



Original article

On the comparative study integro – Differential equations using difference numerical methods

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ABSTRACT

In this paper, we use the Adomian decomposition Sumudu transform method with the Pade approximant (ADST - PA method) to obtain closed form solutions of nonlinear integro-differential equations, and perform a comparative study between the present method and three different numerical methods, namely; the Adomian decomposition Sumudu transform method (ADSTM), the homotopy perturbation method (HPM), and the variational iteration method (VIM). Our results show that in comparison with other existing methods, the (ADST - PA method) gives a better approximation for solving a large number of nonlinear integro-differential equations than existing methods.

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1. Introduction

Ever since long time, integro differential equations have played an important part in all facets of science applications such as ice-shaping operation, heat transformer, neutron diffusion, and biological species coexisting together with increasing and falling rates of generating and diffusion process in general. In addition, it also can be found in applied mathematics, physics, and engineering applications, as well as this in models dealing with advanced integral equations such as (Golberg, 1979; Jerri, 1971; Kanwal, 1971; Miller, 1967).

The homotopy perturbation method (HPM) was a consequence of some pioneering ideas beginning in 1999 by His (He, 1999). Since then it has evolved into a fully -fledged theory, which was the contribution from many researchers. The HPM method was found to be an uncomplicated and accurate method to resolve a great number of nonlinear problems.

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Many sources for different cases have obtained some exact and numerical solutions of the integro-differential equation (see Avudainayagam and Vani, 2000; Bahugna et al., 2009; Abu Arqub and Al-samdi, 2013; Abu Arqub and Al-samdi, 2014; Momani et al., 2014; El-ajou et al., 2012; Eltayeb et al., 2014; Ahmed and Elzaki, 2013).

In the present study, we consider the nonlinear integro- differential equation of the following type:

$$V'(x) = f(x) + \int_0^x K(t, v(t), v'(t))dt, \quad (1)$$

with the initial condition;

$$v(0) = \alpha, \quad 0 \leq x \leq 1; \quad (2)$$

where $f(x)$ is known as the source term and $K(t, v(t), v'(t))$ is a linear or nonlinear function depending on the problem discussed.

The main objective of this paper is to introduce a comparative study to solve integro-differential equation (1) by using four of the most recently developed methods, namely, the Adomian decomposition Sumudu transform method (ADSTM), the homotopy perturbation method (HPM), the variational iteration method (VIM) and the Adomian decomposition Sumudu transform method with Pade approximant (ADSTM-PA).

This paper is orderly as follows. in Section 2, we show all the numerical methods in short. Two numerical experiences are intro-

duced in Section 3 for demonstrating the complete research. Concluding remarks are given in the last section.

2. Analysis of numerical methods

2.1. The Adomian decomposition Sumudu transform method (ADSTM)

In this section, Adomian decomposition Sumudu transform method (Kumar et al., 2012) is applied to the following classes of non-linear integro-differential equation (1).

The method depends of first applying the Sumudu transformation to both sides of Eq. (1);

$$S[v'(x)] = S[f(x)] + S\left[\int_0^x K(t, v(t), v'(t))dt\right]. \quad (3)$$

Using the formulas of the Sumudu transform, we get:

$$u^{-1} S[v(x)] - u^{-1} v(0) = S[f(x)] + S\left[\int_0^x K(t, v(t), v'(t))dt\right]. \quad (4)$$

Using the initial condition (2), we have:

$$S[v(x)] = \alpha + uS[f(x)] + uS\left[\int_0^x K(t, v(t), v'(t))dt\right] \quad (5)$$

If we apply the inverse operator S^{-1} to both sides of the Eq. (5), we obtain:

$$v(x) = \alpha + S^{-1}[uS[f(x)]] + S^{-1}\left\{uS\left[\int_0^x K(t, v(t), v'(t))dt\right]\right\}. \quad (6)$$

In the Adomian decomposition Sumudu transform method we assume the solution as an infinite series, given as follows:

$$v = \sum_{n=0}^{\infty} v_n, \quad (7)$$

where the terms v_n are to be recursively computed. Also, the nonlinear term $K(t, v(t), v'(t))$ is decomposed as an infinite series of Adomian polynomials (see Adomian, 1990, 1984, 1986):

$$K(t, v(t), v'(t)) = \sum_{n=0}^{\infty} A_n, \quad (8)$$

where $A_n = A_n(v_1, v_2, v_3, \dots, v_n)$ are determined by the following recursive relation:

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left[\sum_{i=0}^{\infty} (\lambda^i v_i) \right] \right]_{\lambda=0}. \quad (9)$$

Using (7) and (8), we rewrite (6) as:

$$\sum_{n=0}^{\infty} v_n(x) = \alpha + S^{-1}[uS[f(x)]] + S^{-1}\left\{uS\left[\int_0^x \sum_{n=0}^{\infty} A_n dt\right]\right\}. \quad (10)$$

Applying the linearity of the Sumudu transform, we have:

$$\sum_{n=0}^{\infty} v_n(x) = \alpha + S^{-1}[uS[f(x)]] + S^{-1}\left\{u\left[\int_0^x S\left[\sum_{n=0}^{\infty} A_n\right]dt\right]\right\}. \quad (11)$$

Now we define the following iterative algorithm:

$$v_0 = \alpha + S^{-1}[uS[f(x)]], \quad (12)$$

$$v_1 = S^{-1}\left\{u\left[\int_0^x S[A_0]dt\right]\right\}. \quad (13)$$

In general,

$$v_{k+1} = S^{-1}\left\{u\left[\int_0^x S[A_k]dt\right]\right\}, \quad k \geq 1. \quad (14)$$

As the result, the components $(v_1, v_2, v_3, \dots, v_n)$ are identified and the series solution is thus entirely determined.

However, in many cases the exact solution in the closed form may also be obtained.

2.2. The Pade approximant

Here we will investigate the construction of the Pade approximates for the functions studied. The main advantage of the Pade approximation gives a better approximation of the function than truncating its Taylor series. The Pade approximation of a function is given by the ratio of two polynomials. The coefficients of the polynomial in both the numerator and the denominator are determined by using the coefficients in the Taylor series expansion of the function.

The Pade approximation of a function, symbolized by $[m/n]$, is a rational function defined by;

$$[m/n] = \frac{a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m}{1 + b_1 x + b_2 x^2 + \dots + b_n x^n} \quad (15)$$

where we considered $b_0 = 1$, and numerator, denominator have no common factors.

In The (ADSTM-PA) we use the method of the Pade approximation as an after – treatment method to the solution obtained by the (ADSTM). This after – treatment method improves the accuracy of the proposed method.

2.3. Basic idea of the (HPM)

To explain (HPM), we consider (1) as:

$$L(v) = v'(x) - f(x) - \int_0^x K(t, v(t), v'(t))dt = 0, \quad (16)$$

with solution $f(x)$. Now, we can define homotopy $H(v, p)$ by,

$$H(v, 0) = F(v), \quad H(v, 1) = L(v), \quad (17)$$

where $F(v)$ is a functional operator with a solution r_0 , obtained easily. Now, we choose a convex homotopy from:

$$H(v, p) = (1 - p)F(v) + pL(v) = 0 \quad (18)$$

and continuously trace an implicitly defined curve from a starting point $H(r_0, 0)$ to a solution function $H(f, 1)$. Here the parameter p is monotonically increasing from zero to unit along – with the trivial problem $F(v) = 0$ is continuously deformed to the original problem $L(v) = 0$.

The (HPM) uses the homotopy parameter p as an expending parameter to obtain (see Nayfeh, 1985),

$$v = r_0 + pr_1 + p^2 r_2 + p^3 r_3 + \dots \quad (19)$$

when $p \rightarrow 1$, Eq. (17), becomes the approximate solution of (16), i.e.

$$f = \lim_{p \rightarrow 1} r_0 + r_1 + r_2 + \dots \quad (20)$$

Series (20) is convergent for most cases, and the rate of convergence depends on $L(v)$.

2.4. Basic idea of the (VIM)

To clarify the basic ideas of (VIM) (Abassya et al., 2007), we consider Eq. (1) as correction functional as follows;

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\zeta) ((v'_n(\zeta) - f(\zeta)) - \int_0^\zeta K(r, v(r), v'(r)) dr) d\zeta. \quad (21)$$

where λ is general Lagrange multiplier which can be identified optimally via integrated by parts. we used here $\lambda = -1$ for first order integro- differential equation, substituting this value of the Lagrange multiplier into the functional (21), gives the iteration formula:

$$v_{n+1}(x) = v_n(x) - \int_0^x (v'_n(\zeta) - f(\zeta)) - \int_0^\zeta K(r, v(r), v'(r)) dr d\zeta. \quad (22)$$

we can use the initial condition to select $v_0(x) = v(0) = \alpha$. Using this selection into above the correction functional, gives the following successive approximations:

$$v_0(x) = v(0) = \alpha, \quad (23)$$

$$v_{n+1}(x) = v_n(x) - \int_0^x (v'_n(\zeta) - f(\zeta)) - \int_0^\zeta K(r, v(r), v'(r)) dr d\zeta. \quad (24)$$

Consequently, the exact solution may be obtained by using:

$$v(x) = \lim_{n \rightarrow \infty} v_n. \quad (25)$$

3. Application

In this section, we demonstrate the analysis of all the numerical methods by applying the methods to the following two integro-differential equations. A comparison of all methods is also given in the forms of graphs and tables, presented here.

Example 1: Consider the following integro- differential equation:

$$v'(x) = -1 + \int_0^x v^2(t) dt, \quad (26)$$

With the initial condition;

$$v(0) = 0, \quad 0 \leq x \leq 1. \quad (27)$$

Solution: Taking the Sumudu transform on both the sides of (26) gives:

$$u^{-1} S[v(x)] - u^{-1} v(0) = -1 + S \left[\int_0^x v^2(t) dt \right]. \quad (28)$$

Using the initial condition (27) implies,

$$S[v(x)] = -u + uS \left[\int_0^x v^2(t) dt \right]. \quad (29)$$

If we apply the inverse operator S^{-1} to both sides of the Eq. (29), we obtain:

$$v(x) = S^{-1}[-u] + S^{-1} \left\{ uS \left[\int_0^x v^2(t) dt \right] \right\}. \quad (30)$$

By the assumption (7) and (8), we rewrite (30) as,

$$\sum_{n=0}^{\infty} v_n(x) = S^{-1}[-u] + S^{-1} \left\{ uS \left[\int_0^x \sum_{n=0}^{\infty} A_n dt \right] \right\} \quad (31)$$

where the nonlinear term $K(t, v(t), v'(t)) = v^2$ is decomposed in terms of the Adomian polynomials as suggested in (8). Few terms of the Adomian polynomials for v^2 are given as follows:

$$\begin{aligned} A_0 &= v_0^2, \\ A_1 &= 2 v_0 v_1, \\ A_2 &= 2 v_0 v_2 + v_1^2, \\ A_3 &= 2 v_0 v_3 + 2 v_1 v_2 \end{aligned}$$

And so on. Following the Adomian decomposition Sumudu transform method, we define an iterative scheme,

$$v_0 = S^{-1}[-u], \quad (32)$$

$$v_{k+1} = S^{-1} \left\{ u \left[\int_0^x S[A_k] dt \right] \right\}, \quad k \geq 0. \quad (33)$$

Consequently, we obtain:

$$v_0 = S^{-1}[-u] = -x, \quad (34)$$

$$v_1 = S^{-1} \left[uS \left[\int_0^x A_0 dt \right] \right] = \frac{x^4}{12}, \quad (35)$$

$$v_2 = S^{-1} \left[uS \left[\int_0^x A_1 dt \right] \right] = -\frac{x^7}{252}, \quad (36)$$

$$v_3 = S^{-1} \left[uS \left[\int_0^x A_2 dt \right] \right] = \frac{x^{10}}{6048}, \quad (37)$$

$$v_4 = S^{-1} \left[uS \left[\int_0^x A_3 dt \right] \right] = -\frac{x^{13}}{157248}. \quad (38)$$

Similarly, we can also find other components. Finally, the solution takes the following form:

$$v(x) = v_0 + v_1 + v_2 + v_3 + \dots, \quad (39)$$

or,

$$v(x) = -x + \frac{x^4}{12} - \frac{x^7}{252} + \frac{x^{10}}{6048} - \frac{x^{13}}{157248} + \dots \quad (40)$$

Notes on (ADSTM):

From the previous analysis, we can observe that:

- (ADSTM) can obtain a series solution, not converge, which must be truncated. The truncated series solution is an inaccurate solution in that region, which will greatly restrict the application area of the method.
- (ADSTM) needs some modification to overcome the Taylor series does not converge.

To overcome these disadvantages of ADSTM, the following ADSTM - PA method is suggested.

The [m/n] Pade approximant of the infinite series (40), with $m \geq 4$ and $n \geq 4$, which gives the following rational fraction approximation to the solution:

$$v(x) = \frac{-x + \frac{x^2}{28}}{1 + \frac{x^3}{21}}. \quad (41)$$

Example 2: Consider the following integro- differential equation (Figs. 1 and 2):

$$v'(x) = 1 + \int_0^x v(t) v'(t) dt, \quad (42)$$

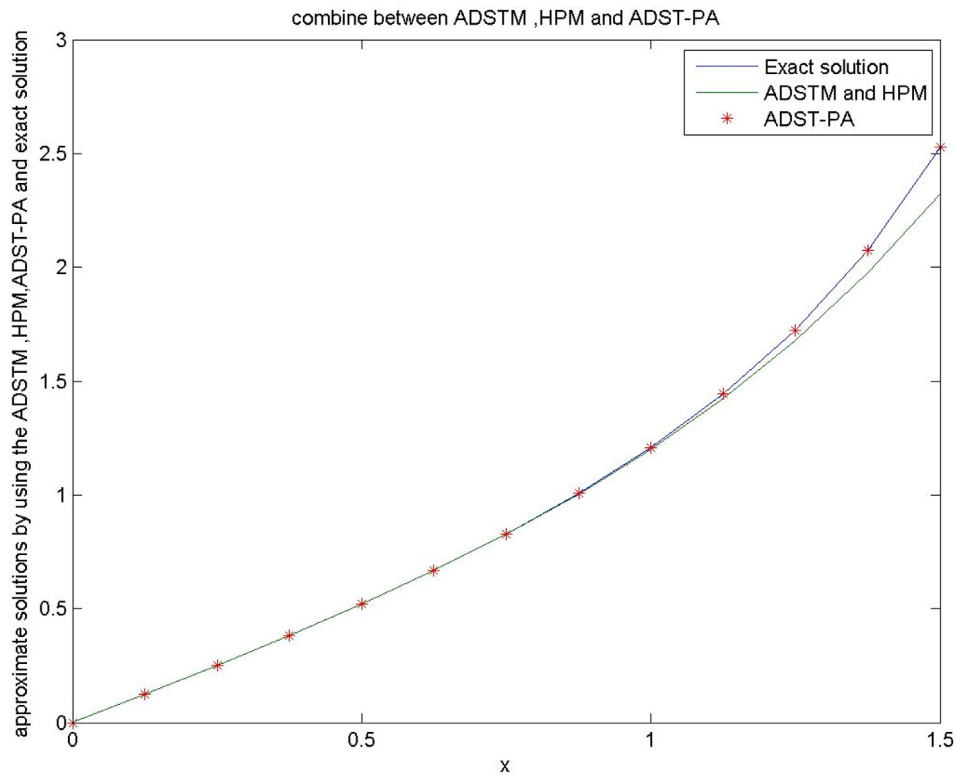


Fig. 1. Comparison between (ADSTM), (HPM) and (ADST-PA), for Example 2.

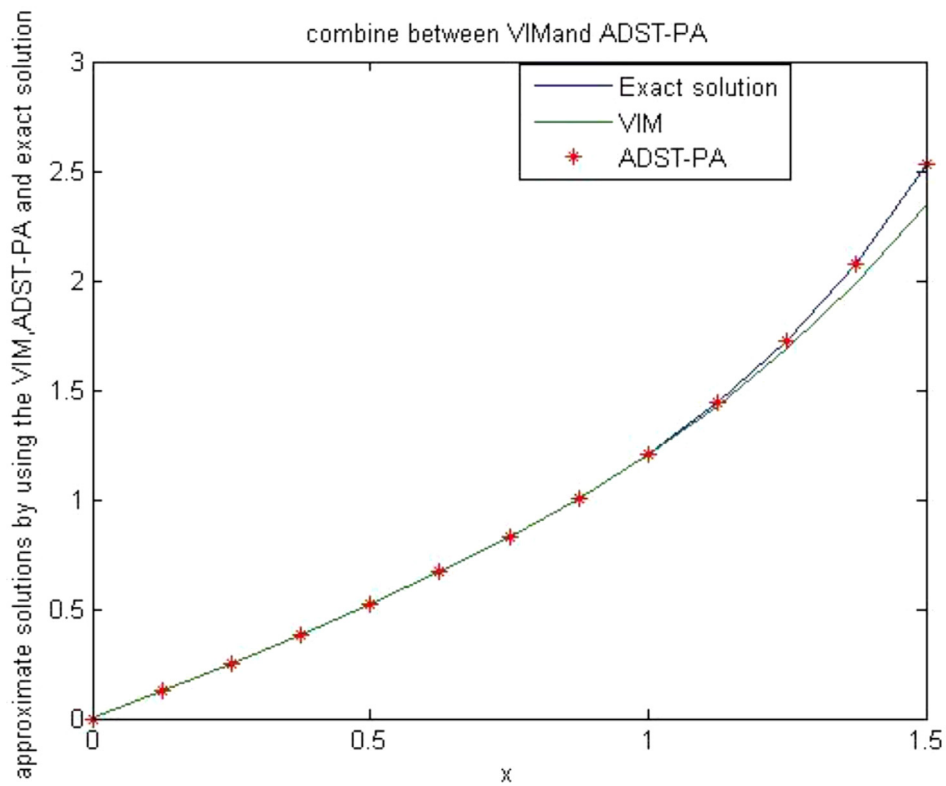


Fig. 2. Comparison between (ADST-PA) and (VIM), for Example 2.

Table 1
Comparison between (ADSTM), (HPM) and (ADST - PA method), for Example 2.

Step Size	Exact Sol.	Error(ADSTM)	Error (ADST-PA)	Error (HPM)
0.0000	0.0000	0.0000	0.0000	0.0000
0.1250	0.1253	0.0000	0.0000	0.0000
0.2500	0.2526	0.0000	0.0000	0.0000
0.3750	0.3840	0.0000	0.0000	0.0000
0.5000	0.5219	0.0001	0.0000	0.0001
0.6250	0.6691	0.0003	0.0000	0.0003
0.7500	0.8292	0.0010	0.0000	0.0010
0.8750	1.0069	0.0030	0.0000	0.0030
1.0000	1.2085	0.0081	0.0000	0.0081
1.1250	1.4431	0.0198	0.0000	0.0198
1.2500	1.7243	0.0452	0.0000	0.0452
1.3750	2.0737	0.0979	0.0000	0.0979
1.5000	2.5275	0.2051	0.0001	0.2051

Table 2
Comparison between (ADST - PA method) and (VIM), for Example 2.

Step Size	Exact Sol.	Error (ADST-PA)	Error (VIM)
0.0000	0.0000	0.0000	0.0000
0.1250	0.1253	0.0000	0.0000
0.2500	0.2526	0.0000	0.0000
0.3750	0.3840	0.0000	0.0000
0.5000	0.5219	0.0000	0.0000
0.6250	0.6691	0.0000	0.0000
0.7500	0.8292	0.0000	0.0001
0.8750	1.0069	0.0000	0.0002
1.0000	1.2085	0.0000	0.0009
1.1250	1.4431	0.0000	0.0029
1.2500	1.7243	0.0000	0.0087
1.3750	2.0737	0.0000	0.0242
1.5000	2.5275	0.0001	0.0644

Given the initial condition;

$$v(0) = 0 \text{ , } 0 \leq x \leq . \tag{43}$$

With the exact solution:

$$v(x) = \sqrt{2} \tan \left(\frac{x}{\sqrt{2}} \right). \tag{44}$$

Solution: Proceeding as in Example 1, Eq. (42) becomes:

$$\sum_{n=0}^{\infty} v_n(x) = S^{-1}[u] + S^{-1} \left\{ uS \left[\int_0^x \sum_{n=0}^{\infty} B_n dt \right] \right\}. \tag{45}$$

where the nonlinear term $K(t, v(t), v'(t)) = v(t)v'(t)$ is decomposed in terms of the Adomian polynomials as suggested in (8). We have a few terms of the Adomian polynomials of $v(t)v'(t)$ which are given by:

$$\begin{aligned} B_0 &= v_0 v_{0t} \text{ ,} \\ B_1 &= v_0 v_{1t} + v_1 v_{0t} \text{ ,} \\ B_2 &= v_0 v_{2t} + v_1 v_{1t} + v_2 v_{0t} \text{ ,} \\ B_3 &= v_0 v_{3t} + v_3 v_{0t} + v_1 v_{2t} + v_2 v_{1t} . \end{aligned}$$

And so on. Following the Adomain decomposition Sumudu transform method, we define an iterative scheme;

$$v_0 = S^{-1}[u] \tag{46}$$

$$v_{k+1} = S^{-1} \left\{ u \left[\int_0^x S[B_k]dt \right] \right\} \text{ , } k \geq 0. \tag{47}$$

Now, we obtain the following components:

$$v_0 = S^{-1}[u] = x, \tag{48}$$

$$v_1 = S^{-1} \left[uS \left[\int_0^x B_0 dt \right] \right] = \frac{x^3}{6} \tag{49}$$

$$v_2 = S^{-1} \left[uS \left[\int_0^x B_1 dt \right] \right] = \frac{x^5}{30}, \tag{50}$$

$$v_3 = S^{-1} \left[uS \left[\int_0^x B_2 dt \right] \right] = \frac{17x^7}{2520}, \tag{51}$$

Similarly, we can also find other components, then the solution takes the following form;

$$v(x) = v_0 + v_1 + v_2 + v_3 + \dots , \tag{52}$$

or

$$v(x) = x + \frac{x^3}{6} + \frac{x^5}{30} + \frac{17x^7}{2520} + \dots \tag{53}$$

The [m/n] Pade approximant of the infinite series (53), with $m \geq 4$ and $n \geq 4$, which gives the following rational fraction approximation to the solution:

$$v(x) = \frac{x - \frac{x^3}{21}}{1 - \frac{3x^2}{14} + \frac{x^4}{420}}. \tag{54}$$

Numerical outcomes shown in Tables 1 and 2 and Figs. 1 and 2 illustrate the importance of (ADST - PA method) over other numerical methods.

4. Concluding remarks

In this report, we have examined a few recent familiar numerical methods for solving integro-differential equations. The numerical studies showed that all the methods give highly accurate solutions for given equations. The (ADSTM), the (HPM) and the (VIM) are uncomplicated and comfortable. Despite this, they are not converging to a closed form. Since the method of the (ADSTM) is based on an approximation of the solution function in this study by the truncating of approximation the solution, this kind of approximation is an inaccurate solution, which will greatly restrict the application area of the method. To get to the more dependable of these demerits, we use the Pade approximations. This fact is indicated in the second example.

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