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Original article

Estimation of step-stress life testing model using time-censoring: A Bayesian approach

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A B S T R A C T

Accelerated life tests (ALTs) of highly reliable products or materials are effective testing techniques to gather failure data rapidly in a limited time period. Also, partially accelerated life tests (PALTs) can enable us to achieve this goal without putting all test units under severe conditions. This article considers both frequent and Bayesian estimations of the step-stress PALTs model using time-censored data from generalized exponential distribution (GED). The maximum likelihood and Bayesian estimates of the model parameters are obtained. The posterior means and posterior variances are computed under the squared error (SE) loss function using Lindley's procedure. The performance of the estimators is evaluated numerically for different parameter values and different sample sizes via their mean squared error (MSE). In addition, the average confidence intervals lengths (ACIL) of the model parameters are also obtained. For illustrative purposes, a simulation study is given.

1. Introduction

"Accelerated life testing of materials or products is used to collect failure data quickly when the lifetime of a specimen under use conditions is too long. Such method is called fully accelerated life tests (FALTs) or simply accelerated life tests (ALTs) where all the test units are put to run under severe conditions. Accelerated test stresses involve higher than usual temperature, voltage, pressure, load, humidity, etc., or some combination of them. Sometimes, only some of test units (not all) are put under severe conditions. In such cases, the testing method is called partially accelerated life tests (PALTs). In PALTs the test items are run under both normal (design/use)- and high-stresses. In this respect readers can refer, for example, to (Ismail, 2020; 2022).

As indicated by (Nelson, 1990), there are two common ways to apply stresses. They are step-stress and constant-stress. The step-stress technique will be adopted in this article. Under step-stress PALTs (SSPALTs), a test unit is first run at use condition and, if it does not fail for a specified time, then it is run at accelerated condition until failure occurs or the test is terminated. Accordingly, one of many advantages of PALTs is to collect more failure data in a shorted time without necessarily using high stresses to all test units".

"In this article, the main aim is to find Bayesian estimates (BEs) of SSPALTs model based on time-censored data from generalized exponential distribution (GED) and to compare it with the maximum likelihood estimates (MLEs) using Monte Carlo simulation studies".

"The GED has been introduced by (Gupta and Kundu, 2003b). Even though it is generally believed that the Weibull distribution is the obvious generalization of the exponential distribution, (Gupta and Kundu, 2003a) observed that in many situations the GED provides a better fit than a Weibull distribution. Further

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Nomenclature

Notation

ALTs	"accelerated life tests"
PALTs	"partially accelerated life tests"
SSPALTs	"step-stress partially accelerated life tests"
GED	"generalized exponential distribution"
MLEs	"maximum likelihood estimates/estimators"
MSE	"mean square error"
ACIL	"average confidence interval length"
NIP	"non-informative prior"
$1 - \gamma$	"confidence level"
n	"total number of test units in a PALT"
n_u, n_a	"number of units failed at normal (use) and accelerated conditions, respectively"
r	$n_u + n_a$
n_c	"number of censored units (n-r)"
$\wedge$	"denotes maximum likelihood estimate"
β	"acceleration factor ($\beta > 1$)"
α	"GE shape parameter ($\alpha > 0$)"
λ	"GE scale parameter ($\lambda > 0$)"
T	"lifetime of a unit at normal (use) condition"
Y	"total lifetime of a unit in SSPALTs"
y	"observed value of the total lifetime Y_i of unit i, $i = 1, \ldots, n$"
τ	"stress change-point in SSPALTs"
η	"the time at which the test is terminated"
"$y_{(1)} \leq \ldots \leq y_{(n_u)} \leq \tau \leq y_{(n_u+1)} \leq \ldots \leq \eta$	ordered failure times"

properties of the GED have been studied by (Gupta & Kundu, 1999; 2001a; 2001b; 2002). The simple mathematical structure of the GED enables it to be used effectively for modeling various lifetime data types with possible censoring or grouping, (Baklizi, 2007)".

"The two-parameter GED family has the distribution function"

$$F(y;\alpha,\lambda) = \left(1 - e^{-\lambda y}\right)^{\alpha}, y > 0 \tag{1.1}$$

"The corresponding density function is"

$$f(y;\alpha,\lambda) = \alpha\lambda e^{-\lambda y}\left(1 - e^{-\lambda y}\right)^{\alpha-1},\ y > 0,\ \alpha > 0,\ \lambda > 0, \tag{1.2}$$

"where α and λ are the shape and scale parameters, respectively". "When $\alpha = 1$ it coincides with the exponential distribution with mean $1/\lambda$. When $\alpha \leq 1$ the density function is strictly decreasing and for $\alpha > 1$ it has a unimodal shape. These densities are illustrated in (Gupta and Kundu; 1999). It is witnessed that the GED density functions are always right skewed. So, the GED can be used quite effectively to analyze skewed data sets".

"The hazard rate function of the GED is"

$$h(y;\alpha,\lambda) = \frac{\alpha\lambda e^{-\lambda y}\left(1 - e^{-\lambda y}\right)^{\alpha-1}}{1 - \left(1 - e^{-\lambda y}\right)^{\alpha}} \tag{1.3}$$

"and the mean time to failure (MTTF) can take the following form"

$$MTTF = \frac{1}{\lambda}\{\psi(\alpha+1) - \psi(1)\} \tag{1.4}$$

"where $\psi(.)$ is the digamma function".

"The GED can have increasing and decreasing hazard rates depending on the shape parameter α. The hazard rate increases from 0 to λ if $\alpha > 1$ and if $\alpha < 1$ it decreases from ∞ to λ. This property leads to good ability of using this distribution in reliability and life testing, (Abuammoh and Sarhan, 2007)".

"In this article, assuming GED, the goal of Bayesian statistical inference on SSPALTs model is to estimate the failure behavior of the specimens under use condition using the failure data obtained under severe condition. To achieve this goal, proper statistical models are required. The pioneering work for SSPALTs modeling is by (DeGroot and Goel, 1979). They proposed a tampered random variable model. This model will be described in Section 3″.

"For an overview of the literature on SSPALTs, readers can, for example, refer to (Goel, 1971; Bai and Chung, 1992; Ismail, 2014; 2016a; 2016b; Ismail and Al-Habardi, 2017; among others). For those who interested in constant-stress PALTs, they can, for example, refer to (Ismail 2013; 2014b; 2015; 2017; Ismail and Al Tamimi, 2017; 2019; Ismail and Al-Harbi, 2019)".

"The rest of this article can be structured as follows: In Section 2 the test procedure and its necessary assumptions are presented. MLEs of the model parameters are considered in Section 3. Section 4 presents BEs of the model parameters using Lindley's technique. In Section 5 simulation studies are provided to demonstrate the theoretical results given in this article. Finally, Section 6 concludes the article".

2. Test procedure and its assumptions

"The test procedure of SSPALTs and its assumptions are described as follows"

2.1. Test procedure

"Each of the n test units is first run at use condition".

"If it does not fail at use condition by a pre-specified time τ, then it is put on accelerated condition and run until either it fails or the test is terminated".

2.2. Assumptions

1. "The lifetimes of the n test units are independent and identically distributed random variables (i.i.d. r.v.'s)".
2. "The lifetimes of test units are assumed to follow the GED with pdf given by equation (1.2)".

3. Estimation process

"According to (Balakrishnan and Zhu, 2014), the MLEs of the model parameters are unique when they exist. The method of proof is based on the monotonicity property of the likelihood function".

"The MLEs of the GED parameters uniquely exist. Due to the invariance property of MLEs, the existence and uniqueness of the MLEs of the model parameters follow. Both point and interval estimates using the Maximum Likelihood method will be considered in the next two subsections".

3.1. The maximum likelihood estimates

"The lifetime of a test unit under SSPALTs can be written as"

$$Y = \begin{cases} T if T \leq \tau \\ \tau + (T-\tau)/\beta if T > \tau \end{cases}$$

"where T is the lifetime of the unit under use condition, τ is the stress change time and β is the acceleration factor; β > 1. This model is called tampered random variable (TRV) model. It was proposed by (DeGroot & Goel, 1979). Therefore, the pdf of Y under SSPALTs can be given by"

$$Y = \begin{cases} 0, y \le 0, \\ f_1(y) = \alpha\lambda e^{-\lambda y}(1 - e^{-\lambda y})^{\alpha-1}, 0 < y \le \tau \\ f_2(y) = \beta\alpha\lambda e^{-\lambda[\tau+\beta(y-\tau)]}\left(1 - e^{-\lambda[\tau+\beta(y-\tau)]}\right)^{\alpha-1}, y > \tau > 0 \end{cases} \tag{3.2}$$

"where f1(y) is given by equation (1.2) and f2(y) is obtained by the transformation-variable technique using f1(y) and the model presented by equation (3.1)".

"The observed values of the total lifetime Y are given by"

y(1) ≤ ... ≤ y(nu) ≤ τ ≤ y(nu + 1) ≤ ... ≤ η.

"Since the total lifetimes Y1, ..., Yn of n test units are i.i.d. r.v.'s, then the total likelihood function for them can be written as"

$$L(\beta,\ \alpha,\ \lambda) \propto \prod_{i=1}^{n_u} \alpha\lambda e^{-\lambda y_i}\left(1 - e^{-\lambda y_i}\right)^{\alpha-1}$$
$$\times \prod_{i=1}^{n_u+n_a} \beta\alpha\lambda e^{-\lambda[\tau+\beta(y_i-\tau)]}\left(1 - e^{-\lambda[\tau+\beta(y_i-\tau)]}\right)^{\alpha-1}$$
$$\times \prod_{i=n_u+n_a+1}^{n_u+n_a+n_c} \left\{1 - \left(1 - e^{-\lambda[\tau+\beta(\eta-\tau)]}\right)^{\alpha}\right\} \tag{3.3}$$

"The natural logarithm of the above likelihood function is given by"

$$\begin{aligned} \ln L = {} & (n_u + n_a)[ln\alpha + ln\lambda] + n_a \ln\beta \\ & - \lambda\left(\sum_{i=1}^{n_u} y_i + \sum_{i=n_u+1}^{n_u+n_a} [\tau + \beta(y_i - \tau)]\right) + (\alpha - 1) \\ & \times \left[\sum_{i=1}^{n_u} \ln\left(1 - e^{-\lambda y_i}\right) + \sum_{i=n_u+1}^{n_u+n_a} \ln\left(1 - e^{-\lambda[\tau+\beta(y_{i\_}\tau)]}\right)\right] \\ & + n_c \ln\left(1 - \left[1 - e^{-\lambda[\tau+\beta(\eta\_\tau)]}\right]\alpha\right) \end{aligned} \tag{3.4}$$

"By taking the partial derivatives of the natural logarithm of likelihood function with respect to β, α and λ, respectively, we get"

$$\begin{aligned} \frac{\partial lnL}{\partial\beta} = {} & \frac{n_a}{\beta} - \lambda\sum_{i=n_u+1}^{n_u+n_a}(y_i - \tau) + \lambda(\alpha - 1)\sum_{i=n_u+1}^{n_u+n_a}\frac{(y_i - \tau)e^{-\lambda\psi_i}}{1 - e^{-\lambda\psi_i}} \\ & - \frac{n_c\alpha(\eta - \tau)\lambda e^{-\lambda\psi_r}(1 - e^{-\lambda\psi_r})^{\alpha-1}}{1 - (1 - e^{-\lambda\psi_r})^{\alpha}} \end{aligned} \tag{3.5}$$

Where

$\psi_i = \tau + \beta(y_{i\_}\tau)$ and $\psi_r = \tau + \beta(\eta\_\tau)$,

$$\begin{aligned} \frac{\partial lnL}{\partial\alpha} = {} & \frac{n_u + n_a}{\alpha} + \sum_{i=1}^{n_u} ln\left(1 - e^{-\lambda y_i}\right) + \sum_{i=n_u+1}^{n_u+n_a} ln\left(1 - e^{-\lambda\psi_i}\right) \\ & - \frac{n_c(1 - e^{-\lambda\psi_r})^{\alpha} ln(1 - e^{-\lambda\psi_r})}{1 - (1 - e^{-\lambda\psi_r})^{\alpha}} \end{aligned} \tag{3.6}$$

$$\begin{aligned} \frac{\partial lnL}{\partial\lambda} = {} & \frac{n_u + n_a}{\lambda} + \sum_{i=1}^{n_u} y_i - \sum_{i=n_u+1}^{n_u+n_a}\psi_i \\ & + (\alpha - 1)\left[\sum_{i=1}^{n_u}\frac{y_i e^{-\lambda y_i}}{1 - e^{-\lambda y_i}} + \sum_{i=n_u+1}^{n_u+n_a}\frac{\psi_i e^{-\lambda\psi_i}}{1 - e^{-\lambda\psi_i}}\right] \\ & - \frac{n_c\alpha\psi_r e^{-\lambda\psi_r}(1 - e^{-\lambda\psi_r})^{\alpha-1}}{1 - (1 - e^{-\lambda\psi_r})^{\alpha}} \end{aligned} \tag{3.7}$$

"Now, we have a system of three nonlinear equations in three unknowns β, α and λ. It is clear that a closed form solution is very difficult to obtain. Therefore, iterative procedure such as Newton-Raphson can be used to find a numerical solution of the above nonlinear system".

3.2. Asymptotic Confidence Intervals

"In this subsection, we obtain the confidence intervals of the model parameters based on the asymptotic distribution of the MLEs of the parameters".

"It is known that the asymptotic distribution of the MLEs of the elements of the vector of unknown parameters θ = (β, α, λ) is given by (Miller, 1981) as"

$$\left(\left(\widehat{\beta} - \beta\right), \left(\widehat{\alpha} - \alpha\right), \left(\widehat{\lambda} - \lambda\right)\right) N\left(0, \boldsymbol{I}^{-1}(\beta, \alpha, \lambda)\right)$$

"where $\boldsymbol{I}^{-1}(\beta, \alpha, \lambda))$ is the variance–covariance matrix of the unknown parameters θ = (β, α, λ). The elements of the 3 × 3 matrix $\boldsymbol{I}^{-1}$, I_{ij} (β, α, λ), i, j = 1, 2, 3; can be approximated by $I_{ij}\left(\widehat{\beta}, \widehat{\alpha}, \widehat{\lambda}\right)$), where"

$$I_{ij}\left(\widehat{\theta}\right) = -\frac{\partial^2 lnL(\theta)}{\partial\theta_i\partial\theta_j} \downarrow \theta = \widehat{\theta}$$

"From equation (3.4), we get the following"

$$\frac{\partial^2 lnL}{\partial\beta^2} = -\frac{n_a}{\beta^2} - \lambda^2(\alpha - 1)\sum_{i=n_u+1}^{n_u+n_a}\frac{(y_i - \tau)^2\left[e^{-\lambda\psi_i}(1 - e^{-\lambda\psi_i}) + e^{-2\lambda\psi_i}\right]}{(1 - e^{-\lambda\psi_i})^2} - \frac{\varphi_1\varphi_2 + \varphi_3\varphi_4}{\varphi_2^2}, \tag{3.8}$$

Where

$$\varphi_1 = n_c\alpha\lambda^2(\eta - \tau)^2 e^{-\lambda\psi_r}\left(1 - e^{-\lambda\psi_r}\right)^{\alpha-1}\left[(\alpha - 1)e^{-\lambda\psi_r}\left(1 - e^{-\lambda\psi_r}\right)^{-1} - 1\right]$$

$$\varphi_2 = 1 - \left(1 - e^{-\lambda\psi_r}\right)^{\alpha}$$

$$\varphi_3 = n_c\alpha(\eta - \tau)\lambda e^{-\lambda\psi_r}\left(1 - e^{-\lambda\psi_r}\right)^{\alpha-1}$$

$$\varphi_4 = -\alpha\lambda(\eta - \tau)e^{-\lambda\psi_r}\left(1 - e^{-\lambda\psi_r}\right)^{\alpha-1}$$

$$\frac{\partial^2 lnL}{\partial\alpha^2} = -\frac{n_u + n_a}{\alpha^2} - \frac{\left(\frac{n_c}{\alpha}\right)(1 - \varphi_2)[ln(1 - \varphi_2)]\left\{\left(\frac{\varphi_2}{\alpha}\right)[ln(1 - \varphi_2)] + \left(\frac{1-\varphi_2}{\alpha}\right)ln(1 - \varphi_2)\right\}}{\varphi_2^2} \tag{3.9}$$

$$\begin{aligned} \frac{\partial^2 lnL}{\partial\lambda^2} = {} & -\frac{n_u + n_a}{\lambda^2} - (\alpha - 1)\left[\sum_{i=1}^{n_u}\frac{y_i^2 e^{-\lambda y_i}}{(1 - e^{-\lambda y_i})^2} + \sum_{i=n_u+1}^{n_u+n_a}\frac{\psi_i^2 e^{-\lambda\psi_i}}{(1 - e^{-\lambda\psi_i})^2}\right] \\ & + n_c\alpha\psi_r^2 e^{-\lambda\psi_r}\left(1 - e^{-\lambda\psi_r}\right)^{\alpha-1} \\ & \times \left[\frac{\left(1 - (\alpha - 1)e^{-\lambda\psi_r}(1 - e^{-\lambda\psi_r})^{-1}\right)\varphi_2 + \alpha e^{-\lambda\psi_r}(1 - e^{-\lambda\psi_r})^{\alpha-1}}{\varphi_2^2}\right] \end{aligned} \tag{3.10}$$

$$\begin{aligned} \frac{\partial^2 lnL}{\partial\beta\partial\alpha} = {} & \lambda\sum_{i=n_u+1}^{n_u+n_a}\frac{(y_i - \tau)e^{-\lambda\psi_i}}{1 - e^{-\lambda\psi_i}} - \frac{n_c\lambda(\eta - \tau)e^{-\lambda\psi_r}(1 - e^{-\lambda\psi_r})^{\alpha-1}}{\varphi_2^2} \\ & \times \left\{\left[1 + \alpha ln\left(1 - e^{-\lambda\psi_r}\right)\right]\varphi_2 + \alpha\left(1 - e^{-\lambda\psi_r}\right)^{\alpha} ln\left(1 - e^{-\lambda\psi_r}\right)\right\}, \end{aligned} \tag{3.11}$$

$$\frac{\partial^2 lnL}{\partial\beta\partial\lambda} = -\sum_{i=n_u+1}^{n_u+n_a}(y_i-\tau) + (\alpha-1)\sum_{i=n_u+1}^{n_u+n_a} \times \frac{(y_i-\tau)e^{-\lambda\psi_i}[(1-\lambda\psi_i)(1-e^{-\lambda\psi_i})-\lambda\psi_i e^{-\lambda\psi_i}]}{(1-e^{-\lambda\psi_i})^2} - \frac{n_c\alpha(\eta-\tau)e^{-\lambda\psi_r}(1-e^{-\lambda\psi_r})^{\alpha-1}\left[(1-\lambda\psi_r)+(\alpha-1)\lambda\psi_r e^{-\lambda\psi_r}(1-e^{-\lambda\psi_r})^{-1}\right]\varphi_2}{\varphi_2^2} - \frac{n_c\alpha^2(\eta-\tau)\lambda\psi_r e^{-2\lambda\psi_r}(1-e^{-\lambda\psi_r})^{2(\alpha-1)}}{\varphi_2^2}, \tag{3.12}$$

and

$$\frac{\partial^2 lnL}{\partial\alpha\partial\lambda} = \sum_{i=1}^{n_u}\frac{y_i e^{-\lambda y_i}}{(1-e^{-\lambda y_i})} + \sum_{i=n_u+1}^{n_u+n_a}\frac{\psi_i e^{-\lambda\psi_i}}{(1-e^{-\lambda\psi_i})} - \frac{n_c\psi_r e^{-\lambda\psi_r}(1-e^{-\lambda\psi_r})^{\alpha-1}\{[1+\alpha ln(1-e^{-\lambda\psi_r})]\varphi_2+\alpha(1-e^{-\lambda\psi_r})^{\alpha}ln(1-e^{-\lambda\psi_r})\}}{\varphi_2^2} \tag{3.13}$$

"Thus, the approximate 100(1 - γ) % two sided confidence intervals for β, α, and λ are, respectively, given by"

$$\widehat{\beta} \pm Z_{\frac{\gamma}{2}}\sqrt{\mathrm{I}_{11}^{-1}\left(\widehat{\beta}\right)}, \widehat{\alpha} \pm Z_{\frac{\gamma}{2}}\sqrt{\mathrm{I}_{22}^{-1}(\widehat{\alpha})} \text{ and } \widehat{\lambda} \pm Z_{\frac{\gamma}{2}}\sqrt{\mathrm{I}_{33}^{-1}\left(\widehat{\lambda}\right)}$$

"where $Z_{\gamma/2}$ is the upper (γ/2)th percentile of a standard normal distribution".

4. Bayesian estimates

"In this section, the squared error loss function is considered. Then, the Bayes estimator of a parameter is its posterior expectation. The Bayes estimators can't be expressed in explicit forms. Approximate Bayes estimates will be obtained under the assumption of non-informative priors (NIP) using Lindley's technique".

"In many practical situations, the information about the parameters are available in an independent manner, see (Basu et al., 1999). Thus, it is assumed that the parameters are independent a priori and let the NIP for each parameter be represented by the limiting form of the appropriate natural conjugate prior".

"It follows that a NIP for the acceleration factor β is given by"

$$\pi_1(\beta) \propto \beta^{-1}, \beta > 1.$$

"Also, the NIP's for the scale parameter λ and the shape parameter α are respectively as"

$\pi_2(\lambda) \propto \lambda^{-1}, \lambda > 0$ and $\pi_3(\alpha) \propto \alpha^{-1}, \alpha > 0$.

"Therefore, the joint NIP of the three parameters can be expressed by"

$$\pi(\beta,\lambda,\alpha) \propto (\beta\lambda\alpha)^{-1}, \beta > 1, \lambda > 0, \alpha > 0. \tag{4.1}$$

"Forming the product of (4.1) and (3.3), the joint posterior density function of β, α and λ given the data can be written as"

$$\pi^*(\beta,\lambda,\alpha|data) \propto L(\beta,\lambda,\alpha).\pi(\beta,\lambda,\alpha) \propto \beta^{n_a-1}(\lambda\alpha)^{n_u+n_a-1}\prod_{i=1}^{n_u}e^{-\lambda y_i}\left(1-e^{-\lambda y_i}\right)^{\alpha-1} \times \prod_{i=n_u+1}^{n_u+n_a}e^{-\lambda[\tau+\beta(y_i-\tau)]}\left(1-e^{-\lambda[\tau+\beta(y_i-\tau)]}\right)^{\alpha-1} \times \prod_{i=n_u+n_a+1}^{n_u+n_a+n_c}\left[1-\left(1-e^{-\lambda[\tau+\beta(\eta-\tau)]}\right)^{\alpha}\right] \tag{4.2}$$

"As stated earlier, under a squared error loss function, the Bayes estimator of a parameter is its posterior expectation. To obtain the posterior means and posterior variances of β, α and λ non-tractable integrals will be confronted. It is not possible to compute them analytically. The marginal posteriors are somewhat unwieldy and require a numerical integration that may not converge. Instead, an approximation due to (Lindley, 1980) via an asymptotic expansion of the ratio of two non-tractable integrals is used to obtain the approximate Bayes estimates. Lindley's approximation is evaluated at the MLEs of the model parameters".

"Now, let Θ be a set of parameters $\{\Theta_1, \Theta_2, \cdots, \Theta_m\}$, where m is the number of parameters, then the posterior expectation of an arbitrary function $u(\Theta)$ can be asymptotically estimated by"

$$E[u(\Theta)] = \frac{\int u(\Theta)\pi(\Theta)e^{lnL(y|\Theta)}d\Theta}{\int \pi(\Theta)e^{lnL(y|\Theta)}d\Theta} \approx \left\{u + \left(\frac{1}{2}\right)\left[\sum_{i,j}\left(u_{ij}^{(2)} + 2u_i^{(1)}\rho_j^{(1)}\right)\sigma_{ij} + \sum_{i,j,k,s}L_{ijk}^{(3)}\sigma_{ij}\sigma_{ks}u_s^{(1)}\right]\right\} \downarrow \widehat{\Theta} \tag{4.3}$$

"Which is the Bayes estimator of $u(\Theta)$ under a squared error loss function, where $\pi(\Theta)$ is the prior distribution of Θ, $u \equiv u(\Theta)$, $L \equiv L(\Theta)$ is the likelihood function, $\rho \equiv \rho(\Theta) = log\pi(\Theta)$, σ_{ij} are the elements of the inverse of the asymptotic Fisher's information matrix of Θ and"

$u_i^{(1)} = \frac{\partial u}{\partial\Theta_i}$, $u_{ij}^{(2)} = \frac{\partial^2 u}{\partial\Theta_i\partial\Theta_j}$, $\rho_j^{(1)} = \frac{\partial \log\pi(\Theta)}{\partial\Theta_j}$ and.$L_{ijk}^{(3)} = \frac{\partial^3 lnL(y|\Theta)}{\partial\Theta_i\partial\Theta_j\partial\Theta_k}$

"Such an approximation is easy to use and does not require innovative programming and extensive computer time. According to (Green, 1980), the linear Bayes estimator in (4.3) is a "very good and operational approximation for the ratio of multi-dimension integrals". As indicated by (Sinha, 1986), it has led to many useful applications. The Derivations of posterior means and posterior variances are presented in the Appendix".

5. Simulation studies

"This section presents a simulation study to demonstrate the theoretical results introduced in this article and to evaluate the performance of both MLEs and BEs of the model parameters via their mean squared errors (MSE) and the associated average confidence intervals lengths (ACIL). The posterior means and variances of the model parameters β, α and λ are obtained assuming NIP for each parameter under a squared error loss function using type-I censored data from GED. Since the BEs of the model param-

Table 1
"Average values of the MLEs and BEs with associated MSE and ACIL using β = 3.5, α = 0.8, λ = 2 when τ = 5 and η = 9".

"n	parameter	Method	estimate	MSE	ACIL
25	β	ML	3.8498	0.3248	3.9363
		Bayes	3.5732	0.2736	3.2034
	α	ML	1.0907	0.1483	3.1721
		Bayes	1.0540	0.1046	1.9973
	λ	ML	2.2090	0.2391	3.4038
		Bayes	2.1713	0.2097	3.1092
50	β	ML	3.6641	0.2433	3.3034
		Bayes	3.5363	0.2008	2.7345
	α	ML	0.9201	0.0768	2.6651
		Bayes	0.9042	0.0361	1.4959
	λ	ML	2.1024	0.1509	2.8593
		Bayes	2.0860	0.1216	2.4869
75	β	ML	3.6055	0.1772	2.4559
		Bayes	3.5209	0.1523	1.9041
	α	ML	0.8764	0.0569	1.9799
		Bayes	0.8532	0.0311	1.2136
	λ	ML	2.0611	0.0609	2.1251
		Bayes	2.0501	0.0482	1.6458
100	β	ML	3.5748	0.1451	1.9521
		Bayes	3.5130	0.1257	1.5989
	α	ML	0.8217	0.0243	1.5679
		Bayes	0.8079	0.0154	0.9233
	λ	ML	2.0455	0.0438	1.6843
		Bayes	2.0374	0.0277	0.4431"

Table 2
"Average values of the MLEs and BEs with associated MSE and ACIL using β = 3.5, α = 1.5, λ = 2 when τ = 5 and η = 9".

"n	parameter	Method	estimate	MSE	ACIL
25	β	ML	3.8572	0.2391	4.1164
		Bayes	3.5504	0.2014	3.3521
	α	ML	1.6341	0.1092	3.3113
		Bayes	1.5829	0.0773	2.0769
	λ	ML	2.2192	0.1759	3.5532
		Bayes	2.1798	0.1544	3.2562
50	β	ML	3.6621	0.1792	3.4505
		Bayes	3.5320	0.1477	2.8586
	α	ML	1.5600	0.0565	2.7847
		Bayes	1.5355	0.0266	1.5628
	λ	ML	2.0953	0.1143	2.9872
		Bayes	2.0788	0.0894	2.5974
75	β	ML	3.6007	0.1305	2.5667
		Bayes	3.5156	0.1119	1.9901
	α	ML	1.5412	0.0418	2.0685
		Bayes	1.5253	0.0229	1.2677
	λ	ML	2.0638	0.0448	2.2208
		Bayes	2.0524	0.0353	1.7196
100	β	ML	3.5791	0.1035	2.0379
		Bayes	3.5171	0.0926	1.6712
	α	ML	1.5309	0.0147	1.6374
		Bayes	1.5188	0.0113	0.9649
	λ	ML	2.0480	0.0322	1.7631
		Bayes	2.0396	0.0204	1.5087"

eters can't be obtained analytically, approximate BEs are obtained numerically using the technique of Lindley. Different sample sizes and Different parameter values are used to compare the performance of both MLEs and BEs of the model parameters".

"As shown from the simulation results reported in Tables 1 and 2, the BEs perform better than the MLEs. That is, the BEs have smaller MSE than that of MLEs. In addition, the confidence intervals of the model parameters obtained using Lindley's technique at confidence level 95 % are narrower than those by MLEs. These results coincide with the note of (Achcar, 1994). That is, the use of approximate Bayesian methods could be a good alternative for the usual asymptotically classical methods in accelerated life testing. However, as expected, the performance of both BEs and MLEs become better when the sample size increases".

6. Conclusion

"In this article both MLEs and BEs of the parameters of GED and the acceleration factor have been considered under SSPALTs model using time-censored data. The MLEs have been obtained numerically using Newton-Raphson method as an iterative technique. Under the assumptions of squared error loss functions and non-informative priors BEs have been calculated. The performance of the estimators has been evaluated in terms of MSE and ACIL numerically using different parameter values and different sample sizes".

"BEs using Lindley's technique produces smaller MSE than that of the MLEs. In addition, the confidence ACIL of the model parameters obtained using Lindley's technique with confidence level 95 % are narrower than those by MLEs. That is, the BEs performs better than the MLEs. It can be concluded that the intrinsic appeal of that method can be expressed in is its being a sort of adjustment to the maximum likelihood approach to reduce variability".

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

"Here, there are three parameters in the model. That is, m = 3. Let the subscripts 1, 2 and 3 refer to β, α and λ, respectively". "Therefore, the posterior means (BEs) of the three parameters can be expressed by"

$$"\beta^* = E[\beta|y] = \left[\beta - \left(\frac{\sigma_{11}}{\beta} + \frac{\sigma_{12}}{\lambda} + \frac{\sigma_{13}}{\alpha}\right) + \left(\frac{1}{2}\right)(\sigma_{11}\psi_1 + \sigma_{12}\psi_2 + \sigma_{13}\psi_3)\right] \downarrow \widehat{\Theta}" \tag{4.4}$$

$$"\lambda^* = E[\lambda|y] = \left[\lambda - \left(\frac{\sigma_{21}}{\beta} + \frac{\sigma_{22}}{\lambda} + \frac{\sigma_{23}}{\alpha}\right) + \left(\frac{1}{2}\right)(\sigma_{21}\psi_1 + \sigma_{22}\psi_2 + \sigma_{23}\psi_3)\right] \downarrow \widehat{\Theta}" \tag{4.5}$$

$$"\alpha^* = E[\alpha|y] = \left[\alpha - \left(\frac{\sigma_{31}}{\beta} + \frac{\sigma_{32}}{\lambda} + \frac{\sigma_{33}}{\alpha}\right) + \left(\frac{1}{2}\right)(\sigma_{31}\psi_1 + \sigma_{32}\psi_2 + \sigma_{33}\psi_3)\right] \downarrow \widehat{\Theta}" \tag{4.6}$$

"Thus, the posterior variances can be obtained by"

$$"Var(\beta|y) = E[\beta^2|y] - (\beta^*)^2 = \sigma_{11} - \left[\left(\frac{\sigma_{11}}{\beta} + \frac{\sigma_{12}}{\lambda} + \frac{\sigma_{13}}{\alpha}\right) - \left(\frac{1}{2}\right)(\sigma_{11}\psi_1 + \sigma_{12}\psi_2 + \sigma_{13}\psi_3)\right]^2 \downarrow \widehat{\Theta}" \tag{4.7}$$

$$"Var(\lambda|y) = E[\lambda^2|y] - (\lambda^*)^2$$

$$= \sigma_{22} - \left[\left(\frac{\sigma_{21}}{\beta} + \frac{\sigma_{22}}{\lambda} + \frac{\sigma_{23}}{\alpha}\right) - \left(\frac{1}{2}\right)(\sigma_{21}\psi_1 + \sigma_{22}\psi_2 + \sigma_{23}\psi_3)\right]^2 \downarrow \widehat{\Theta}" \tag{4.8}$$

$$"Var(\alpha|y) = E[\alpha^2|y] - (\alpha^*)^2$$

$$= \sigma_{33} - \left[\left(\frac{\sigma_{31}}{\beta} + \frac{\sigma_{32}}{\lambda} + \frac{\sigma_{33}}{\alpha}\right) - \left(\frac{1}{2}\right)(\sigma_{31}\psi_1 + \sigma_{32}\psi_2 + \sigma_{33}\psi_3)\right]^2 \downarrow \widehat{\Theta}" \tag{4.9}$$

where

$$\psi_1 = \sum_{i,j} \sigma_{ij} L^{(3)}_{ij1} \equiv \sigma_{11}L^{(3)}_{111} + 2\sigma_{12}L^{(3)}_{121} + 2\sigma_{13}L^{(3)}_{131} + \sigma_{22}L^{(3)}_{221} + 2\sigma_{23}L^{(3)}_{231} + \sigma_{33}L^{(3)}_{331},$$

$$\psi_2 = \sum_{i,j} \sigma_{ij} L^{(3)}_{ij2} \equiv \sigma_{11}L^{(3)}_{112} + 2\sigma_{12}L^{(3)}_{122} + 2\sigma_{13}L^{(3)}_{132} + \sigma_{22}L^{(3)}_{222} + 2\sigma_{23}L^{(3)}_{232} + \sigma_{33}L^{(3)}_{332}$$

and

$\psi_3 = \sum_{i,j} \sigma_{ij} L^{(3)}_{ij3} \equiv \sigma_{11}L^{(3)}_{113} + 2\sigma_{12}L^{(3)}_{123} + 2\sigma_{13}L^{(3)}_{133} + \sigma_{22}L^{(3)}_{223} + 2\sigma_{23}L^{(3)}_{233} + \sigma_{33}L^{(3)}_{333}$;
for i, j = 1,2,3.

"To compute the posterior means and the posterior variances of the three parameters β, α and λ, the third derivatives of the natural logarithm of the likelihood function are required. The third derivatives of ln L with respect to β, α and λ can be represented as follows".

$$L_{ijk}^{(3)} = \frac{\partial^3 lnL(y|\Theta)}{\partial\Theta_i \partial\Theta_j \partial\Theta_k}.$$

"That is,

$$"L_{111}^{(3)} = \frac{\partial^3 \ln L}{\partial\beta^3}, L_{222}^{(3)} = \frac{\partial^3 \ln L}{\partial\alpha^3}, L_{333}^{(3)} = \frac{\partial^3 \ln L}{\partial\lambda^3},$$

$$L_{112}^{(3)} = L_{121}^{(3)} = L_{211}^{(3)} = \frac{\partial^3 \ln L}{\partial\beta^2 \partial\alpha},$$

$$L_{113}^{(3)} = L_{131}^{(3)} = L_{311}^{(3)} = \frac{\partial^3 \ln L}{\partial\beta^2 \partial\lambda},$$

$$L_{123}^{(3)} = L_{213}^{(3)} = L_{132}^{(3)} = L_{231}^{(3)} = L_{312}^{(3)} = L_{321}^{(3)} = \frac{\partial^3 \ln L}{\partial\beta \partial\alpha \partial\lambda},$$

$$L_{221}^{(3)} = L_{212}^{(3)} = L_{122}^{(3)} = \frac{\partial^3 \ln L}{\partial\alpha^2 \partial\beta},$$

$$L_{223}^{(3)} = L_{232}^{(3)} = L_{322}^{(3)} = \frac{\partial^3 \ln L}{\partial\alpha^2 \partial\lambda},$$

$$L_{331}^{(3)} = L_{313}^{(3)} = L_{133}^{(3)} = \frac{\partial^3 \ln L}{\partial\lambda^2 \partial\beta},$$

$$L_{332}^{(3)} = L_{323}^{(3)} = L_{233}^{(3)} = \frac{\partial^3 \ln L}{\partial\lambda^2 \partial\alpha}."$$

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