literature are generalized or expanded. An illustration is provided to highlight the utility of our findings. Finally, for illustrative applications, we discuss the qualitative properties of the solutions for a fractional boundary value problem in the Caputo sense, and we study the dynamic economic equilibrium problem.

1. Introduction

Since the last century, numerous results on fixed point theory have been constructed because of its various applications in areas like mathematics, computer science, and economics. Eldered and Veeramani (2006) proposed the idea of cyclic contraction mapping and established best proximity point theorems in a uniformly convex Banach space. Raj (2013) proposed the idea of weakly contractive mapping. He introduced a notion called P-property and used it to prove adequate conditions to make certain existence of the best proximity point. Altun et al. (2020) proposed the ideas of p-proximal contraction and p-proximal contractive mappings and established best proximity point theorems on metric spaces($\mathcal{MS}$). Aslantas et al. (2021) proposed the idea of cyclic p-contraction pair for single-valued mappings and established best proximity point theorems. Moreover, he gave the existence and uniqueness results for the solution of a system of second-order boundary value problems. In addition, Basha (2011) proposed the idea of non-self-proximal contractions and established best proximity point theorems that were applied to check the existence of the best approximation answers to equations and it is sensible that it has no solution. Additionally, a methodology was constructed to find such an optimal approximate solution. *Is there any point* y_0 *in the metric space* $(\mathfrak{M}, \psi)$ *satisfying* $\psi(y_0, \Pi y_0) = \psi(\Omega, \Gamma)$ *where* Ω, Γ *are non-empty subsets of* $\mathfrak{M}$, $\Pi : \Omega \to \Gamma$ *is a non-self mapping and* $\psi(\Omega, \Gamma) = \inf\{\psi(\mathfrak{e}, \mathfrak{g}) : \mathfrak{e} \in \Omega, \mathfrak{g} \in \Gamma\}$?. The point $y_0 \in \mathfrak{M}$ is called the *best proximity point*($\mathcal{BPP}$).

Wardowski (2012) proposed the idea of F-contraction mapping and established fixed point theorems that generalize the Banach contraction principle. Cosentino and Vetro (2014) published Hardy–Rogers-type fixed point results for self-mappings on entire ordered metric spaces, corresponding with this research direction. Moreover, Omidvari et al. (2014) proved the existence of the best proximity point for $F-$contractive non-self mappings and presented two types of $F-$proximal contraction. Beg et al. (2021) extended the notion of F-proximal contraction maps and established certain best proximity point theorems for non-self mappings in a complete metric space. Gordji et al. (2017) introduced orthogonal metric space and proved Banach's fixed point theorem with applications to the existence of a solution for a first-order ordinary differential equation. Gunaseelan et al. (2021) proved fixed theorems under orthogonal O-contractions on b-complete

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metric space. Applications of some fixed-point theorems in orthogonal extended S-metric spaces are given in Basha and Veeramani (2000), Khalehoghli et al. (2020), Rezaei et al. (2021) and Touail and Moutawakil (2021). Other concepts and applications of orthogonal contractions and fixed point theorems are apparent in Gnanaprakasam et al. (2023), Sawangsup et al. (2020) and Charoensawan et al. (2023). An orthogonally complete metric space is often defined in terms of a completeness property related to orthogonality. For instance, a space might be said to be orthogonally complete if every orthogonal decomposition (or a related structure) converges in the metric space. Saleem et al. (2021) discussed on some coincidence best proximity point results. Younis et al. (2024) discussed on best proximity points for multivalued mappings and equation of motion. Ahmad (2024) presented the usefulness of contraction mappings in mathematics and their wide range of applications in nonlinear differential equations.

In the aforementioned piece of work, we propose a new idea of orthogonal F -proximal contractions($\mathcal{FPC}$) (of the first and second kind), generalized orthogonal F -proximal contractions (of the first and second kind), and then prove $\mathcal{BPP}$ results on $\mathcal{OCMS}$.

2. Preliminaries

In this section, we present the idea of a control function given in Wardowski (2012). Consider $\mathcal{J}$ to be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that:

($F1$) the mapping F is strictly non-decreasing;

($F2$) for each positive sequence $\{\alpha_\nu\}$, one has

$$\lim_{\nu\to\infty} \alpha_\nu = 0 \iff \lim_{\nu\to\infty} F(\alpha_\nu) = -\infty;$$

($F3$) we can find $\theta \in (0,1)$ satisfying $\lim_{\alpha\to 0^+} \alpha^\theta F(\alpha) = 0$.

Now, we introduce several examples of above mappings:

Example 2.1 (*Wardowski, 2012*). Let $F_1, F_2, F_3, F_4 : \mathbb{R}^+ \to \mathbb{R}$ by:

(1) $F_1(x) = \ln x, \ \forall x > 0$;
(2) $F_2(x) = x + \ln x, \ \forall x > 0$;
(3) $F_3(x) = -\frac{1}{\sqrt{x}}, \ \forall x > 0$;
(4) $F_4(x) = \ln(x^2 + x), \ \forall x > 0$.

Then, $F_1, F_2, F_3, F_4 \in \mathcal{J}$.

Definition 2.2 (*Wardowski, 2012*). A mapping $\Pi : \mathfrak{M} \to \mathfrak{M}$ on a $\mathcal{MS}$ $\mathfrak{M}$ is said to be an F -contraction if we can find $F \in \mathcal{J}$ and $\tau \in \mathbb{R}^+$ satisfying

$$\tau + F(\psi(\Pi \mathrm{y}, \Pi\vartheta)) \le F(\psi(\mathrm{y}, \vartheta)),$$

for all $\mathrm{y}, \vartheta \in \mathfrak{M}$ with $\psi(\Pi\mathrm{y}, \Pi\vartheta) > 0$.

Gordji et al. (2017) established the notion of an orthogonal set (or $\mathcal{O}$-set), with various properties and examples as follows:

Definition 2.3 (*Gordji et al., 2017*). Let $\mathfrak{M} \neq \phi$. Consider $\perp \subseteq \mathfrak{M}\times\mathfrak{M}$ to be a binary relation. If $\perp$ verifies the next property:

$$\exists \mathrm{y}_0 \in \mathfrak{M} : (\forall \mathrm{y} \in \mathfrak{M}, \mathrm{y}\perp\mathrm{y}_0) \quad \text{or} \quad (\forall \mathrm{y}\in\mathfrak{M}, \mathrm{y}_0\perp\mathrm{y}).$$

Then, $\mathfrak{M}$ is called an orthogonal set . We denote this $\mathcal{O}$-set by $(\mathfrak{M}, \perp)$.

Example 2.4 (*Gordji et al., 2017*). Let $\mathfrak{M} = [0,\infty)$ and define $\tilde{y}\perp\mathrm{y}$ if $\tilde{y}\ \mathrm{y} \in \{\tilde{y}, \mathrm{y}\}$. Then, by setting $\tilde{y}_0 = 0$ or $\tilde{y}_0 = 1$, $(\mathfrak{M},\perp)$ is an $\mathcal{O}$-set.

Definition 2.5 (*Gordji et al., 2017*). Assume that $(\mathfrak{M},\perp)$ is an $\mathcal{O}$-set. We say that $\{\mathrm{y}_\nu\}$ is an orthogonal sequence (briefly, $\mathcal{O}$-sequence) if

$$(\forall \nu\in\mathbb{N}, \mathrm{y}_\nu\perp\mathrm{y}_{\nu+1}) \quad or \quad (\forall\nu\in\mathbb{N}, \mathrm{y}_{\nu+1}\perp\mathrm{y}_\nu).$$

Definition 2.6 (*Gordji et al., 2017*). The triplet $(\mathfrak{M},\perp,\psi)$ is said to be an orthogonal $\mathcal{MS}$ ($\mathcal{OMS}$)if $(\mathfrak{M},\perp)$ is an $\mathcal{O}$-set and $(\mathfrak{M},\psi)$ is a $\mathcal{MS}$.

Definition 2.7 (*Gordji et al., 2017*). Consider $(\mathfrak{M},\perp,\psi)$ be an $\mathcal{OMS}$. Then, an operator $\Pi : \mathfrak{M}\to\mathfrak{M}$ is called an orthogonally continuous in $\mathrm{y}\in\mathfrak{M}$, if for all $\mathcal{O}$-sequence $\{\mathrm{y}_\nu\}$ in $\mathfrak{M}$ with $\mathrm{y}_\nu \to \mathrm{y}$ as $\nu\to\infty$, one has $\Pi(\mathrm{y}_\nu)\to\Pi(\mathrm{y})$ as $\nu\to\infty$. Moreover, Π is called $\perp$-continuous on $\mathfrak{M}$, if Π is $\perp$-continuous in every $\mathrm{y}\in\mathfrak{M}$.

Definition 2.8 (*Gordji et al., 2017*). Let $(\mathfrak{M},\perp,\psi)$ be an $\mathcal{OMS}$. Then, $\mathfrak{M}$ is called as $\mathcal{OCMS}$, if every $\mathcal{O}$-Cauchy sequence is convergent.

Remark 2.9 (*Gordji et al., 2017*). Every complete $\mathcal{MS}$ is $\mathcal{OCMS}$ and the converse need not be a true.

Example 2.10. Let $\mathfrak{M} = [0,1)$ and suppose that

$$\mathrm{y}\perp\vartheta \iff \begin{cases} \mathrm{y}\le\vartheta\le\frac{2}{5}; \\ \text{or } \mathrm{y}=0. \end{cases}$$

Then $(\mathfrak{M},\perp)$ is an $\mathcal{O}$-set. Clearly, $\mathfrak{M}$ with the Euclidean metric is not complete $\mathcal{MS}$, but it is $\mathcal{OCMS}$. In fact, if $\{\mathrm{y}_\nu\}$ is an $\mathcal{O}$-Cauchy sequence in $\mathfrak{M}$, then we can find a subsequence $\{\mathrm{y}_{\nu_k}\}$ of $\{\mathrm{y}_\nu\}$ for which $\mathrm{y}_{\nu_k} = 0\ \forall\ \nu\ge 1$ or we can find a monotone subsequence $\{\mathrm{y}_{\nu_k}\}$ of $\{\mathrm{y}_\nu\}$ for which $\mathrm{y}_{\nu_k}\le\frac{2}{5}\ \forall\ \nu\ge 1$. It follows that $\{\mathrm{y}_{\nu_k}\}$ converges to a point $\mathrm{y}\in[0,\frac{2}{5}]\subseteq\mathfrak{M}$. Already, we know that every Cauchy sequence with a convergent subsequence is convergent. Furthermore, $\{\mathrm{y}_\nu\}$ is convergent.

Let Ω and Γ be non-void subsets of $\mathfrak{M}$, then

$$\psi(\mathfrak{e},\Gamma) := \inf\{\psi(\mathfrak{e},\mathfrak{g}) : \mathfrak{g}\in\Gamma,\ \mathfrak{e}\in\Omega\},$$
$$\Omega_0 := \{\mathfrak{e}\in\Omega : \psi(\mathfrak{e},\mathfrak{g}) = \psi(\Omega,\Gamma) \text{ for some } \mathfrak{g}\in\Gamma\},$$
$$\Gamma_0 := \{\mathfrak{g}\in\Gamma : \psi(\mathfrak{e},\mathfrak{g}) = \psi(\Omega,\Gamma) \text{ for some } \mathfrak{e}\in\Omega\}.$$

If $\psi(\Omega,\Gamma) > 0$ for two closed subsets of a normed space, then Ω_0 and Γ_0 are contained in the limits of Ω and Γ, respectively (Basha and Veeramani, 2000).

Definition 2.11. A mapping $\Pi : \Omega\to\Gamma$ is called an orthogonal $\mathcal{FPC}$ of first kind ($\mathcal{OPCOFK}$) if we can find $F\in\mathcal{J}$ and $\tau > 0$ satisfying

$$\left.\begin{matrix} \vartheta_1\perp\vartheta_2 \\ \psi(\mathrm{y}_1,\Pi\vartheta_1) = \psi(\Omega,\Gamma) \\ \psi(\mathrm{y}_2,\Pi\vartheta_2) = \psi(\Omega,\Gamma)\end{matrix}\right\} \Rightarrow \tau + F(\psi(\mathrm{y}_1,\mathrm{y}_2)) \le F\big(\psi(\vartheta_1,\vartheta_2)\big),$$

for all $\mathrm{y}_1,\mathrm{y}_2,\vartheta_1,\vartheta_2$ in Ω and $\mathrm{y}_1\neq\mathrm{y}_2$.

Definition 2.12. A mapping $\Pi : \Omega\to\Gamma$ is called an orthogonal $\mathcal{FPC}$ of second kind ($\mathcal{OPCOSK}$) if we can find $F\in\mathcal{J}$ and $\tau>0$ satisfying

$$\left.\begin{matrix} \vartheta_1\perp\vartheta_2 \\ \psi(\mathrm{y}_1,\Pi\vartheta_1) = \psi(\Omega,\Gamma) \\ \psi(\mathrm{y}_2,\Pi\vartheta_2) = \psi(\Omega,\Gamma)\end{matrix}\right\} \Rightarrow \tau + F(\psi(\Pi\mathrm{y}_1,\Pi\mathrm{y}_2)) \le F\big(\psi(\Pi\vartheta_1,\Pi\vartheta_2)\big),$$

for all $\mathrm{y}_1,\mathrm{y}_2,\vartheta_1,\vartheta_2$ in Ω and $\Pi\mathrm{y}_1\neq\Pi\mathrm{y}_2$.

Definition 2.13. A mapping $\Pi : \Omega\to\Gamma$ is called a generalized orthogonal $\mathcal{FPC}$ of first kind ($\mathcal{GOPCOFK}$) if we can find $F\in\mathcal{J}$ and $\sigma,\mathfrak{t},\ell,\beta\ge 0$, $\tau>0$ with $\sigma+\mathfrak{t}+\ell+2\beta = 1$, $\ell\neq 1$ satisfying

$$\left.\begin{matrix} \vartheta_1\perp\vartheta_2 \\ \psi(\mathrm{y}_1,\Pi\vartheta_1) = \psi(\Omega,\Gamma) \\ \psi(\mathrm{y}_2,\Pi\vartheta_2) = \psi(\Omega,\Gamma)\end{matrix}\right\} \Rightarrow \tau + F(\psi(\mathrm{y}_1,\mathrm{y}_2)) \le F\big(\sigma\psi(\vartheta_1,\vartheta_2) + \mathfrak{t}\psi(\mathrm{y}_1,\vartheta_1) + \ell\psi(\mathrm{y}_2,\vartheta_2) + \beta(\psi(\vartheta_1,\mathrm{y}_2) + \psi(\vartheta_2,\mathrm{y}_1))\big),$$

for all $\mathrm{y}_1,\mathrm{y}_2,\vartheta_1,\vartheta_2$ in Ω and $\mathrm{y}_1\neq\mathrm{y}_2$.

Definition 2.14. A mapping $\Pi : \Omega \to \Gamma$ is called a generalized orthogonal $\mathcal{FPC}$ of second kind ($\mathcal{GOPCOGK}$) if $\exists\, F \in \mathcal{J}$ and $\sigma, \mathrm{t}, \ell, \psi \geq 0, \tau > 0$ with $\sigma + \mathrm{t} + \ell + 2\beta = 1$, $\ell \neq 1$ such that the conditions

$$\left.\begin{array}{r}\vartheta_1 \perp \vartheta_2\\ \psi(\mathrm{y}_1, \Pi\vartheta_1) = \psi(\Omega, \Gamma)\\ \psi(\mathrm{y}_2, \Pi\vartheta_2) = d(\Omega, \Gamma)\end{array}\right\} \Rightarrow \tau + F(\psi(\Pi\mathrm{y}_1, \Pi\mathrm{y}_2)) \leq F\big(\sigma\psi(\Pi\vartheta_1, \Pi\vartheta_2) + \mathrm{t}\psi(\Pi\mathrm{y}_1, \Pi\vartheta_1) + \ell\psi(\Pi\mathrm{y}_2, \Pi\vartheta_2) + \beta(\psi(\Pi\vartheta_1, \Pi\mathrm{y}_2) + \psi(\Pi\vartheta_2, \Pi\mathrm{y}_1))\big),$$

for all $\mathrm{y}_1, \mathrm{y}_2, \vartheta_1, \vartheta_2$ in Ω and $\Pi\mathrm{y}_1 \neq \Pi\mathrm{y}_2$.

Definition 2.15. A mapping $\Pi : \Omega \to \Gamma$ is called an $\perp$-proximally preserving if

$$\left.\begin{array}{r}\vartheta_1 \perp \vartheta_2\\ \psi(\mathrm{y}_1, \Pi\vartheta_1) = \psi(\Omega, \Gamma)\\ \psi(\mathrm{y}_2, \Pi\vartheta_2) = \psi(\Omega, \Gamma)\end{array}\right\} \Rightarrow \mathrm{y}_1 \perp \mathrm{y}_2,$$

for all $\mathrm{y}_1, \mathrm{y}_2, \vartheta_1, \vartheta_2$ in Ω.

Definition 2.16. Let $(\mathfrak{M}, \perp, \psi)$ be a $\mathcal{O}$-$\mathcal{MS}$ and (Ω, Γ) be a non-void closed subsets of $(\mathfrak{M}, \perp, \psi)$. The pair (Ω, Γ) satisfies the $\perp$-$\mathcal{Q}$ property if

$$\left.\begin{array}{r}\mathrm{y}_1 \perp \mathrm{y}_2, \vartheta_1 \perp \vartheta_2\\ \psi(\mathrm{y}_1, \vartheta_1) = \psi(\Omega, \Gamma)\\ \psi(\mathrm{y}_2, \vartheta_2) = \psi(\Omega, \Gamma)\end{array}\right\} \Rightarrow \psi(\mathrm{y}_1, \mathrm{y}_2) = \psi(\vartheta_1, \vartheta_2),$$

for all $\mathrm{y}_1, \mathrm{y}_2, \vartheta_1, \vartheta_2$ in Ω.

Definition 2.17. Let $(\mathfrak{M}, \perp, \psi)$ be a $\mathcal{O}$-$\mathcal{MS}$. A set Γ is said to be an relatively compact in context with Ω if every sequence $\{\mathrm{y}_\nu\}$ of Γ with $\psi(\mathrm{y}, \mathrm{y}_\nu) \to \psi(\mathrm{y}, \Gamma)$ for some $\mathrm{y} \in \Omega$ has a convergent subsequence.

3. Main results

Throughout this part,we present some basic result.

Lemma 3.1. *Let $(\mathfrak{M}, \perp, \psi)$ be an orthogonal $\mathcal{MS}$ and (Ω, Γ) be non-void closed subsets pair of $(\mathfrak{M}, \perp, \psi)$. Let $\Pi : \Omega \to \Gamma$ satisfy the following conditions:*

(L1) $\Pi(\Omega_0) \subseteq \Gamma_0$ and (Ω, Γ) satisfies the $\perp$-$\mathcal{Q}$ property;
(L2) Π is $\perp$-proximally preserving;
(L3) there exists $\mathrm{y}_0, \mathrm{y}_1 \in \Omega_0$ such that

$$\psi(\mathrm{y}_1, \Pi\mathrm{y}_0) = \psi(\Omega, \Gamma),$$

and $\mathrm{y}_0 \perp \mathrm{y}_1$.

Then, we can find a $\mathrm{y} \in \Omega$ implies that $\psi(\mathrm{y}, \Pi\mathrm{y}) = \psi(\Omega, \Gamma)$.

Proof. By condition (L3), there exists $\mathrm{y}_0, \mathrm{y}_1 \in \Omega_0$ such that

$$\psi(\mathrm{y}_1, \Pi\mathrm{y}_0) = \psi(\Omega, \Gamma),$$

and $\mathrm{y}_0 \perp \mathrm{y}_1$. Since $\Pi\mathrm{y}_1 \in \Gamma_0$, there exists $\mathrm{y}_2 \in \Omega_0$ such that

$$\psi(\mathrm{y}_2, \Pi\mathrm{y}_1) = \psi(\Omega, \Gamma),$$

Since $\perp$-proximally preserving, we get

$$\mathrm{y}_1 \perp \mathrm{y}_2.$$

Likewise, we can construct an $\mathcal{O}$-sequence

$$\mathrm{y}_0 \perp \mathrm{y}_1, \mathrm{y}_1 \perp \mathrm{y}_2, \mathrm{y}_2 \perp \mathrm{y}_3, \ldots, \mathrm{y}_\nu \perp \mathrm{y}_{\nu+1}, \ldots.$$

Then, $\{\mathrm{y}_\nu\}$ is an $\mathcal{O}$-sequence with

$$\psi(\mathrm{y}_{\nu+1}, \Pi\mathrm{y}_\nu) = \psi(\Omega, \Gamma),$$

for all $\nu \in \mathbb{N}$. By $\perp$-$\mathcal{Q}$-property, we have

$$\psi(\mathrm{y}_\nu, \mathrm{y}_{\nu+1}) = \psi(\Pi\mathrm{y}_{\nu-1}, \Pi\mathrm{y}_\nu),$$

for all $\nu \in \mathbb{N}$. If for some ν_0, $\psi(\mathrm{y}_{\nu_0}, \mathrm{y}_{\nu_0+1}) = 0$, consequently

$$\psi(\Pi\mathrm{y}_{\nu_0-1}, \Pi\mathrm{y}_{\nu_0}) = 0.$$

Therefore, $\Pi\mathrm{y}_{\nu_0-1} = \Pi\mathrm{y}_{\nu_0}$. Hence, $\psi(\Omega, \Gamma) = \psi(\mathrm{y}_{\nu_0}, \Pi\mathrm{y}_{\nu_0})$. Thus the conclusion is immediate. □

Now, we give our best proximity result on $\mathcal{OPCOFK}$.

Theorem 3.2. *Let $(\mathfrak{M}, \perp, \psi)$ be a $\mathcal{OCMS}$ and (Ω, Γ) be non-void closed subsets of $(\mathfrak{M}, \perp, \psi)$. Let $\Pi : \Omega \to \Gamma$ satisfy the following conditions:*

(B1) $\Pi(\Omega_0) \subseteq \Gamma_0$ and (Ω, Γ) satisfies the $\perp$-$\mathcal{Q}$ property;
(B2) Π is $\perp$-proximally preserving;
(B3) Π is an $\mathcal{OPCOFK}$;
(B4) there exists $\mathrm{y}_0, \mathrm{y}_1 \in \Omega_0$ such that

$$\psi(\mathrm{y}_1, \Pi\mathrm{y}_0) = \psi(\Omega, \Gamma),$$

and $\mathrm{y}_0 \perp \mathrm{y}_1$;

(B5) Π is $\perp$-continuous.

Then, there is one $\mathrm{y} \in \Omega$, where $\psi(\mathrm{y}, \Pi\mathrm{y}) = \psi(\Omega, \Gamma)$.

Proof. By Lemma 3.1, we have $\psi(\mathrm{y}, \Pi\mathrm{y}) = \psi(\Omega, \Gamma)$. So let for any $\nu \geq 0$, $\psi(\mathrm{y}_\nu, \mathrm{y}_{\nu+1}) > 0$. Since Π is an $\mathcal{OPCOFK}$, we have that

$$\tau + F(\psi(\mathrm{y}_\nu, \mathrm{y}_{\nu+1})) \leq F\big(\psi(\mathrm{y}_{\nu-1}, \mathrm{y}_\nu)\big).$$

Consequently,

$$\tau + F(\psi(\mathrm{y}_\nu, \mathrm{y}_{\nu+1})) \leq F(\psi(\mathrm{y}_\nu, \mathrm{y}_{\nu-1})), \ \forall \nu \in \mathbb{N}.$$

It implies

$$F(\psi(\mathrm{y}_\nu, \mathrm{y}_{\nu+1})) \leq F(\psi(\mathrm{y}_\nu, \mathrm{y}_{\nu-1})) - \tau \leq \cdots \leq F(\psi(\mathrm{y}_0, \mathrm{y}_1)) - \nu\tau, \ \forall \nu \in \mathbb{N}. \quad (3.1)$$

Put $\lambda_\nu := \psi(\mathrm{y}_\nu, \mathrm{y}_{\nu+1})$. From (3.1) $\lim_{\nu\to\infty} F(\lambda_\nu) = -\infty$. By the property $(F1)$, we get that

$$\lambda_\nu \to 0 \text{ as } \nu \to \infty.$$

Now, let $\theta \in (0, 1)$ such that $\lim_{\nu\to\infty} \lambda_\nu^\theta F(\lambda_\nu) = 0$. By (3.1), for all $\nu \in \mathbb{N}$:

$$\lambda_\nu^\theta F(\lambda_\nu) - \lambda_\nu^\theta F(\lambda_0) \leq -\nu\lambda_\nu^\theta \tau \leq 0. \quad (3.2)$$

Letting $\theta \to \infty$ in (3.2), we have

$$\lim_{\nu\to\infty} \nu\lambda_\nu^\theta = 0.$$

Thus, $\lim_{\nu\to\infty} \nu^{\frac{1}{\theta}} \lambda_\nu = 0$, then series $\sum_{\nu=1}^{\infty} \lambda_\nu$ is convergent. Therefore $\{\mathrm{y}_\nu\}$ is a $\mathcal{O}$-Cauchy sequence in Ω. Since, Ω is a closed subset of $(\mathfrak{M}, \perp, \psi)$ and the space is $\mathcal{OCMS}$, so $\mathrm{y} \in \Omega$ with $\{\mathrm{y}_\nu\} \to \mathrm{y}$ as $\nu \to \infty$. Since Π is $\perp$-continuous, $\Pi\mathrm{y}_\nu \to \Pi\mathrm{y}$, which means that $\psi(\mathrm{y}_\nu, \Pi\mathrm{y}_\nu) \to \psi(\mathrm{y}, \Pi\mathrm{y})$ as $\nu \to \infty$. Hence,

$$\psi(\mathrm{y}, \Pi\mathrm{y}) = \psi(\Omega, \Gamma).$$

Assume that,

$$\psi(\mathrm{y}^*, \Pi\mathrm{y}^*) = \psi(\Omega, \Gamma).$$

Since Π is $\perp$-proximally preserving, we get

$$\mathrm{y} \perp \mathrm{y}^*.$$

Since Π is an $\mathcal{OPCOFK}$, we get

$$\tau + F(\psi(\mathrm{y}, \mathrm{y}^*)) \leq F\big(\psi(\mathrm{y}, \mathrm{y}^*)\big).$$

Since F strictly increasing, then

$$\psi(\mathrm{y}, \mathrm{y}^*) \leq \psi(\mathrm{y}, \mathrm{y}^*).$$

Therefore, $\mathrm{y} = \mathrm{y}^*$. Hence, Π has a unique $\mathcal{BPP}$. □

Example 3.3. Let $\mathfrak{M} = \mathbb{R}^2$ and Euclidean metric ψ with $\perp$ defined by $(y_1, y_2)\perp(\vartheta_1, \vartheta_2)$, if $y_1, \vartheta_1, y_2, \vartheta_2 \geq 0$. Clearly, $(\mathfrak{M}, \perp, \psi)$ is a $\mathcal{OCMS}$. Let $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\sigma) = \ln \sigma$. To verify the axioms $(F1)$–$(F3)$. Let $\sigma_1 < \sigma_2$. Then

$$F(\sigma_1) < F(\sigma_2). \tag{3.3}$$

Therefore, F is strictly increasing. Assume that

$$\lim_{\nu\to\infty} \sigma_\nu = 0.$$

Then $\sigma_\nu = \frac{1}{\nu}$. Now

$$\lim_{\nu\to\infty} F(\sigma_\nu) = -\infty.$$

Conversely, assume that

$$\lim_{\nu\to\infty} F(\sigma_\nu) = -\infty.$$

Then $\sigma_\nu = \frac{1}{\nu}$. Now

$$\lim_{\nu\to\infty} \sigma_\nu = 0.$$

Thus, F satisfies axiom $(F2)$. We can find that $\theta \in (0,1)$ implies that

$$\lim_{\nu\to\infty} \sigma^\theta F(\sigma) = 0.$$

Therefore, F fulfilled the axioms $(F1)$–$(F3)$. Let $\Omega = \{(\mathfrak{e}, 0) : \mathfrak{e} \geq 0\}$ and $\Gamma = \{(\mathfrak{e}, 1) : \mathfrak{e} \geq 0\}$. We have $\Omega = \Omega_0$ and $\Gamma = \Gamma_0$. Let $\Pi : \Omega \to \Gamma$ by

$$\Pi(\mathfrak{e}, 0) = \left(\frac{\mathfrak{e}}{2}, 1\right),$$

for each $(\mathfrak{e}, 0) \in \Omega$. It is clear that (Ω, Γ) satisfies Π is $\perp$-continuous, $\perp$-proximally preserving and $\Pi(\Omega_0) \subseteq \Gamma_0$. Let $y_1, \vartheta_1, y_2, \vartheta_2 \in \Omega$ such that $\psi(y_1, \Pi\vartheta_1) = \psi(y_2, \Pi\vartheta_2) = \psi(\Omega, \Gamma) = 1$. Take $\vartheta_1 = (\mathfrak{e}_1, 0)$, $\vartheta_2 = (\mathfrak{e}_2, 0)$, $y_1 = (\frac{\mathfrak{e}_1}{2}, 0)$ and $y_2 = (\frac{\mathfrak{e}_2}{2}, 0)$ for some $\mathfrak{e}_1, \mathfrak{e}_2 \geq 0$. Then (Ω, Γ) satisfies the $\perp$-$\mathcal{Q}$-property. Clearly,

$$\psi(y_1, y_2) \leq e^{-\tau}\psi(\vartheta_1, \vartheta_2).$$

Consequently, Π is an $\mathcal{OPCOFK}$ with $e^{-\tau} = \frac{16}{7}$ or $\tau = \ln\frac{7}{16}$. From Theorem 3.2 of all axioms are verified. Hence, Π has a unique $\mathcal{BPP}$ $(0,0)$.

If $\Omega = \Gamma$, then our result reduces to Theorem 3.3 in Sawangsup et al. (2020).

Corollary 3.4. *Let $(\mathfrak{M}, \perp, \psi)$ be a $\mathcal{OCMS}$ and Ω be a non-void closed subset of $(\mathfrak{M}, \perp, \psi)$. Let $\Pi : \Omega \to \Omega$ satisfy the following conditions:*

(CB1) *Π is $\perp$-preserving;*
(CB2) *we can find that $\tau > 0$ implies that*

$$\tau + F(\psi(\Pi\vartheta_1, \Pi\vartheta_2)) \leq F\big(\psi(\vartheta_1, \vartheta_2)\big),$$

with $\vartheta_1 \perp \vartheta_2$ and $\vartheta_1 \neq \vartheta_2$;
(CB3) *Π is $\perp$-continuous.*

Then, Π possesses one fixed point.

Now, we give our best proximity result on $\mathcal{OPCOSK}$.

Theorem 3.5. *Let $(\mathfrak{M}, \perp, \psi)$ be a $\mathcal{OCMS}$ and (Ω, Γ) be non-void closed subsets of $(\mathfrak{M}, \perp, \psi)$. Let Ω is relatively compact in context with Γ and $\Pi : \Omega \to \Gamma$ satisfy as follows:*

(CH1) *$\Pi(\Omega_0) \subseteq \Gamma_0$ and (Ω, Γ) satisfies the $\perp$-$\mathcal{Q}$-property;*
(CH2) *Π is $\perp$-proximally preserving;*
(CH3) *Π is an $\mathcal{OPCOSK}$;*
(CH4) *there exists $y_0, y_1 \in \Omega_0$ such that*

$$\psi(y_1, \Pi y_0) = \psi(\Omega, \Gamma),$$

and $y_0 \perp y_1$;
(CH5) *Π is $\perp$-continuous.*

Then, $\exists$ a unique $y \in \Omega$ such that $\psi(y, \Pi y) = \psi(\Omega, \Gamma)$.

Proof. By Lemma 3.1, we have $\psi(y, \Pi y) = \psi(\Omega, \Gamma)$. So let for any $\nu \geq 0$, $\psi(y_\nu, y_{\nu+1}) > 0$. Since, Π is an $\mathcal{OPCOSK}$, we derive that

$$\tau + F\big(\psi(\Pi y_\nu, \Pi y_{\nu+1})\big) \leq F\big(\psi(\Pi y_{\nu-1}, \Pi y_\nu)\big).$$

Therefore,

$$\tau + F(\psi(\Pi y_\nu, \Pi y_{\nu+1})) \leq F(\psi(\Pi y_\nu, \Pi y_{\nu-1})), \ \forall \nu \in \mathbb{N}.$$

Consequently,

$$\begin{aligned} F(\psi(\Pi y_\nu, \Pi y_{\nu+1})) &\leq F(\psi(\Pi y_\nu, \Pi y_{\nu-1})) - \tau \\ &\ \vdots \\ &\leq F(\psi(\Pi y_0, \Pi y_1)) - \nu\tau, \ \forall \nu \in \mathbb{N}. \end{aligned} \tag{3.4}$$

Put $\delta_\nu := \psi(\Pi y_\nu, \Pi y_{\nu+1})$. From (3.4) $\lim_{\nu\to\infty} F(\delta_\nu) = -\infty$. By the property $(F1)$, we get that

$$\delta_\nu \to 0 \text{ as } \nu \to \infty.$$

Now, let $\theta \in (0,1)$ such that $\lim_{\nu\to\infty} \delta_\nu^\theta F(\delta_\nu) = 0$. By (3.4), for all $\nu \in \mathbb{N}$:

$$\delta_\nu^\theta F(\delta_\nu) - \delta_\nu^\theta F(\delta_0) \leq -\nu\delta_\nu^\theta \tau \leq 0. \tag{3.5}$$

As $\theta \to \infty$ in (3.5), we deduce that

$$\lim_{\nu\to\infty} \nu\delta_\nu^\theta = 0.$$

Thus $\lim_{\nu\to\infty} \nu^{\frac{1}{\theta}}\delta_\nu = 0$, then series $\sum_{\nu=1}^{\infty}\delta_\nu$ is convergent. Therefore, $\{\Pi y_\nu\}$ is a $\mathcal{O}$-Cauchy sequence in Γ. Since, Γ is a closed subset of $(\mathfrak{M}, \perp, \psi)$ and the space is $\mathcal{OCMS}$, so $\{\Pi y_\nu\}$ converges to some element v in Γ. Also,

$$\begin{aligned} \psi(v, \Omega) &\leq \psi(v, \Pi y_\nu) \\ &\leq \psi(v, y_{\nu+1}) + \psi(y_{\nu+1}\Pi y_\nu) \\ &= \psi(v, y_{\nu+1}) + \psi(\Omega, \Gamma) \\ &\leq \psi(v, y_{\nu+1}) + \psi(y, \Omega). \end{aligned}$$

Therefore, $\psi(v, \Pi y_\nu) \to \psi(v, \Omega)$. Since Ω is relatively compact in context with Γ, the sequence $\{y_\nu\}$ has a subsequence y_{ν_k} converges to $y \in \Omega$. Hence,

$$\psi(y, v) = \lim_{\nu\to\infty} \psi(y_{\nu_{k+1}}, \Pi y_{\nu_k}) = \psi(\Omega, \Gamma).$$

Since Π is a $\perp$-continuous mapping,

$$\psi(y, \Pi y) = \lim_{\nu\to\infty} \psi(y_{\nu+1}, \Pi y_\nu) = \psi(\Omega, \Gamma).$$

Assume that, y^*, so that

$$\psi(y^*, \Pi y^*) = \psi(\Omega, \Gamma).$$

Since Π is $\perp$-proximally preserving, we get

$$y \perp y^*.$$

Since Π is an $\mathcal{OPCOSK}$,

$$\tau + F(\psi(\Pi y, \Pi y^*)) \leq F\big((\psi(\Pi y, \Pi y^*))\big).$$

Since F strictly increasing,

$$\psi(\Pi y, \Pi y^*) \leq \psi(\Pi y, \Pi y^*).$$

Thus, $\Pi y = \Pi y^*$. Hence, Π has a unique $\mathcal{BPP}$. □

Example 3.6. Let $\mathfrak{M} = \mathbb{R}^2$ and the metric

$$\psi((y_1, y_2), (\vartheta_1, \vartheta_2)) = |y_1 - \vartheta_1| + |y_2 - \vartheta_2|.$$

with $\perp$ defined by $(y_1, y_2)\perp(\vartheta_1, \vartheta_2)$, if $y_1, \vartheta_1, y_2, \vartheta_2 \geq 0$. Clearly, $(\mathfrak{M}, \perp, \psi)$ is a $\mathcal{OCMS}$. Let $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\sigma) = \ln\sigma$. Clearly, for any $\theta \in (0,1)$, F fulfills the axioms $(F1)$–$(F3)$. Let $\Omega = \{(0, \mathfrak{e}) : \mathfrak{e} \in \mathbb{R}\}$ and $\Gamma = \{(2, \mathfrak{e}) : \mathfrak{e} \in \mathbb{R}\}$. We have $\Omega_0 = \Omega$ and $\Gamma_0 = \Gamma$. Clearly, Ω is relatively compact in context with Γ. Let $\Pi : \Omega \to \Gamma$ by

$$\Pi(0, \mathfrak{e}) = \left(2, \frac{\mathfrak{e}}{2}\right),$$

for each $(0,\mathfrak{e}) \in \Omega$. It is clear that (Ω,Γ) satisfies Π is $\perp$-continuous, $\perp$-proximally preserving and $\Pi(\Omega_0) \subseteq \Gamma_0$. Let $y_1, \vartheta_1, y_2, \vartheta_2 \in \Omega$ such that $\psi(y_1, \Pi\vartheta_1) = \psi(y_2, \Pi\vartheta_2) = \psi(\Omega,\Gamma) = 2$. Take $y_1 = (0,1)$, $\vartheta_1 = (0,2)$, $y_2 = (0,4)$ and $\vartheta_2 = (0,8)$. Then (Ω,Γ) satisfies the $\perp$-$\mathcal{Q}$-property. Clearly,

$$\psi(\Pi y_1, \Pi y_2) \le e^{-\tau}\psi(\Pi\vartheta_1, \Pi\vartheta_2).$$

Consequently, Π is an $\mathcal{OPCOSK}$ with $e^{-\tau} = \frac{17}{7}$ or $\tau = \ln\frac{7}{17}$. From Theorem 3.5 of all axioms are verified. Hence, Π has a unique $\mathcal{BPP}$ $(0,0)$.

In what follows, we present our best proximity result on $\mathcal{GOPCOFK}$.

Theorem 3.7. *Let $(\mathfrak{M},\perp,\psi)$ be a $\mathcal{OCMS}$ and (Ω,Γ) be non-void closed subsets of $(\mathfrak{M},\perp,\psi)$. Let $\Pi : \Omega \to \Gamma$ satisfy the following conditions:*

(H1) $\Pi(\Omega_0) \subseteq \Gamma_0$ and (Ω,Γ) satisfies the $\perp$-$\mathcal{Q}$-property;
(H2) Π is $\perp$-proximally preserving;
(H3) Π is a $\mathcal{GOPCOFK}$;
(H4) there exists $y_0, y_1 \in \Omega_0$ such that

$$\psi(y_1, \Pi y_0) = \psi(\Omega,\Gamma),$$

and $y_0 \perp y_1$;
(H5) Π is $\perp$-continuous.

Then, $\exists$ a unique $y \in \Omega$ such that $\psi(y, \Pi y) = \psi(\Omega,\Gamma)$.

Proof. By Lemma 3.1, we have $\psi(y, \Pi y) = \psi(\Omega,\Gamma)$. So let for any $\nu \ge 0$, $\psi(y_\nu, y_{\nu+1}) > 0$. Since, Π is a $\mathcal{GOPCOFK}$, we have that

$$\begin{aligned}\tau + F(\psi(y_\nu, y_{\nu+1})) &\le F\big(\sigma\psi(y_{\nu-1}, y_\nu) + \mathfrak{t}\psi(y_{\nu-1}, y_\nu) + \ell\psi(y_\nu, y_{\nu+1}) \\ &\quad + \beta\psi(y_{\nu-1}, y_{\nu+1})\big) \\ &\le F\big(\sigma\psi(y_{\nu-1}, y_\nu) + \mathfrak{t}\psi(y_{\nu-1}, y_\nu) + \ell\psi(y_\nu, y_{\nu+1}) \\ &\quad + \beta[\psi(y_{\nu-1}, y_\nu) + \psi(y_\nu, y_{\nu+1})]\big) \\ &= F\big((\sigma + \mathfrak{t} + \beta)\psi(y_{\nu-1}, y_\nu) + (\ell + \beta)\psi(y_\nu, y_{\nu+1})\big).\end{aligned}$$

Since F is strictly non-decreasing, we deduce

$$\psi(y_\nu, y_{\nu+1}) \le (\sigma + \mathfrak{t} + \beta)\psi(y_{\nu-1}, y_\nu) + (\ell + \beta)\psi(y_\nu, y_{\nu+1}).$$

Thus

$$\psi(y_\nu, y_{\nu+1}) \le \left(\frac{\sigma + \mathfrak{t} + \beta}{1 - \ell - \beta}\right)\psi(y_\nu, y_{\nu-1}), \ \forall \nu \in \mathbb{N}.$$

From $\sigma + \mathfrak{t} + \ell + 2\beta = 1$ and $\ell \ne 1$, we have that $1 - \ell - \beta > 0$, and so

$$\psi(y_\nu, y_{\nu+1}) \le \left(\frac{\sigma + \mathfrak{t} + \beta}{1 - \ell - \beta}\right)\psi(y_\nu, y_{\nu-1}) = \psi(y_\nu, y_{\nu-1}), \ \forall \nu \in \mathbb{N}.$$

Consequently,

$$\tau + F(\psi(y_\nu, y_{\nu+1})) \le F(\psi(y_\nu, y_{\nu-1})), \ \forall \nu \in \mathbb{N}.$$

It implies

$$F(\psi(y_\nu, y_{\nu+1})) \le F(\psi(y_\nu, y_{\nu-1})) - \tau \le \cdots \le F(\psi(y_0, y_1)) - \nu\tau, \ \forall \nu \in \mathbb{N}. \quad (3.6)$$

Put $\lambda_\nu := \psi(y_\nu, y_{\nu+1})$. From (5.1) $\lim_{\nu\to\infty} F(\lambda_\nu) = -\infty$. By the properties $(F1)$, we get that

$$\lambda_\nu \to 0 \text{ as } \nu \to \infty.$$

Now, let $\theta \in (0,1)$ such that $\lim_{\nu\to\infty} \lambda_\nu^\theta F(\lambda_\nu) = 0$. By (5.1), for all $\nu \in \mathbb{N}$:

$$\lambda_\nu^\theta F(\lambda_\nu) - \lambda_\nu^\theta F(\lambda_0) \le -\nu\lambda_\nu^\theta\tau \le 0. \quad (3.7)$$

Letting $\theta \to \infty$ in (5.2), we have

$$\lim_{\nu\to\infty} \nu\lambda_\nu^\theta = 0.$$

Thus, $\lim_{\nu\to\infty} \nu^{\frac{1}{\theta}}\lambda_\nu = 0$, then series $\sum_{\nu=1}^{\infty}\lambda_\nu$ is convergent. Therefore $\{y_\nu\}$ is a $\mathcal{O}$-Cauchy sequence in Ω. Since, Ω is a closed subset of $(\mathfrak{M},\perp,\psi)$ and the space is $\mathcal{OCMS}$, so $y \in \Omega$ with $\{y_\nu\} \to y$ as $\nu \to \infty$. Since Π is $\perp$-continuous, $\Pi y_\nu \to \Pi y$, which means that $\psi(y_\nu, \Pi y_\nu) \to \psi(y, \Pi y)$ as $\nu \to \infty$. Hence,

$$\psi(y, \Pi y) = \psi(\Omega,\Gamma).$$

Assume that,

$$\psi(y^*, \Pi y^*) = \psi(\Omega,\Gamma).$$

Since Π is $\perp$-proximally preserving, we get

$$y \perp y^*.$$

Since Π is a $\mathcal{GOPCOFK}$, we get

$$\tau + F(\psi(y, y^*)) \le F\big((\sigma + 2\beta)\psi(y, y^*)\big).$$

Since F strictly increasing,

$$\psi(y, y^*) \le (\sigma + 2\beta)\psi(y, y^*).$$

Therefore, $y = y^*$. Hence, Π has a unique $\mathcal{BPP}$. □

Example 3.8. Let $\mathfrak{M} = \mathbb{R}^2$ and Euclidean metric ψ with $\perp$ defined by $(y_1, y_2) \perp (\vartheta_1, \vartheta_2)$, if $y_1, \vartheta_1, y_2, \vartheta_2 \ge 0$. Clearly, $(\mathfrak{M},\perp,\psi)$ is a $\mathcal{OCMS}$. Let $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\sigma) = \ln\sigma$. Clearly, for any $\theta \in (0,1)$, F fulfills the axioms $(F1)$–$(F3)$. Let $\Omega = \{(\mathfrak{e}, 0) : \mathfrak{e} \ge 0\}$ and $\Gamma = \{(\mathfrak{e}, 1) : \mathfrak{e} \ge 0\}$. We have $\Omega = \Omega_0$ and $\Gamma = \Gamma_0$. Let $\Pi : \Omega \to \Gamma$ by

$$\Pi(\mathfrak{e}, 0) = \left(\frac{\mathfrak{e}}{3}, 1\right),$$

for each $(\mathfrak{e}, 0) \in \Omega$. It is clear that (Ω,Γ) satisfies Π is $\perp$-continuous, $\perp$-proximally preserving and $\Pi(\Omega_0) \subseteq \Gamma_0$. Let $y_1, \vartheta_1, y_2, \vartheta_2 \in \Omega$ such that $\psi(y_1, \Pi\vartheta_1) = \psi(y_2, \Pi\vartheta_2) = \psi(\Omega,\Gamma) = 2$. Take $\vartheta_1 = (\mathfrak{e}_1, 0)$, $\vartheta_2 = (\mathfrak{e}_2, 0)$, $y_1 = (\frac{\mathfrak{e}_1}{3}, 0)$ and $y_2 = (\frac{\mathfrak{e}_2}{3}, 0)$ for some $\mathfrak{e}_1, \mathfrak{e}_2 \ge 0$. Then (Ω,Γ) satisfies the $\perp$-$\mathcal{Q}$-property. Clearly,

$$\psi(y_1, y_2) \le e^{-\tau}(\sigma\psi(\vartheta_1, \vartheta_2) + \psi(\vartheta_2, y_1)).$$

Consequently, Π is an $\mathcal{GOPCOFK}$ with $e^{-\tau} = \frac{13}{5}$ or $\tau = \ln\frac{5}{13}$, $\sigma = 1$ and $\mathfrak{t} = \ell = \beta = 0$. From Theorem 3.7 of all axioms are verified. Hence, Π has a unique $\mathcal{BPP}$ $(0,0)$.

Next, we present our best proximity result on $\mathcal{GOPCOSK}$.

Theorem 3.9. *Let $(\mathfrak{M},\perp,\psi)$ be a $\mathcal{OCMS}$ and (Ω,Γ) be non-void closed subsets of $(\mathfrak{M},\perp,\psi)$. Let $\Pi : \Omega \to \Gamma$ satisfy the following conditions:*

(C1) $\Pi(\Omega_0) \subseteq \Gamma_0$ and (Ω,Γ) satisfies the $\perp$-$\mathcal{Q}$-property;
(C2) Π is $\perp$-proximally preserving;
(C3) Π is $\mathcal{GOPCOSK}$;
(C4) there exists $y_0, y_1 \in \Omega_0$ such that

$$\psi(y_1, \Pi y_0) = \psi(\Omega,\Gamma),$$

and $y_0 \perp y_1$;
(C5) Π is $\perp$-continuous.

Then, we can find a unique $y \in \Omega$ implies that $\psi(y, \Pi y) = \psi(\Omega,\Gamma)$.

Proof. By Lemma 3.1, we have $\psi(y, \Pi y) = \psi(\Omega,\Gamma)$. So let for any $\nu \ge 0$, $\psi(y_\nu, y_{\nu+1}) > 0$. Since, Π is a $\mathcal{GOPCOSK}$, we derive that

$$\begin{aligned}\tau + F\big(\psi(\Pi y_\nu, \Pi y_{\nu+1})\big) &\le F\big(\sigma\psi(\Pi y_{\nu-1}, \Pi y_\nu) + \mathfrak{t}\psi(\Pi y_{\nu-1}, \Pi y_\nu) \\ &\quad + \ell\psi(\Pi y_\nu, \Pi y_{\nu+1}) + \beta\psi(\Pi y_{\nu-1}, \Pi y_{\nu+1})\big) \\ &\le F\big(\sigma\psi(\Pi y_{\nu-1}, \Pi y_\nu) + \mathfrak{t}\psi(\Pi y_{\nu-1}, \Pi y_\nu) \\ &\quad + \ell\psi(\Pi y_\nu, \Pi y_{\nu+1}) + \beta[\psi(\Pi y_{\nu-1}, \Pi y_\nu) \\ &\quad + \psi(\Pi y_\nu, \Pi y_{\nu+1})]\big) \\ &\le F\big((\sigma + \mathfrak{t} + \beta)\psi(\Pi y_{\nu-1}, \Pi y_\nu) + (\ell + \beta)\psi(\Pi y_\nu, \Pi y_{\nu+1})\big).\end{aligned}$$

Since F strictly increasing,

$$\psi(\Pi y_\nu, \Pi y_{\nu+1}) \le (\sigma + \mathfrak{t} + \beta)\psi(\Pi y_{\nu-1}, \Pi y_\nu) + (\ell + \beta)\psi(\Pi y_\nu, \Pi y_{\nu+1}),$$

and thus

$$\psi(\Pi y_\nu, \Pi y_{\nu+1}) \le \left(\frac{\sigma + \mathfrak{t} + \beta}{1 - \ell - \beta}\right)\psi(\Pi y_\nu, \Pi y_{\nu-1}), \ \forall \nu \in \mathbb{N}.$$

From $\sigma+\mathfrak{t}+\ell+2\beta=1$ and $\ell\neq 1$, we deduce that $1-\ell-\beta>0$ and so

$$\psi(\Pi y_\nu,\Pi y_{\nu+1})\le\left(\frac{\sigma+\mathfrak{t}+\beta}{1-\ell-\beta}\right)\psi(\Pi y_\nu,\Pi y_{\nu-1})=\psi(\Pi y_\nu,\Pi y_{\nu-1}),\ \forall\nu\in\mathbb{N}.$$

Therefore,

$$\tau+F(\psi(\Pi y_\nu,\Pi y_{\nu+1}))\le F(\psi(\Pi y_\nu,\Pi y_{\nu-1})),\ \forall\nu\in\mathbb{N}.$$

Consequently,

$$\begin{aligned}F(\psi(\Pi y_\nu,\Pi y_{\nu+1}))&\le F(\psi(\Pi y_\nu,\Pi y_{\nu-1}))-\tau\\&\ \vdots\\&\le F(\psi(\Pi y_0,\Pi y_1))-\nu\tau,\ \forall\nu\in\mathbb{N}.\end{aligned}\tag{3.8}$$

Put $\delta_\nu:=\psi(\Pi y_\nu,\Pi y_{\nu+1})$. From (3.8) $\lim_{\nu\to\infty}F(\delta_\nu)=-\infty$. By the properties $(F1)$, we get that

$$\delta_\nu\to 0 \text{ as } \nu\to\infty.$$

Now, let $\theta\in(0,1)$ such that $\lim_{\nu\to\infty}\delta_\nu^\theta F(\delta_\nu)=0$. By (3.8), for all $\nu\in\mathbb{N}$:

$$\delta_\nu^\theta F(\delta_\nu)-\delta_\nu^\theta F(\delta_0)\le-\nu\delta_\nu^\theta\tau\le 0.\tag{3.9}$$

As $\theta\to\infty$ in (3.9), we deduce that

$$\lim_{\nu\to\infty}\nu\delta_\nu^\theta=0.$$

Thus, $\lim_{\nu\to\infty}\nu^{\frac{1}{\theta}}\delta_\nu=0$, then series $\sum_{\nu=1}^\infty\delta_\nu$ is convergent. Since, Γ is a closed subset. Therefore, $\{\Pi y_\nu\}$ is a $\mathcal{O}$-Cauchy sequence in Γ. Hence, $\{\Pi y_\nu\}$ converges to v in Γ. Also,

$$\begin{aligned}\psi(v,\Omega)&\le\psi(v,\Pi y_\nu)\\&\le\psi(v,y_{\nu+1})+\psi(y_{\nu+1}\Pi y_\nu)\\&=\psi(v,y_{\nu+1})+\psi(\Omega,\Gamma)\\&\le\psi(v,y_{\nu+1})+\psi(y,\Omega).\end{aligned}$$

Therefore, $\psi(v,\Pi y_\nu)\to\psi(v,\Omega)$. Since Ω is relatively compact in context with Γ, the sequence $\{y_\nu\}$ has a subsequence y_{ν_k} converges to $y\in\Omega$. Hence,

$$\psi(y,v)=\lim_{\nu\to\infty}\psi(y_{\nu_{k+1}},\Pi y_{\nu_k})=\psi(\Omega,\Gamma).$$

Since Π is a $\perp$-continuous mapping,

$$\psi(y,\Pi y)=\lim_{\nu\to\infty}\psi(y_{\nu+1},\Pi y_\nu)=\psi(\Omega,\Gamma).$$

Assume that,

$$\psi(y^*,\Pi y^*)=\psi(\Omega,\Gamma).$$

Since Π is $\mathcal{GOPCOSK}$,

$$\tau+F(\psi(\Pi y,\Pi y^*))\le F\big((\sigma+2\beta)\psi(\Pi y,\Pi y^*)\big).$$

Since F strictly increasing,

$$\psi(\Pi y,\Pi y^*)\le(\sigma+2\beta)\psi(\Pi y,\Pi y^*).$$

Thus $\Pi y=\Pi y^*$. Hence, Π has a unique $\mathcal{BPP}$. □

Example 3.10. Let $\mathfrak{M}=\mathbb{R}^2$ and the metric

$$\psi((y_1,y_2),(\vartheta_1,\vartheta_2))=|y_1-\vartheta_1|+|y_2-\vartheta_2|.$$

with $\perp$ defined by $(y_1,y_2)\perp(\vartheta_1,\vartheta_2)$, if $y_1,\vartheta_1,y_2,\vartheta_2\ge 0$. Clearly, $(\mathfrak{M},\perp,\psi)$ is a $\mathcal{OCMS}$. Let $F:\mathbb{R}^+\to\mathbb{R}$ by $F(\sigma)=\ln\sigma$. Clearly, for any $\theta\in(0,1)$, F fulfills the axioms $(F1)$–$(F3)$. Let $\Omega=\{(0,\mathfrak{e}):\mathfrak{e}\in\mathbb{R}\}$ and $\Gamma=\{(2,\mathfrak{e}):\mathfrak{e}\in\mathbb{R}\}$. We have $\Omega_0=\Omega$ and $\Gamma_0=\Gamma$. Clearly, Ω is relatively compact in context with Γ. Let $\Pi:\Omega\to\Gamma$ by

$$\Pi(0,\mathfrak{e})=\left(2,\frac{\mathfrak{e}}{3}\right),$$

for each $(0,\mathfrak{e})\in\Omega$. It is clear that (Ω,Γ) satisfies Π is $\perp$-continuous, $\perp$-proximally preserving and $\Pi(\Omega_0)\subseteq\Gamma_0$. Let $y_1,\vartheta_1,y_2,\vartheta_2\in\Omega$ such that $\psi(y_1,\Pi\vartheta_1)=\psi(y_2,\Pi\vartheta_2)=\psi(\Omega,\Gamma)=2$. Take $y_1=(0,1)$, $\vartheta_1=(0,3)$, $y_2=(0,2)$ and $\vartheta_2=(0,6)$. Then (Ω,Γ) satisfies the $\perp$-$\mathcal{Q}$-property. Clearly,

$$\psi(\Pi y_1,\Pi y_2)\le e^{-\tau}(\sigma\psi(\Pi\vartheta_1,\Pi\vartheta_2)+\psi(\Pi\vartheta_2,\Pi y_1)).$$

Consequently, Π is an $\mathcal{GOPCOSK}$ with $e^{-\tau}=\frac{15}{7}$ or $\tau=\ln\frac{7}{15}$, $\sigma=1$ and $\mathfrak{t}=\ell=\beta=0$. From Theorem 3.9 of all axioms are verified. Hence, Π has a unique $\mathcal{BPP}$ $(0,0)$.

4. Application to fractional differential equations

Fractional differential equations could be the perfect way to model complex systems: powerful and versatile. Their description of memory effects, non-local behavior, and anomalous diffusion has made them irreplaceable in various fields. Moreover, FDEs can describe systems possessing non-local behavior. In such systems, the behavior at one point depends on values at other points in the domain. Such properties generally induce complex dynamics with nonlinear behaviors. FDEs can describe such systems more precisely compared to their conventional integer-order models, for example see Rezapour et al. (2024), Thabet et al. (2023), Boutiara et al. (2023) and Abdeljawad et al. (2023).

Consider the Caputo fractional derivative using fractional differential equation.

$$\mathbb{D}_{0+}^{\delta}\nu(z)+h(z,\nu(z))=0,\quad 0<z<1,\tag{4.1}$$

where, $1<\delta\le 2,\nu(0)+\nu'(0)=0,\nu(1)+\nu'(1)=0$ are the boundary conditions with $h:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous. Let $\mathfrak{M}=C([0,1],\mathbb{R})$. Define

$$\psi(\nu,\mu)=\sup_{z\in[0,1]}|\nu(z)-\mu(z)|,$$

$\forall\ \nu,\mu\in\mathfrak{M}$ with $\perp$ is defined as

$$\nu\perp\mu\iff\nu(z)\mu(z)\ge\nu(z)\text{ or }\nu(z)\mu(z)\ge\mu(z),$$

for all $z\in[0,1]$. Then $(\mathfrak{M},\perp,\psi)$ is a complete $\mathcal{OCMS}$. Let $\Omega=C([0,1],\mathbb{R}^+)$. Note that $\nu\in\Omega$ solves (5.1) whenever $\nu\in\Omega$ is the solution of

$$\begin{aligned}\nu(z)=&\frac{1}{\Gamma(\delta)}\int_0^1(1-t)^{\delta-1}(1-z)h(t,\nu(t))dt\\&+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-t)^{\delta-2}(1-z)h(t,\nu(t))dt\\&+\frac{1}{\Gamma(\delta)}\int_0^z(z-t)^{\delta-1}h(t,\nu(t))dt.\end{aligned}$$

Theorem 4.1. *Let the mapping $\Pi:\Omega\to\Omega$ as:*

$$\begin{aligned}\Pi\nu(z)=&\frac{1}{\Gamma(\delta)}\int_0^1(1-t)^{\delta-1}(1-z)h(t,\nu(t))dt\\&+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-t)^{\delta-2}(1-z)h(t,\nu(t))dt\\&+\frac{1}{\Gamma(\delta)}\int_0^z(z-t)^{\delta-1}h(t,\nu(t))dt,\end{aligned}$$

suppose the conditions:

(i) for all $\nu,\mu\in\Omega,h:[0,1]\times[0,\infty)\to[0,\infty)$, and $\tau>0$ satisfies

$$|h(t,\nu(t))-h(t,\mu(t))|\le e^{-\tau}|\nu(t)-\mu(t)|,$$

$$\sup_{z\in[0,1]}\left|\frac{1-z}{\Gamma(\delta+1)}+\frac{1-z}{\Gamma(\delta)}+\frac{z^\delta}{\Gamma(\delta+1)}\right|=\eta<1,$$

holds. Then, Eq. (5.1) has a unique solution.

(ii) **Proof.** Clearly, Π is $\perp$-preserving and $\perp$-continuous. Let $\nu,\mu\in\Omega$ and consider

$$\begin{aligned}&|\Pi\nu(z)-\Pi\mu(z)|\\&=\left|\frac{1}{\Gamma(\delta)}\int_0^1(1-t)^{\delta-1}(1-z)(h(t,\nu(t)-h(t,\mu(t))))dt\right.\\&+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-t)^{\delta-2}(1-z)(h(t,\nu(t)-h(t,\mu(t))))dt\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\delta)}\int_0^z (z-t)^{\delta-1}(h(t,\nu(t)-h(t,\mu(t))))dt\Big| \\
&\le \frac{1}{\Gamma(\delta)}\int_0^1 (1-t)^{\delta-1}(1-z)\Big|(h(t,\nu(t)-h(t,\mu(t))))\Big|dt \\
&+ \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-t)^{\delta-2}(1-z)\Big|(h(t,\nu(t)-h(t,\mu(t))))\Big|dt \\
&+ \frac{1}{\Gamma(\delta)}\int_0^z (z-t)^{\delta-1}\Big|(h(t,\nu(t)-h(t,\mu(t))))\Big|dt \\
&\le \frac{1}{\Gamma(\delta)}\int_0^1 (1-t)^{\delta-1}(1-z)e^{-\tau}|\nu(t)-\mu(t)|dt \\
&+ \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-t)^{\delta-2}(1-z)e^{-\tau}|\nu(t)-\mu(t)|dt \\
&+ \frac{1}{\Gamma(\delta)}\int_0^z (z-t)^{\delta-1}e^{-\tau}|\nu(t)-\mu(t)|dt \\
&= e^{-\tau}|\nu(z)-\mu(z)|\Big(\frac{1}{\Gamma(\delta)}\int_0^1 (1-t)^{\delta-1}(1-z)dt \\
&+ \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-t)^{\delta-2}(1-z)dt + \frac{1}{\Gamma(\delta)}\int_0^z (z-t)^{\delta-1}dt\Big) \\
&= e^{-\tau}|\nu(z)-\mu(z)|\Big(\frac{1-z}{\Gamma(\delta+1)} + \frac{1-z}{\Gamma(\delta)} + \frac{z^\delta}{\Gamma(\delta+1)}\Big) \\
&\le e^{-\tau}|\nu(z)-\mu(z)|\sup_{z\in[0,1]}\Big(\frac{1-z}{\Gamma(\delta+1)} + \frac{1-z}{\Gamma(\delta)} + \frac{z^\delta}{\Gamma(\delta+1)}\Big) \\
&= \eta e^{-\tau}|\nu(z)-\mu(z)| \\
&\le e^{-\tau}|\nu(z)-\mu(z)|,
\end{aligned}
$$

so, we have

$$\Big|\Pi\nu(z) - \Pi\mu(z)\Big| \le e^{-\tau}|\nu(z)-\mu(z)|,$$

i.e.,

$$\sup_{z\in[0,1]}\Big|\Pi\nu(z) - \Pi\mu(z)\Big| \le e^{-\tau}\sup_{z\in[0,1]}|\nu(z)-\mu(z)|,$$

thus, we have

$$\tau + F(\psi(\Pi\nu(z), \Pi\mu(z))) \le F(\psi(\nu(z), \mu(z))),$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\sigma) = \ln\sigma$. From Corollary 3.4, Eq. (5.1) has a unique solution. □

5. Application in production-consumption equilibrium

Throughout this part, we discuss the existence & uniqueness of solutions to integral equations by using Corollary 3.4.

Our results are applied to the dynamic market equilibrium problem, an important economics topic, where we solve an initial value problem and develop a mathematical model. Daily pricing trends and prices show an important effect on markets for both production ℓ_τ and consumption ℓ_c, despite price movements. Consequently, the economist is interested in knowing the current price $\nu(z)$. Let us consider

$$
\begin{aligned}
\ell_\tau &= t_1 + \varphi_1\nu(z) + \eta_1\frac{d\nu(z)}{dz} + \sigma_1\frac{d^2\nu(z)}{dz^2}, \\
\ell_c &= t_2 + \varphi_2\nu(z) + \eta_2\frac{d\nu(z)}{dz} + \sigma_2\frac{d^2\nu(z)}{dz^2},
\end{aligned}
$$

initially $\nu(0) = 0, \frac{d\nu}{dz}(0) = 0$, where $t_1, t_2, \varphi_1, \varphi_2, \eta_1, \eta_2, \sigma_1$ and σ_2 are constants. A state of dynamic economic equilibrium occurs when market forces are in balance, meaning that the current gap between production and consumption stabilizes, that is, $\ell_\tau = \ell_c$. Thus,

$$t_1 + \varphi_1\nu(z) + \eta_1\frac{d\nu(z)}{dz} + \sigma_1\frac{d^2\nu(z)}{dz^2} = t_2 + \varphi_1\nu(z) + \eta_2\frac{d\nu(z)}{dz} + \sigma_2\frac{d^2\nu(z)}{dz^2},$$

$$(t_1 - t_2) + (\varphi_1 - \varphi_2)\nu(z) + (\eta_1 - \eta_2)\frac{d\nu(z)}{dz} + (\sigma_1 - \sigma_2)\frac{d^2\nu(z)}{dz^2} = 0,$$

$$\sigma\frac{d^2\nu(z)}{dz^2} + \eta\frac{d\nu(z)}{dz} + \varphi\nu(z) = -t,$$

$$\frac{d^2\nu(z)}{dz^2} + \frac{\eta}{\sigma}\frac{d\nu(z)}{dz} + \frac{\varphi}{\sigma}\nu(z) = -\frac{t}{\sigma},$$

where $t = t_1 - t_2, \varphi = \varphi_1 - \varphi_2, \eta = \eta_1 - \eta_2$, and $\sigma = \sigma_1 - \sigma_2$.

Our initial value problem is now represented as follows:

$$\nu''(z) + \frac{\eta}{\sigma}\nu'(z) + \frac{\varphi}{\sigma}\nu(z) = -\frac{t}{\sigma}, \text{ with } \nu(0) = 0 \text{ and } \nu'(0) = 0. \tag{5.1}$$

Now, we study the production and consumption duration time $\mathfrak{w}$, problem (5.1) is equivalent to

$$\nu(z) = \int_0^{\mathfrak{w}} \mathcal{G}(z, z^*)\mathcal{K}(z^*, z, \nu(z))dz, \tag{5.2}$$

where Green function $\mathcal{G}(z, z^*)$ is

$$\mathcal{G}(z, z^*) = \begin{cases} z\hbar^{\frac{\varphi}{2\eta}}(z^* - z), & 0 \le z \le \mathfrak{s} \le \mathfrak{w}, \\ \mathfrak{s}\hbar^{\frac{\varphi}{2\eta}}(z - z^*), & 0 \le \mathfrak{s} \le z \le \mathfrak{w}, \end{cases}$$

and $\mathcal{K} : [0, \mathfrak{w}] \times \mathcal{U}^2 \to \mathbb{R}$ is a continuous function. Let $\mathfrak{M} = C([0, \mathfrak{w}], \mathbb{R})$. Define

$$\psi(\nu, \mu) = \sup_{z\in[0,1]}|\nu(z) - \mu(z)|.$$

for all $\nu, \mu \in \mathfrak{M}$ with $\perp$ is defined as

$$\nu\perp\mu \iff \nu(z)\mu(z) \ge \nu(z) \text{ or } \nu(z)\mu(z) \ge \mu(z),$$

for all $z \in [0, \mathfrak{w}]$. Then $(\mathfrak{M}, \perp, \psi)$ is a complete $\mathcal{OCMS}$. Let $\Omega = C([0, \mathfrak{w}], \mathbb{R}^+)$.

Define $\Pi : \Omega \to \Omega$ is given by

$$\Pi(\nu(z)) = \int_0^{\mathfrak{w}} \mathcal{G}(z, z^*)\mathcal{K}(z^*, z, \nu(z))dz. \tag{5.3}$$

Let us consider, the solution to the dynamic market equilibrium problem, which is represented as (5.1), is a fixed point of Π(5.3). Now, the current price $\nu(z)$ is given by (5.1).

Theorem 5.1. *Consider the operator $\Pi : \Omega \to \Omega$ (5.3) in a complete $\mathcal{OCMS}$ $(\mathfrak{M}, \perp, \psi)$, satisfying*

(i) we can find $z \in [0, \mathfrak{w}]$, $\tau > 0$ and $a, a' \in \Omega$ such that

$$|\mathcal{K}(z^*, z, \nu_1(z)) - \mathcal{K}(z^*, z, \nu_2(z))| \le \frac{e^{-\tau}}{\mathfrak{w}}|\nu_1(z) - \nu_2(z)|;$$

(ii) a continuous function $\mathcal{G} : \mathcal{U}^2 \to \mathbb{R}$ that satisfies

$$\sup_{\mathfrak{s}\in[0,\mathfrak{w}]}\int_0^{\mathfrak{w}} \mathcal{G}(z, z^*)dz \le 1.$$

Then, the dynamic market equilibrium problem (5.1) has exactly one solution.

Proof. Clearly, Π is $\perp$-preserving and $\perp$-continuous. Now

$$
\begin{aligned}
&|\Pi(\nu_1(z)) - \Pi(\nu_2(z))| \\
&= \Big|\int_0^{\mathfrak{w}} \mathcal{G}(z, z^*)\mathcal{K}(z^*, z, \nu_1(z))dz - \int_0^{\mathfrak{w}} \mathcal{G}(z, z^*)\mathcal{K}(z^*, z, \nu_2(z))dz\Big| \\
&\le \int_0^{\mathfrak{w}} \mathcal{G}(z, z^*)dz\int_0^{\mathfrak{w}}\Big|\mathcal{K}(z^*, z, \nu_1(z)) - \mathcal{K}(z^*, z, \nu_2(z))\Big|dz \\
&\le e^{-\tau}|\nu_1(z) - \nu_2(z)|.
\end{aligned}
$$

So, we have

$$\Big|\Pi\nu(z) - \Pi\mu(z)\Big| \le e^{-\tau}|\nu(z)-\mu(z)|,$$

i.e.,

$$\sup_{z\in[0,1]}\Big|\Pi\nu(z) - \Pi\mu(z)\Big| \le e^{-\tau}\sup_{z\in[0,1]}|\nu(z)-\mu(z)|,$$

thus, we have

$$\tau + F(\psi(\Pi\nu(z), \Pi\mu(z))) \le F(\psi(\nu(z), \mu(z))),$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\sigma) = \ln\sigma$. From Corollary 3.4, Eq. (5.2) has a unique solution. □

6. Conclusions

In this article, we presented the notion of orthogonal F -proximal contractions (of the first and second kind), generalized orthogonal F -proximal contractions (of the first and second kind), then established $\mathcal{BPP}$ results on $\mathcal{OCMS}$. Moreover, we presented best examples of our outcome results. Moreover, an application to the fractional boundary value problem in the Caputo sense and Production-Consumption Equilibrium problem was carried out to highlight the utility of our results.

Khalehoghli et al. (2020) proved fixed point theorems in R-$\mathcal{MS}$s. In the future, it is an open problem to prove the best proximity theorems on R-$\mathcal{MS}$.

CRediT authorship contribution statement

Gunaseelan Mani: Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Raman Thandavarayan Tirukalathi:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Sabri T.M. Thabet:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis. **Miguel Vivas-Cortez:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Funding acquisition.

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Declaration of competing interest

The authors declare that they have no competing interests.

Data availability

No data were used to support the findings of this study.

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