



## Original article

Existence theorems for a nonlinear second-order distributional differential equation<sup>☆</sup>Wei Liu<sup>a</sup>, Guoju Ye<sup>a</sup>, Dafang Zhao<sup>a,b</sup>, Delfim F.M. Torres<sup>c,d,\*</sup><sup>a</sup> College of Science, Hohai University, Nanjing 210098, PR China<sup>b</sup> School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, PR China<sup>c</sup> Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal<sup>d</sup> African Institute for Mathematical Sciences (AIMS-Cameroon), P.O. Box 608, Limbe, Cameroon

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## ABSTRACT

In this work, we are concerned with existence of solutions for a nonlinear second-order distributional differential equation, which contains measure differential equations and stochastic differential equations as special cases. The proof is based on the Leray–Schauder nonlinear alternative and Kurzweil–Henstock–Stieltjes integrals. Meanwhile, examples are worked out to demonstrate that the main results are sharp. © 2017 The Authors. Production and hosting by Elsevier B.V. on behalf of King Saud University. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

## 1. Introduction

The first-order distributional differential equation (DDE) in the form

$$Dx = f(t, x) + g(t, x)Du, \quad (1.1)$$

where  $Dx$  and  $Du$  stand, respectively, for the distributional derivative of function  $x$  and  $u$  in the sense of Schwartz, has been studied as a perturbed system of the ordinary differential equation (ODE)

$$x' = f(t, x) \quad \left( ' := \frac{d}{dt} \right).$$

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\* Corresponding author at: Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal.

E-mail addresses: [liuw626@hhu.edu.cn](mailto:liuw626@hhu.edu.cn) (W. Liu), [yegji@hhu.edu.cn](mailto:yegji@hhu.edu.cn) (G. Ye), [dafangzhao@163.com](mailto:dafangzhao@163.com) (D. Zhao), [delfim@ua.pt](mailto:delfim@ua.pt), [delfim@aims-cameroon.org](mailto:delfim@aims-cameroon.org) (D.F.M. Torres).

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The DDE (1.1) provides a good model for many physical processes, biological neural nets, pulse frequency modulation systems and automatic control problems (Das and Sharma, 1971, 1972; Leela, 1974). Particularly, when  $u$  is an absolute continuous function, then (1.1) reduces to an ODE. However, in physical systems, one cannot always expect the perturbations to be well-behaved. For example, if  $u$  is a function of boundary variation,  $Du$  can be identified with a Stieltjes measure and will have the effect of suddenly changing the state of the system at the points of discontinuity of  $u$ , that is, the system could be controlled by some impulsive force. In this case, (1.1) is also called a measure differential equation (MDE), see Das and Sharma (1971, 1972), Dhage and Bellale (2009), Federson and Mesquita (2013), Federson et al. (2012), Leela (1974), Antunes Monteiro and Slavík (2016), Satco (2014), Slavík (2013), Slavík (2015). Results concerning existence, uniqueness, and stability of solutions, were obtained in those papers. However, this situation is not the worst, because it is well-known that the solutions of a MDE, if exist, are still functions of bounded variation. The case when  $u$  is a continuous function has also been considered in Liu et al. (2012) and Zhou et al. (2015). The integral there is understood as a Kurzweil–Henstock integral (Krejčí, 2006; Kurzweil, 1957; Lee, 1989; Pelant and Tvrdý, 1993; Schwabik and Ye, 2005; Talvila, 2008; Tvrdý, 1994; Tvrdý, 2002; Ye and Liu, 2016) (or Kurzweil–Henstock–Stieltjes integral, or distributional Kurzweil–Henstock integral), which is a generalization

of the Lebesgue integral. Especially, if  $u$  denotes a Wiener process (or Brownian motion), then (1.1) becomes a stochastic differential equation (SDE), see, for example, Boon and Lam (2011) and Mao (2008). In this case,  $u$  is continuous but pointwise differentiable nowhere, and the Itô integral plays an important role there. As for the relationship between the Kurzweil–Henstock integral and the Itô integral, we refer the interested readers to Boon and Lam (2011), Chew et al. (2001) and Toh and Chew (2012) and references therein.

It is well-known that regulated functions (that is, a function whose one-side limits exist at every point of its domain) contain continuous functions and functions of bounded variation as special cases (Fraňková, 1991). Therefore, it is natural to consider the situation when  $u$  is a regulated function, see Pelant and Tvrdý (1993) and Tvrdý (1994). Denote by  $G[0, 1]$  the space of all real regulated functions on  $[0, 1]$ , endowed with the supremum norm  $\|\cdot\|$ . Since the DDE allows both the inputs and outputs of the systems to be discontinuous, most conventional methods for ODEs are inapplicable, and thus the study of DDEs becomes very interesting and important.

The purpose of our paper is to apply the Leray–Schauder nonlinear alternative and Kurzweil–Henstock–Stieltjes integrals to establish existence of a solution to the second order DDE of type

$$-D^2x = f(t, x) + g(t, x)Du, \quad t \in [0, 1], \tag{1.2}$$

subject to the three-point boundary condition (cf. Sun and Zhao (2015))

$$x(0) = \beta Dx(0), \quad Dx(1) + Dx(\eta) = 0, \tag{1.3}$$

where  $D^2x$  stands for the second order distributional derivative of the real function  $x \in G[0, 1]$ ,  $u \in G[0, 1]$ ,  $\beta$  is a constant, and  $\eta \in [0, 1]$ . This approach is well-motivated since this topic has not yet been addressed in the literature, and by the fact that the Kurzweil–Henstock–Stieltjes integral is a powerful tool for the study of DDEs. We assume that  $f$  and  $g$  satisfy the following assumptions:

- (H<sub>1</sub>)  $f(t, x)$  is Kurzweil–Henstock integrable with respect to  $t$  for all  $x \in G[0, 1]$ ;
- (H<sub>2</sub>)  $f(t, x)$  is continuous with respect to  $x$  for all  $t \in [0, 1]$ ;
- (H<sub>3</sub>) there exist nonnegative Kurzweil–Henstock integrable functions  $k$  and  $h$  such that

$$-k\|x\| - h \leq f(\cdot, x) \leq k\|x\| + h \quad \forall x \in B_r,$$

where  $B_r = \{x \in G[0, 1] : \|x\| \leq r\}$ ,  $r > 0$ ;

- (H<sub>4</sub>)  $g(t, x)$  is a function with bounded variation on  $[0, 1]$  and  $g(0, x) = 0$  for all  $x \in G[0, 1]$ ;
- (H<sub>5</sub>)  $g(t, x)$  is continuous with respect to  $x$  for all  $t \in [0, 1]$ ;
- (H<sub>6</sub>) there exists  $M > 0$  such that

$$\sup_{x \in B_r} \text{var}_g \leq M,$$

where

$$\text{var}_g = \sup_{\{0,1\}} \sum_n |g(s_n, x(s_n)) - g(t_n, x(t_n))|,$$

the supremum taken over every sequence  $\{(t_n, s_n)\}$  of disjoint intervals in  $[0, 1]$ , is called the total variation of  $g$  on  $[0, 1]$ .

Now, we state our main result.

**Theorem 1.1.** [Existence of a solution to problem (1.2) and (1.3)] Suppose assumptions (H<sub>1</sub>)–(H<sub>6</sub>) hold. If

$$(|\beta| + 2) \max_{t \in [0,1]} \left| \int_0^t k(s) ds \right| < 1,$$

then problem (1.2) and (1.3) has at least one solution.

If  $k(t) \equiv 0$  on  $[0, 1]$ , then (H<sub>3</sub>) can be reduced to.

(H'<sub>3</sub>) there exists a nonnegative Kurzweil–Henstock function  $h$  such that

$$-h \leq f(\cdot, x) \leq h \quad \forall x \in B_r.$$

Thus, the following result holds as a direct consequence.

**Corollary 1.2.** Assume that (H<sub>1</sub>), (H<sub>2</sub>), (H'<sub>3</sub>) and (H<sub>4</sub>)–(H<sub>6</sub>) are fulfilled. Then, problem (1.2) and (1.3) has at least one solution.

It is worth to mention that condition (H'<sub>3</sub>), together with (H<sub>1</sub>) and (H<sub>2</sub>), was firstly proposed by Chew and Flordeliza (1991), to deal with first-order Cauchy problems.

The paper is organized as follows. In Section 2, we give two useful lemmas: we prove that under our hypotheses problem (1.2) and (1.3) can be rewritten in an equivalent integral form (Lemma 2.1) and we recall the Leray–Schauder theorem (Lemma 2.2). Then, in Section 3, we prove our existence result (Theorem 1.1). We end with Section 4, providing two illustrative examples. Along all the manuscript, and unless stated otherwise, we always assume that  $x, u \in G[0, 1]$ . Moreover, we use the symbol  $\int_a^b$  to mean  $\int_{[a,b]}$ .

## 2. Auxiliary Lemmas

By (H<sub>1</sub>) and (H<sub>4</sub>), we define

$$F(t, x) = \int_0^t f(s, x(s)) ds, \tag{2.1}$$

$$G_u(t, x) = \int_0^t g(s, x(s)) du(s),$$

for all  $t \in [0, 1]$ .

**Lemma 2.1.** Under the assumptions (H<sub>1</sub>)–(H<sub>6</sub>), problem (1.2) and (1.3) is equivalent to the integral equation

$$x(t) = \frac{t + \beta}{2} (F(1, x) + F(\eta, x) + G_u(1, x) + G_u(\eta, x)) - \int_0^t F(s, x) ds - \int_0^t G_u(s, x) ds \tag{2.2}$$

on  $[0, 1]$ , where  $F$  and  $G_u$  are given in (2.1),  $u \in G[0, 1]$ , and  $\beta$  and  $\eta$  are constants with  $0 \leq \eta \leq 1$ .

**Proof.** For all  $t \in [0, 1]$ ,  $s \in [0, 1]$ , and  $x \in G[0, 1]$ , we have

$$\int_0^t s D^2x(s) ds = \int_0^t s d(Dx(s)) = tDx(t) - x(t) + x(0) \tag{2.3}$$

by the properties of the distributional derivative. Integrating (1.2) once over  $[0, t]$ , we obtain that

$$Dx(t) = Dx(0) - F(t, x) - G_u(t, x). \tag{2.4}$$

Combining with the boundary conditions (1.3), one has

$$Dx(0) = \frac{1}{2} (F(1, x) + F(\eta, x) + G_u(1, x) + G_u(\eta, x)) \tag{2.5}$$

and

$$x(0) = \frac{\beta}{2} (F(1, x) + F(\eta, x) + G_u(1, x) + G_u(\eta, x)). \tag{2.6}$$

It follows from (2.3) and (2.4) that

$$x(t) = tDx(0) + x(0) - \int_0^t (t-s)f(s, x(s)) ds - \int_0^t (t-s)g(s, x(s)) du(s). \tag{2.7}$$

Therefore, by (2.5)–(2.7) and the substitution formula (Theorem 2.3.19, Tvrdý, 2002), one has

$$x(t) = \frac{t + \beta}{2} (F(1, x) + F(\eta, x) + G_u(1, x) + G_u(\eta, x)) - \int_0^t F(s, x) ds - \int_0^t G_u(s, x) ds, \quad t \in [0, 1].$$

It is not difficult to calculate that (1.2) and (1.3) holds by taking the derivative both sides of (2.2). This completes the proof. □

Now, we present the well-known Leray–Schauder nonlinear alternative theorem.

**Lemma 2.2.** [See Deimling (1985)] *Let E be a Banach space, Ω a bounded open subset of E, 0 ∈ Ω, and T : Ω̄ → E be a completely continuous operator. Then, either there exists x ∈ ∂Ω such that T(x) = λx with λ > 1, or there exists a fixed point x\* ∈ Ω̄.*

We prove existence of a solution to problem (1.2) and (1.3) with the help of the preceding two lemmas.

**3. Proof of Theorem 1.1**

Let

$$H(t) = \int_0^t h(s) ds, \tag{3.1}$$

$$K(t) = \int_0^t k(s) ds,$$

t ∈ [0, 1]. Then, by (H<sub>3</sub>), H and K are continuous functions. According to (2.1) and (H<sub>1</sub>), function F is continuous on [0, 1], and

$$\|F\| = \max_{t \in [0, 1]} \left| \int_0^t f(s, x(s)) ds \right| \leq \|K\| \|x\| + \|H\|.$$

On the other hand, by (Tvrdý, 2002, Proposition 2.3.16) and (H<sub>4</sub>), G<sub>u</sub> is regulated on [0, 1]. Further, from (H<sub>6</sub>) and the Hölder inequality (Tvrdý, 2002, Theorem 2.3.8), it follows that

$$\|G_u\| \leq \left( |g(0, x(0))| + |g(1, x(1))| + \text{var}_{[0, 1]} g \right) \|u\| \leq 2M \|u\|.$$

Let

$$r = \frac{(|\beta| + 2)(\|H\| + 2M\|u\|)}{1 - (|\beta| + 2)\|K\|} > 0. \tag{3.2}$$

For each x ∈ B<sub>r</sub> and t ∈ [0, 1], define the operator T : G[0, 1] → G[0, 1] by

$$\begin{aligned} Tx(t) : \\ &= \frac{t + \beta}{2} (F(1, x) + F(\eta, x) + G_u(1, x) + G_u(\eta, x)) \\ &\quad - \int_0^t F(s, x) ds - \int_0^t G_u(s, x) ds. \end{aligned} \tag{3.3}$$

We prove that T is completely continuous in three steps. Step 1: we show that T : B<sub>r</sub> → B<sub>r</sub>. Indeed, for all x ∈ B<sub>r</sub>, one has

$$\begin{aligned} \|Tx\| &\leq (|\beta| + 2)(\|F\| + \|G_u\|) \\ &\leq (|\beta| + 2)(r\|K\| + \|H\| + 2M\|u\|) = r \end{aligned} \tag{3.4}$$

by (3.2) and (3.3). Hence, T(B<sub>r</sub>) ⊆ B<sub>r</sub>. Step 2: we show that T(B<sub>r</sub>) is equiregulated (see the definition in Fraňková (1991)). For t<sub>0</sub> ∈ [0, 1] and x ∈ B<sub>r</sub>, we have

$$\begin{aligned} &|Tx(t) - Tx(t_{0+})| \\ &= \left| \frac{t - (t_{0+})}{2} (F(1, x) + F(\eta, x) + G_u(1, x) + G_u(\eta, x)) \right. \\ &\quad \left. - \int_{t_{0+}}^t F(s, x) + G_u(s, x) ds \right| \\ &\leq 2|t - (t_{0+})|(r\|K\| + \|H\| + 2M\|u\|) \rightarrow 0 \end{aligned}$$

as t → t<sub>0+</sub>. Similarly, we can prove that

$$|Tx(t_{0-}) - Tx(t)| \rightarrow 0 \text{ as } t \rightarrow t_{0-}$$

for each t<sub>0</sub> ∈ (0, 1]. Therefore, T(B<sub>r</sub>) is equiregulated on [0, 1]. In view of Steps 1 and 2 and an Ascoli–Arzelà type theorem (Fraňková, 1991, Corollary 2.4), we conclude that T(B<sub>r</sub>) is relatively compact. Step 3: we prove that T is a continuous mapping. Let x ∈ B<sub>r</sub> and {x<sub>n</sub>}<sub>n ∈ ℕ</sub> be a sequence in B<sub>r</sub> with x<sub>n</sub> → x as n → ∞. By (H<sub>2</sub>) and (H<sub>4</sub>), one has

$$f(\cdot, x_n) \rightarrow f(\cdot, x) \quad \text{and} \quad g(\cdot, x_n) \rightarrow g(\cdot, x)$$

as n → ∞. According to the assumption (H<sub>3</sub>) and the convergence Theorem 4.3 of Lee (1989), we have

$$\lim_{n \rightarrow \infty} \int_0^t f(s, x_n(s)) ds = \int_0^t f(s, x(s)) ds, \quad t \in [0, 1].$$

Moreover, (H<sub>6</sub>), together with the convergence Theorem 1.7 of Krejčí (2006), yields that

$$\lim_{n \rightarrow \infty} \int_0^t g(s, x_n(s)) du(s) = \int_0^t g(s, x(s)) du(s),$$

t ∈ [0, 1]. Hence,

$$\begin{aligned} Tx_n(t) - Tx(t) &= \frac{\beta + t}{2} [(F(1, x_n) + F(\eta, x_n) + G_u(1, x_n) + G_u(\eta, x_n)) \\ &\quad - (F(1, x) + F(\eta, x) + G_u(1, x) + G_u(\eta, x))] \\ &\quad - \int_0^t F(s, x_n(s)) - F(s, x(s)) ds - \int_0^t G_u(s, x_n) - G_u(s, x) ds, \quad t \in [0, 1]. \end{aligned}$$

Therefore, lim<sub>n → ∞</sub> Tx<sub>n</sub>(·) = Tx(·), and thus T is a completely continuous operator. Finally, let

$$\Omega = \{x \in G[0, 1] : \|x\| < r\}$$

and assume that x ∈ ∂Ω such that Tx = λx for λ > 1. Then, by (3.4), one has

$$\lambda r = \lambda \|x\| = \|Tx\| \leq r,$$

which implies that λ ≤ 1. This is a contradiction. Therefore, by Lemma 2.2, there exists a fixed point of T, which is a solution of problem (1.2) and (1.3). The proof of Theorem 1.1 is complete.

**4. Illustrative examples**

We now give two examples to illustrate Theorem 1.1 and Corollary 1.2, respectively. Let g\*(t, x(t)) = 0 if t = 0 and g\*(t, x(t)) = 1 if t ∈ (0, 1] for all x ∈ B<sub>r</sub>. Then, it is easy to see that g\* satisfies hypotheses (H<sub>4</sub>)–(H<sub>6</sub>) with M = 1.

**Example 4.1.** Consider the boundary value problem

$$\begin{cases} -D^2x = \frac{x \sin(x)}{3\sqrt{5+t}} + g^*(t, x) D\mathcal{H}(t - \frac{1}{2}), & t \in [0, 1], \\ x(0) = 4Dx(0), \quad Dx(1) + Dx(\frac{1}{4}) = 0, \end{cases} \tag{4.1}$$

where H is the Heaviside function, i.e., H(t) = 0 if t < 0 and H(t) = 1 if t > 0. It is easy to see that H is of bounded variation, but not continuous. Let f(t, x) =  $\frac{x \sin(x)}{2\sqrt{4+t}}$ , g(t, x) = g\*(t, x), and u(t) = H(t -  $\frac{1}{2}$ ). Then, (H<sub>1</sub>), (H<sub>2</sub>), and (H<sub>4</sub>)–(H<sub>6</sub>) hold. Moreover, there exist HK integrable functions k(t) =  $\frac{1}{3\sqrt{5+t}}$  and h(t) = 1 such that

$$-k\|x\| - h \leq f(\cdot, x) \leq k\|x\| + h \quad \forall x \in G[0, 1],$$

i.e., (H<sub>3</sub>) holds. Further, by (3.1),

$$\|K\| = \frac{2}{3} (\sqrt{6} - \sqrt{5}), \quad \|H\| = 1, \quad \|u\| = \|\mathcal{H}\| = 1.$$

Let β = 4 and η =  $\frac{1}{4}$ . From (3.2), we have

$$r = \frac{(|\beta| + 2)(\|H\| + 2M\|u\|)}{1 - (|\beta| + 2)\|K\|} = \frac{18}{1 - 4(\sqrt{6} - \sqrt{5})}.$$

Therefore, by [Theorem 1.1](#), problem (4.1) has at least one solution  $x^*$  with

$$\|x^*\| \leq \frac{18}{1 - 4(\sqrt{6} - \sqrt{5})}.$$

**Example 4.2.** Consider the boundary value problem

$$\begin{cases} -D^2x = \sin(x) + 2t \sin(t^{-2}) - \frac{2}{t} \cos(t^{-2}) \\ \quad + g^*(t, x)DW, \quad t \in [0, 1], \\ x(0) = -\frac{1}{6}Dx(0), \quad Dx(1) + Dx(\frac{2}{3}) = 0, \end{cases} \quad (4.2)$$

where  $\mathcal{W}$  is the Weierstrass function

$$\mathcal{W}(t) = \sum_{n=1}^{\infty} \frac{\sin 7^n \pi t}{2^n}$$

in [Hardy \(1916\)](#). It is well-known that  $\mathcal{W}(t)$  is continuous but pointwise differentiable nowhere on  $[0, 1]$ , so  $\mathcal{W}(t)$  is not of bounded variation. Let

$$f(t, x) = \sin(x) + 2t \sin(t^{-2}) - \frac{2}{t} \cos(t^{-2}),$$

$$g(t, x) = g^*(t, x),$$

$$u = \mathcal{W}.$$

Then,  $(H_1)$ ,  $(H_2)$  and  $(H_4)$ – $(H_6)$  hold. Moreover, let

$$k(t) = 0, \quad h(t) = 1 + 2t \sin(t^{-2}) - \frac{2}{t} \cos(t^{-2}).$$

Obviously, the highly oscillating function  $h(t)$  is Kurzweil–Henstock integrable but not Lebesgue integrable, and

$$H(t) = \int_0^t h(s)ds = \begin{cases} t + t^2 \sin(t^{-2}), & t \in (0, 1], \\ 0, & t = 0. \end{cases}$$

Moreover, we have

$$-h \leq f(\cdot, x) \leq h \quad \forall x \in G[0, 1],$$

that is,  $(H_3)$  holds. Let  $\beta = -\frac{1}{6}$  and  $\eta = \frac{2}{3}$ . Since

$$0 \leq \|u\| = \|\mathcal{W}\| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1, \quad \|H\|_{\infty} = 1 + \sin(1),$$

we have by (3.2) that

$$\begin{aligned} 3.9899 &\approx \frac{13}{6}(\sin(1) + 1) \\ &\leq r = \frac{(|\beta| + 2)(\|H\| + 2M\|u\|)}{1 - (|\beta| + 2)\|K\|_{\infty}} \\ &\leq \frac{13}{6}(\sin(1) + 3) \\ &\approx 8.3232. \end{aligned}$$

Therefore, by [Corollary 1.2](#), problem (4.2) has at least one solution  $x^*$  with

$$\|x^*\| \leq \frac{13}{6}(\sin(1) + 3).$$

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