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ORIGINAL ARTICLE

Dual generalized order statistics from family of *J*-shaped distribution and its characterization

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Abstract In this paper we derive explicit algebraic expressions and some recurrence relations for both single and product moments of dual generalized order statistics from a family of *J*-shaped distribution. These relations generalize the results given by Zghoul (2010, 2011). Further, a characterization of this distribution through conditional expectation of dual generalized order statistics is given and some computational works are also carried out.

1. Introduction

The concept of lower generalized order statistics was introduced by Pawlas and Szynal (2001). Later Burkschat et al. (2003) extensively studied and discussed it as a dual generalized order statistics (*dgos*) to enable a common approach to descending ordered random variables like reversed order statistics, lower k records and lower Pfeifer records. In this paper we will consider the *dgos* defined as follows:

Let, $n \in N$, $k \geqslant 1$, $m \in \Re$ be the parameters such that

$$\gamma_r = k + (n-r)(m+1) \geqslant 1, \text{ for all } 1 \leqslant r \leqslant n.$$

The random variables, $X^*(1,n,m,k), X^*(2,n,m,k), \ldots, X^*(n,n,m,k)$ are n *dgos* from an absolutely continuous distribution function (*df*) $F(x)$ with probability density function, (*pdf*) $f(x)$ if their joint *pdf* has the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^m f(x_i)\right)[F(x_n)]^{k-1}f(x_n) \tag{1.1}$$

on the cone $F^{-1}(1) > x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n > F^{-1}(0)$.

In view of (1.1), the marginal *pdf* of rth, *dgos*, $X^*(r,n,m,k)$, $1 \leqslant r \leqslant n$ is

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x)). \tag{1.2}$$

and the joint *pdf* of $X^*(r,n,m,k)$ and, $X^*(s,n,m,k)$, $1 \leqslant r < s \leqslant n$ is

$$f_{X^*(r,n,m,k),X^*(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \times [F(x)]^m f(x) g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y), y < x, \tag{1.3}$$

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where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i,\quad h_m(x)=\begin{cases}-\frac{1}{m+1}x^{m+1}, & m\neq-1\\ -\log x, & m=-1\end{cases}$$

and

$$g_m(x)=h_m(x)-h_m(1),\quad x\in[0,1).$$

If $m=0$, $k=1$ then $X^*(r,n,m,k)$ reduces to the $(n-r+1)$th order statistic, $X_{n-r+1:n}$ from the sample $X_1,X_2,\ldots,X_n$ and when, $m=-1$ then $X^*(r,n,m,k)$ reduces to the rth lower k record value. We shall also take $X^*(0,n,m,k)=0$.

Several authors utilized the concept of *dgos* in their work. References may be made to Pawlas and Szynal (2001), Ahsanullah (2004, 2005), Mbah and Ahsanullah (2007), Khan et al. (2010), Khan and Kumar (2010, 2011) among others.

In the present study, we have established explicit expressions and some recurrence relations for single and product moments of *dgos* from a family of *J*-shaped distribution. These relations generalize the results given by Zghoul (2010, 2011). Further, a characterizing result of this distribution through conditional expectation of *dgos* is stated and proved.

A random variable X is said to have J-shaped distribution, if its *pdf* is of the form

$$f(x)=\frac{2\alpha}{\beta}\left(1-\frac{x}{\beta}\right)\left\{\frac{x}{\beta}\left(2-\frac{x}{\beta}\right)\right\}^{\alpha-1},\quad 0\leqslant x\leqslant\beta\quad 0<\alpha<1. \tag{1.4}$$

We will consider in this paper without loss of any generality $\beta=1$, i.e.

$$f(x)=2\alpha(1-x)[x(2-x)]^{\alpha-1},\quad 0\leqslant x\leqslant 1\quad 0<\alpha<1 \tag{1.5}$$

and the corresponding *df*

$$F(x)=[x(2-x)]^{\alpha},\quad 0\leqslant x\leqslant 1\quad 0<\alpha<1. \tag{1.6}$$

It is easy to see that

$$x(2-x)f(x)=2\alpha(1-x)F(x). \tag{1.7}$$

More details on this distribution see, for example, Topp and Leone (1955), Nadarajah and Kotz (2003) and Zghoul (2010, 2011).

2. Single moments and relations

Theorem 2.1. *For J-shaped distribution as given in (1.6) and,* $1\leqslant r\leqslant n$, $k=1,2,\ldots$

$$E[X^{*j}(r,n,m,k)]=\sum_{p=0}^{\infty}D(j,p)\prod_{v=1}^{r}\left(\frac{\alpha\gamma_v}{j+p+\alpha\gamma_v}\right),\quad m\neq-1 \tag{2.1}$$

$$E[X^{*j}(r,n,-1,k)]=E[(Z_r^{(k)})^j]=\sum_{p=0}^{\infty}D(j,p)\left(\frac{\alpha k}{j+p+\alpha k}\right)^r,\quad m=-1 \tag{2.2}$$

where

$$D(j,0)=2^{-j},D(j,1)=2^{-j-2}j \text{ and } D(j,p)=\frac{2^{-j-2p}j}{p!}\prod_{u=1}^{p-1}(j+p+u) \text{ for } p\geqslant 2.$$

Proof. From (1.2), we have

$$E[X^{*j}(r,n,m,k)]=\frac{C_{r-1}}{(r-1)!}\times\int_0^1 x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \tag{2.3}$$

Setting, $t=F(x)$ then $x=F^{-1}(t)=1-\sqrt{1-t^{1/\alpha}}$ in (2.3), we find that

$$E[X^{*j}(r,n,m,k)]=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\times\int_0^1\left(1-\sqrt{1-t^{1/\alpha}}\right)^j t^{\gamma_r-1}\left(1-t^{m+1}\right)^{r-1}dt. \tag{2.4}$$

Making the substitution $t^{m+1}=z$ in (2.4), we get

$$E[X^{*j}(r,n,m,k)]=\frac{C_{r-1}}{(r-1)!(m+1)^r}\times\int_0^1\left(1-\sqrt{1-z^{1/\alpha(m+1)}}\right)^j z^{\frac{\gamma_r}{m+1}-1}(1-z)^{r-1}dz. \tag{2.5}$$

For any real number q and, $|x|<1$ we have Gradshteyn and Ryzhik (2007, p-25)

$$(1+\sqrt{1+x})^q=2^q\left\{1+\frac{q}{1!}\left(\frac{x}{4}\right)+\frac{q(q-3)}{2!}\left(\frac{x}{4}\right)^2+\frac{q(q-4)(q-5)}{3!}\left(\frac{x}{4}\right)^3+\ldots\right\}=\sum_{p=0}^{\infty}C(q,p)x^p, \tag{2.6}$$

where

$$C(q,0)=2^q, C(q,1)=2^{q-2}q \text{ and } C(q,p)=\frac{2^{q-2p}q}{p!}\prod_{u=1}^{p-1}(q-p-u) \text{ for } p\geqslant 2.$$

Therefore,

$$(1-\sqrt{1-x})^q=\sum_{p=0}^{\infty}D(q,p)x^{q+p}, \tag{2.7}$$

where

$$D(q,0)=2^{-q}, D(q,1)=2^{-q-2}q \text{ and } D(q,p)=\frac{2^{-q-2p}q}{p!}\prod_{u=1}^{p-1}(q+p+u) \text{ for } p\geqslant 2.$$

Now on substituting (2.7) in (2.5), we have

$$E[X^{*j}(r,n,m,k)]=\frac{C_{r-1}}{(r-1)!(m+1)^r}\sum_{p=0}^{\infty}D(j,p)\int_0^1 z^{\frac{j+p}{\alpha(m+1)}+\frac{\gamma_r}{m+1}-1}(1-z)^{r-1}dz,$$
$$=\frac{C_{r-1}}{(r-1)!(m+1)^r}\sum_{p=0}^{\infty}D(j,p)B\left(\frac{j+p}{\alpha(m+1)}+\frac{\gamma_r}{(m+1)},r\right), \tag{2.8}$$

where $B(a,b)$ is the beta function.

Applying the well known relation between the beta and gamma functions in (2.8), we have the result given in (2.1).

Now taking m tends to -1 in (2.1), we have the result given in (2.2). □

2.1. Special cases

i. Putting $m=0$ and $k=1$ in (2.1), the exact expression for the single moments of order statistics of the J-shaped distribution can be obtained as

$$E[X^j_{n-r+1:n}]=\frac{n!}{(n-r)!}\sum_{p=0}^{\infty}D(j,p)\frac{\Gamma\left(\frac{j+p}{\alpha}+n-r+1\right)}{\Gamma\left(\frac{j+p}{\alpha}+n+1\right)}.$$

That is

$$E[X^j_{r:n}]=\frac{n!}{(r-1)!}\sum_{p=0}^{\infty}D(j,p)\frac{\Gamma\left(\frac{j+p}{\alpha}+r\right)}{\Gamma\left(\frac{j+p}{\alpha}+n+1\right)}$$

as obtained by Zghoul (2010).

ii. Putting $k=1$ in (2.2), we deduce the explicit expression for the single moment of lower record for J-shaped distribution in the form

$$E[(Z^{(1)}_r)^j]=E[X^j_{L(r)}]=\sum_{p=0}^{\infty}D(j,p)\left(\frac{\alpha}{j+p+\alpha}\right)^r$$

which verify the result of Zghoul (2011) for $r=n+1$.

A recurrence relation for single moments of *dgos* from *df* (1.6) is obtained in the following theorem.

Theorem 2.2. *For the distribution as given in* (1.6) *and*, $1\leqslant r\leqslant n$, $k=1,2,\ldots$

$$\left(1+\tfrac{j+1}{2\alpha\gamma_r}\right)E[X^{*j+1}(r,n,m,k)]=E[X^{*j+1}(r-1,n,m,k)]$$
$$+(j+1)\left(\tfrac{1}{\alpha\gamma_r}+\tfrac{1}{j}\right)E[X^{*j}(r,n,m,k)]-\tfrac{j+1}{j}E[X^{*j}(r-1,n,m,k)].\qquad(2.9)$$

Proof. From Eqs. (1.2) and (1.7), we have

$$2E[X^{*j}(r,n,m,k)]-E[X^{*j+1}(r,n,m,k)]$$
$$=\frac{2\alpha C_{r-1}}{(r-1)!}\left\{\int_0^1 x^{j-1}[F(x)]^{\gamma_r}g_m^{r-1}(F(x))dx-\int_0^1 x^j[F(x)]^{\gamma_r}g_m^{r-1}(F(x))dx\right\}.$$

Integrating above equation by parts and simplifying the resulting expression, we derive the relation given in (2.9). □

Remark 2.1. Putting $m=0$ and $k=1$ in (2.9), the recurrence relation for the single moments of order statistics of the J-shaped distribution can be obtained as

$$\left(1+\tfrac{j+1}{2\alpha(n-r+1)}\right)E(X^{j+1}_{n-r+1:n})=E(X^{j+1}_{n-r+2:n})$$
$$+(j+1)\left(\tfrac{1}{\alpha(n-r+1)}+\tfrac{1}{j}\right)E(X^{j}_{n-r+1:n})-\tfrac{j+1}{j}E(X^{j}_{n-r+2:n}).$$

Replacing $(n-r+1)$ by, $(r-1)$ we get

$$E(X^{j+1}_{r:n})=\left(1+\tfrac{j+1}{2\alpha(r-1)}\right)E(X^{j+1}_{r-1:n})$$
$$-(j+1)\left(\tfrac{1}{\alpha(r-1)}+\tfrac{1}{j}\right)E(X^{j}_{r-1:n})+\tfrac{j+1}{j}E(X^{j}_{r:n})$$

which verify the result of Zghoul (2010) for $r=r+1$.

Table 1 First four moments of order statistics from J-shaped distribution.

n	r	$j=1$			$j=2$		
		$\alpha=0.1$	$\alpha=0.4$	$\alpha=0.7$	$\alpha=0.1$	$\alpha=0.4$	$\alpha=0.7$
1	1	0.0564	0.1824	0.2691	0.0219	0.0791	0.1265
2	1	0.0083	0.0723	0.1407	0.0015	0.0176	0.0412
	2	0.1044	0.2925	0.3975	0.0422	0.1406	0.2118
3	1	0.0018	0.0375	0.0907	0.0001	0.0058	0.0187
	2	0.0213	0.1418	0.2406	0.0042	0.0412	0.0861
	3	0.1460	0.3678	0.4760	0.0613	0.1902	0.2746
4	1	0.0005	0.0225	0.0650	0.0182	0.0024	0.0101
	2	0.0058	0.0826	0.1678	0.0006	0.0161	0.0444
	3	0.0368	0.2010	0.3134	0.0078	0.0663	0.1278
	4	0.1824	0.4234	0.5302	0.0791	0.2315	0.3235
5	1	0.0001	0.0147	0.0497	0.0128	0.0011	0.0061
	2	0.0018	0.0534	0.1263	0.0001	0.0074	0.0262
	3	0.0116	0.1264	0.2300	0.0013	0.0295	0.0718
	4	0.0537	0.2507	0.3691	0.0121	0.0912	0.1651
	5	0.2146	0.4666	0.5704	0.0958	0.2666	0.3631
n	r	$j=3$			$j=4$		
		$\alpha=0.1$	$\alpha=0.4$	$\alpha=0.7$	$\alpha=0.1$	$\alpha=0.4$	$\alpha=0.7$
1	1	0.0119	0.0447	0.0738	0.0075	0.0288	0.0485
2	1	0.0005	0.0065	0.0165	0.0002	0.0029	0.0079
	2	0.0233	0.0829	0.1311	0.0148	0.0547	0.0890
3	1	0.0248	0.0015	0.0056	0.0075	0.0005	0.0021
	2	0.0014	0.0165	0.0383	0.0006	0.0079	0.0196
	3	0.0342	0.1161	0.1775	0.0219	0.0781	0.1237
4	1	0.0094	0.0005	0.0024	0.0076	0.0001	0.0007
	2	0.0001	0.0047	0.0153	0.0285	0.0017	0.0062
	3	0.0027	0.0282	0.0614	0.0012	0.0140	0.0329
	4	0.0447	0.1453	0.2162	0.0288	0.0994	0.1539
5	1	0.0045	0.0002	0.0012	0.0027	0.0244	0.0003
	2	0.0114	0.0016	0.0073	0.0096	0.0005	0.0024
	3	0.0003	0.0092	0.0273	0.0672	0.0036	0.0119
	4	0.0043	0.0409	0.0841	0.0019	0.0210	0.0471
	5	0.0548	0.1714	0.2493	0.0355	0.1190	0.1807

Remark 2.2. Setting $m=-1$ and $k\geqslant 1$ in (2.9), the relation for single moment of lower k record values is deduced in the form

$$E[(Z^{(k)}_r)^{j+1}]=\left(\tfrac{2\alpha k}{2\alpha k+j+1}\right)E[(Z^{(k)}_{r-1})^{j+1}]+\left(\tfrac{2(j+1)(j+\alpha k)}{2\alpha k+j+1}\right)E[(Z^{(k)}_r)^j]$$
$$-\left(\tfrac{2\alpha k(j+1)}{j(2\alpha k+j+1)}\right)E[(Z^{(k)}_{r-1})^j]$$

and hence for lower records $(k=1)$

$$E[X^{j+1}_{L(r)}]=\left(\tfrac{2\alpha}{2\alpha+j+1}\right)E[X^{j+1}_{L(r-1)}]+\left(\tfrac{2(j+1)(j+\alpha)}{2\alpha+j+1}\right)E[X^j_{L(r)}]$$
$$-\left(\tfrac{2\alpha(j+1)}{j(2\alpha+j+1)}\right)E[X^j_{L(r-1)}]$$

as obtained by Zghoul (2011).

In the Table 1, it may be noted that the well known property of order statistics, $\sum_{i=1}^n E(X_{i:n})=nE(X)$ (David and Nagaraja (2003)) is satisfied.

3. Product moments and relations

Theorem 3.1. *For the distribution as given in* (1.6) *and*, $1\leqslant r<s\leqslant n$, $k=1,2,\ldots$

$$E[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)] = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} D(j,p)D(i,q) \times \prod_{u=r+1}^{s}\left(\frac{\alpha\gamma_u}{j+p+\alpha\gamma_u}\right) \times \prod_{v=1}^{r}\left(\frac{\alpha\gamma_v}{i+j+p+q+\alpha\gamma_v}\right) \quad m \neq -1 \tag{3.1}$$

$$E[(Z_r^{(k)})^i (Z_s^{(k)})^j] = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} D(j,p)D(i,q)\left(\frac{\alpha k}{j+p+\alpha k}\right)^{s-r} \times \left(\frac{\alpha k}{i+j+p+q+\alpha k}\right)^{r}, \quad m=-1 \tag{3.2}$$

where, $D(j,p), D(i,q)$ are as defined in Theorem 2.1.

Proof. From (1.3), we have

$$E[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \times \int_0^1 \int_0^x x^i y^j [F(x)]^m f(x) g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1} f(y)\,dy\,dx. \tag{3.3}$$

On expanding $[h_m(F(y)) - h_m(F(x))]^{s-r-1}$ binomially in (3.3), we get

$$E[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-r-1}} \times \sum_{a=0}^{s-r-1}(-1)^a\binom{s-r-1}{a}\int_0^1 x^i [F(x)]^{(s-r-1-a)(m+1)+m} f(x) g_m^{r-1} \times (F(x)) I(x)\,dx, \tag{3.4}$$

where

$$I(x) = \int_0^x y^j [F(y)]^{\gamma_{s-1}+a(m+1)} f(y)\,dy \tag{3.5}$$

By setting, $u = F(y)$ then $y = F^{-1}(u) = 1-\sqrt{1-u^{1/\alpha}}$ in (3.5), we get

$$I(x) = \int_0^{F(x)} \left(1-\sqrt{1-u^{1/\alpha}}\right)^j u^{\gamma_{s-1}+a(m+1)}\,du. \tag{3.6}$$

Making use of (2.7) in (3.6) and simplifying the resulting expression, we obtain

$$I(x) = \sum_{p=0}^{\infty}\frac{D(j,p)}{\left(\frac{j+p}{\alpha}+\gamma_s+a(m+1)\right)}[F(x)]^{\frac{j+p}{\alpha}+\gamma_s+a(m+1)}.$$

On substituting the above expression of $I(x)$ in (3.4), we find that

$$E[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)] = A\int_0^1 x^i [F(x)]^{\frac{j+p}{\alpha}+\gamma_s+m+(s-r-1)(m+1)} f(x) g_m^{r-1}(F(x))\,dx, \tag{3.7}$$

where

$$A = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-r-1}} \times \sum_{p=0}^{\infty}\sum_{a=0}^{s-r-1}(-1)^a\binom{s-r-1}{a}\frac{D(j,p)}{\frac{j+p}{\alpha}+\gamma_s+a(m+1)}.$$

Again by setting, $v = F(x)$ then $x = F^{-1}(v) = 1-\sqrt{1-v^{1/\alpha}}$ in (3.7), we obtain

$$E[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)] = \frac{A}{(m+1)^{r-1}} \times \int_0^1 \left(1-\sqrt{1-v^{1/\alpha}}\right)^i v^{\frac{j+p}{\alpha}+\gamma_s+m+(s-r-1)(m+1)}\left(1-v^{m+1}\right)^{r-1}dv. \tag{3.8}$$

Making the substitution $z = v^{m+1}$ in (3.8) and then using (2.7) in resulting expression, we get

$$E[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^s} \times \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{a=0}^{s-r-1}(-1)^a\binom{s-r-1}{a}\frac{D(j,p)D(i,q)}{\left(\frac{j+p}{\alpha(m+1)}+\frac{\gamma_s}{m+1}+a\right)} \times \left(\frac{i+j+p+q}{\alpha(m+1)}+\frac{\gamma_s}{m+1}+(s-r), r\right). \tag{3.9}$$

We have Gradshteyn and Ryzhik (2007, p-6) for real positive, k, c and a positive integer, b

$$\sum_{a=0}^{b}(-1)^a\binom{b}{a}B(a+k,c) = B(k,c+b), \tag{3.10}$$

where $B(a,b)$ is the beta function.

On using (3.10) in (3.9) and then applying the well known relation between the beta and gamma functions, we get the result given in (3.1).

Now taking m tends to -1 in (3.1), we have the result in (3.2). □

3.1. Special cases

i. Setting $m = 0$ and $k = 1$ in (3.1), the explicit expression for the product moments of order statistics of the *J*-shaped distribution can be obtained as

$$E[X^i_{n-r+1:n}X^j_{n-s+1:n}] = \frac{n!}{(n-s)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}D(j,p)D(i,q) \times \frac{\Gamma\left(\frac{j+p}{\alpha}+n-s+1\right)}{\Gamma\left(\frac{j+p}{\alpha}+n-r+1\right)} \times \frac{\Gamma\left(\frac{i+j+p+q}{\alpha}+n-r+1\right)}{\Gamma\left(\frac{i+j+p+q}{\alpha}+n+1\right)}$$

or equivalently

$$E[X^i_{n-r+1:n}X^j_{n-s+1:n}] = C_{r,s:n}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{a=0}^{s-r-1}(-1)^a\binom{s-r-1}{a}\frac{D(j,p)D(i,q)}{\frac{j+p}{\alpha}+(n-s+1)+a}B\left(\frac{i+j+p+q}{\alpha}+(n-r+1), r\right).$$

That is

$$E[X^i_{r:n}X^j_{s:n}] = C_{r,s:n}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{a=0}^{s-r-1}(-1)^a\binom{s-r-1}{a} \times \frac{D(j,p)D(i,q)}{\frac{j+p}{\alpha}+r+a}B\left(\frac{i+j+p+q}{\alpha}+s, r\right),$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

as obtained by Zghoul (2010).

ii. Putting $k = 1$ in (3.2), we deduce the explicit expression for the product moments of lower record values for J-shaped distribution in the form

$$E[X^i_{L(r)} X^j_{L(s)}] = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} D(j,p)D(i,q)\left(\frac{\alpha}{j+p+\alpha}\right)^{s-r} \times \left(\frac{\alpha}{i+j+p+q+\alpha}\right)^{r}.$$

Theorem 3.2. *For the given J-shaped distribution and for,* $1 \leqslant r < s \leqslant n$, $k = 1,2,\ldots$, $i,j \geqslant 0$

$$\begin{aligned}\left(1+\tfrac{j+1}{2\alpha\gamma_s}\right)&E[X^{*i}(r,n,m,k)X^{*j+1}(s,n,m,k)]\\ &= E[X^{*i}(r,n,m,k)X^{*j+1}(s-1,n,m,k)]\\ &+(j+1)\left(\tfrac{1}{\alpha\gamma_s}+\tfrac{1}{j}\right)E[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)]\\ &-\tfrac{j+1}{j}E[X^{*i}(r,n,m,k)X^{*j}(s-1,n,m,k)].\end{aligned} \tag{3.11}$$

Proof. From (1.3), we have

$$\begin{aligned}2E[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)] &- E[X^{*i}(r,n,m,k)X^{*j+1}(s,n,m,k)]\\ &= \tfrac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^1 x^i[F(x)]^m f(x) g_m^{r-1}(F(x))G(x)dx,\end{aligned} \tag{3.12}$$

where

$$G(x) = \int_0^x y^j(2-y)[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dy. \tag{3.13}$$

Making use of relation in (1.7) and splitting the integral according with form, we have

$$G(x) = 2\alpha[G_{j-1}(x) - G_j(x)], \tag{3.14}$$

where

$$G_t(x) = \int_0^x y^t[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s}dy. \tag{3.15}$$

Integrating by parts treating y^t for integration and the rest of the integrand for differentiation yields

$$\begin{aligned}G_t(x) &= \tfrac{s-r-1}{t+1}\int_0^x y^{t+1}[h_m(F(y)) - h_m(F(x))]^{s-r-2}[F(y)]^{\gamma_s+m}f(y)dy\\ &-\tfrac{\gamma_s}{t+1}\int_0^x y^{t+1}[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dy.\end{aligned}$$

Upon substituting for $G_j(x)$ and $G_{j-1}(x)$ in Eq. (3.14) and then substituting the resulting expression for $G(x)$ in (3.12) and simplifying, we derive the relations in (3.11). □

Remark 3.1. At $j = 0$ in (3.1), we have

$$E[X^{*i}(r,n,m,k)] = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} D(0,p)D(i,q)\prod_{u=r+1}^{s}\left(\frac{\alpha\gamma_u}{p+\alpha\gamma_u}\right) \times \prod_{v=1}^{r}\left(\frac{\alpha\gamma_v}{i+p+q+\alpha\gamma_v}\right),$$

where

$D(0,p) = 1$ for,$p = 0$ and $D(0,p) = 0$ for $p > 0$.

Therefore,

$$E[X^{*i}(r,n,m,k)] = \sum_{q=0}^{\infty} D(i,q)\prod_{v=1}^{r}\left(\frac{\alpha\gamma_v}{i+q+\alpha\gamma_v}\right)$$

which is the relation for exact moment of single moment as given in (2.1).

Remark 3.2. At $i = 0$ in (3.11), the recurrence relations for product moments reduces to relations for single moments as obtained in (2.9).

Remark 3.3. Putting $m = 0$ and $k = 1$ in (3.11), we obtain the recurrence relation for the product moments of order statistics of the J-shaped distribution in the form

$$\begin{aligned}\left(1+\tfrac{j+1}{2\alpha(n-s+1)}\right)E(X^i_{n-r+1:n}X^{j+1}_{n-s+1:n}) &= E(X^i_{n-r+1:n}X^{j+1}_{n-s+2:n})\\ &+(j+1)\left(\tfrac{1}{\alpha(n-s+1)}+\tfrac{1}{j}\right)E(X^i_{n-r+1:n}X^j_{n-s+1:n})\\ &-\left(\tfrac{j+1}{j}\right)E(X^i_{n-r+1:n}X^j_{n-s+2:n}).\end{aligned}$$

That is

$$\begin{aligned}E(X^{i+1}_{r:n}X^j_{s:n}) &= \left(1+\frac{i+1}{2\alpha(r-1)}\right)E(X^{i+1}_{r-1:n}X^j_{s:n})\\ &-(i+1)\left(\frac{1}{\alpha(r-1)}+\frac{1}{i}\right)E(X^j_{r-1:n}X^j_{s:n})\\ &+\frac{i+1}{i}E(X^i_{r:n}X^j_{s:n}).\end{aligned}$$

Remark 3.4. Setting $m = -1$ and $k \geqslant 1$ in Theorem 3.2, the relation for product moments of lower k record values is deduced in the form

$$\begin{aligned}\left(1+\frac{j+1}{2\alpha k}\right)E[(Z_r^{(k)})^i(Z_s^{(k)})^{j+1}] &= E[(Z_r^{(k)})^i(Z_{s-1}^{(k)})^{j+1}]\\ &+(j+1)\left(\frac{1}{\alpha k}+\frac{1}{j}\right)E[(Z_r^{(k)})^i(Z_s^{(k)})^j]\\ &-\frac{j+1}{j}E[(Z_r^{(k)})^i(Z_{s-1}^{(k)})^j]\end{aligned}$$

and hence for lower records ($k = 1$)

$$\begin{aligned}\left(1+\frac{j+1}{2\alpha}\right)E(X^i_{L(r)}X^{j+1}_{L(s)}) &= E(X^i_{L(r)}X^{j+1}_{L(s-1)})\\ &+(j+1)\left(\frac{1}{\alpha}+\frac{1}{j}\right)E(X^i_{L(r)}X^j_{L(s)})\\ &-\frac{j+1}{j}E(X^i_{L(r)}X^{j+1}_{L(s-1)}).\end{aligned}$$

4. Characterization

Let, $X^*(r,n,m,k) = 1,2,\ldots,n$ be *dgos* from a continuous distribution with *df* $F(x)$ and, *pdf* $f(x)$ then the conditional *pdf* of $X^*(s,n,m,k)$ given, $X^*(r,n,m,k) = x$, $1 \leqslant r < s \leqslant n$ is

$$f_{X^*(s,n,m,k)|X^*(r,n,m,k)}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}}[F(x)]^{m-\gamma_{r+1}}[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_{s}-1}f(y), \quad y < x, \quad m \neq -1 \tag{4.1}$$

$$f_{Z_s^{(k)}|Z_r^{(k)}}(y|x) = \frac{k^{s-r}}{(s-r-1)!}[\ln F(x) - \ln F(y)]^{s-r-1}\left(\frac{F(y)}{F(x)}\right)^{k-1}\frac{f(y)}{F(x)}, \quad y < x, \quad m = -1 \tag{4.2}$$

Theorem 4.1. *Let X be a non-negative random variable having an absolutely continuous df $F(x)$ with $F(0) = 0$ and $0 \leqslant F(x) \leqslant 1$ for all, $x > 0$ then*

$$E[\xi\{X^*(s,n,m,k)\}|X^*(l,n,m,k) = x] = [x(2-x)]^{\alpha}\prod_{j=1}^{s-l}\left(\frac{\gamma_{l+j}}{\gamma_{l+j}+1}\right), \quad l = r, r+1, \quad m \neq -1 \tag{4.3}$$

$$E[\xi(Z_s^{(k)})|Z_l^{(k)} = x] = [x(2-x)]^{\alpha}\left(\frac{k}{k+1}\right)^{s-l}, \quad l = r, r+1, \quad m = -1 \tag{4.4}$$

if and only if

$$F(x) = [x(2-x)]^{\alpha}, 0 \leqslant x \leqslant 1, 0 < \alpha < 1, \tag{4.5}$$

where

$$\xi(y) = [y(2-y)]^{\alpha}.$$

Proof. When, $m \neq -1$ we have from (4.1) for, $s > r+1$

$$E[\xi X^*(s,n,m,k)|X^*(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \times \int_0^x [y(2-y)]^{\alpha}\left(\frac{F(y)}{F(x)}\right)^{\gamma_{s-1}}\left[1 - \left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1} \times \frac{f(y)}{F(x)}dy. \tag{4.6}$$

By setting $u = \frac{F(y)}{F(x)} = \left(\frac{y(2-y)}{x(2-x)}\right)^{\alpha}$ from (1.6) in (4.6), we obtain

$$E[\xi\{X^*(s,n,m,k)\}|X^*(r,n,m,k) = x] = \frac{C_{s-1}[x(2-x)]^{\alpha}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_0^1 u^{\gamma_s}(1-u^{m+1})^{s-r-1}du. \tag{4.7}$$

Again by setting, $t = u^{m+1}$ in (4.7), we get

$$E[\xi\{X^*(s,n,m,k)\}|X^*(r,n,m,k) = x] = \frac{C_{s-1}[x(2-x)]^{\alpha}}{(s-r-1)!C_{r-1}(m+1)^{s-r}}\int_0^1 t^{\frac{k+1}{m+1}+n-s-1}(1-t)^{s-r-1}dt$$

and hence the necessary part given in (4.3).

To prove sufficient part, we have from (4.1) and (4.3)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \times \int_0^x [y(2-y)]^{\alpha}[(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dy = g_{s|r}(x)[F(x)]^{\gamma_{r+1}}, \tag{4.8}$$

where

$$g_{s|r}(x) = [x(2-x)]^{\alpha}\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+1}\right).$$

Differentiating (4.8) both the sides with respect to, x we get

$$\frac{C_{s-1}[F(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}} \times \int_0^x [y(2-y)]^{\alpha}[(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-2}[F(y)]^{\gamma_s-1}f(y)dy = g'_{s|r}(x)[F(x)]^{\gamma_{r+1}} + \gamma_{r+1}g_{s|r}(x)[F(x)]^{\gamma_{r+1}-1}f(x)$$

or

$$\gamma_{r+1}g_{s|r+1}(x)[F(x)]^{\gamma_{r+2}+m}f(x) = g'_{s|r}(x)[F(x)]^{\gamma_{r+1}} + \gamma_{r+1}g_{s|r}(x)[F(x)]^{\gamma_{r+1}-1}f(x).$$

Therefore,

$$\frac{f(x)}{F(x)} = \frac{g'_{s|r}(x)}{\gamma_{r+1}[g_{s|r+1}(x) - g_{s|r}(x)]} = \frac{2\alpha(1-x)}{x(2-x)},$$

which proves that

$$F(x) = [x(2-x)]^{\alpha}, 0 \leqslant x \leqslant 1, 0 < \alpha < 1.$$

For the case when, $m = -1$ from (4.2) by using the transformation, $u = \frac{F(y)}{F(x)} = \left(\frac{y(2-y)}{x(2-x)}\right)^{\alpha}$ we obtain

$$E[\xi(Z_s^{(k)})|Z_r^{(k)} = x] = \frac{k^{s-r}[x(2-x)]^{\alpha}}{(s-r-1)!}\int_0^1 u^k(-\ln u)^{s-r-1}du. \tag{4.9}$$

We have Gradshteyn and Ryzhik (2007, p-551)

$$\int_0^1 (-\ln x)^{\mu-1}x^{\upsilon-1}dx = \frac{\Gamma\mu}{\upsilon^{\mu}}, \quad \mu > 0 \quad \upsilon > 0. \tag{4.10}$$

On using (4.10) in (4.9), we have the result given in (4.4).

Sufficiency part can be proved on the lines of case $m \neq -1$. □

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