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# Parameter estimation of Weibull-Exponential distribution under Type-I hybrid censored sample

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ABSTRACT

We have studied the problem of estimation of unknown parameters of a Weibull-Exponential distribution (WED) under Type-I hybrid censored data in this paper. The maximum likelihood estimates (MLE's) are derived. Using asymptotic distributions property of MLE's, we constructed asymptotic interval estimates. We derived Bayes estimates based on the squared error loss function using the Lindley method and Metropolis-Hastings (M-H) algorithm. Highest posterior density (HPD) credible intervals are also obtained. In each case, to compare the proposed estimates using simulations and illustrative example is also presented a numerical study is performed. Two data analysis has been performed for illustrative purposes.



## 1. Introduction

In life testing experiments we can not studied the complete data. The most common censoring schemes are Type-I and Type-II censoring in the literature of life testing experiment. Lawless (1982) discussed about these two censoring schemes in detail. Hybrid censoring is the combination of Type-I and Type-II censoring schemes which first introduced by Epstein (1954). In Type-I hybrid censored data, the test is continued until a pre specified number of units, say $r$, have failed or a predetermined time $T$ has been reached. Or we can sat that in Type-I hybrid censoring scheme a life test is terminated at a random time $T_1$ with $T_1 = \min(X_{r:n}, T)$ where $r, (1 \leqslant r \leqslant n)$ and $T(>0)$ are prefixed before the test starts. Hybrid censoring is very beneficial in survival analysis, the limited attention has been paying in analyzing hybrid censored data. A huge literature exists for different lifetime distributions by considering hybrid censored data. One may refer to the review article by Balakrishnan and Kundu (2013). In this review article, the authors described in detail about hybrid censoring and on some of its generalizations. The reference which are cited in this review article are used to study in detail on hybrid censoring scheme. Some recent works on hybrid censoring scheme are Rastogi and Tripathi (2013a, 2013b).

This paper consider, estimation the unknown parameters of a WED based on Type-I hybrid censored data from both classical and Bayesian sight. For a random variable denoted by X, the densities of a three-parameter WED are given by:

$$f_X(x) = \alpha\beta\theta e^{\theta x}(e^{\theta x}-1)^{\beta-1} e^{-\alpha(e^{\theta x}-1)^{\beta}}, \quad x>0, \tag{1.1}$$

$$F_X(x) = 1 - e^{-\alpha(e^{\theta x}-1)^{\beta}}, \quad x>0, \tag{1.2}$$

where $\alpha > 0, \beta > 0$ and $\theta > 0$. $f_X(x)$ and $F_X(x)$ are the probability density and the cumulative distribution functions respectively. For notation purpose, one can say; $X \sim WE(\alpha, \beta, \theta)$.

In literature the three parameter WED was introduced by Oguntunde et al. (2015). It has unimodal and decreasing shapes, various mathematical and structural properties of the distribution has be established and MLE's of model parameters are obtained. The authors discussed two real data set to illustrate its flexibility and potentiality over the exponential distribution. Rastogi (2017) investigated properties of MLE's and Bayesian estimates of unknown parameter and reliability characteristic of a WED using progressive Type-II censoring scheme.

The problem of classical and Bayesian estimation of WED has yet not been studied by any author under hybrid Type-I censored information till now. This paper is written and outlined in the following manner: In the starting, we discussed about WED and hybrid censoring scheme in detail. In Section 2, The maximum likelihood estimates of unknown parameters are obtained. In Section 3, The Bayesian estimates for the unknown parameters are derived

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under squared error loss function. The M-H algorithm and Lindley approximation method are applied for this purpose. The HPD intervals are constructed for the unknown parameters. Provided in Section 4 is a numerical study conducted between the proposed estimates. Two real life applications are provided in Section 5 followed by a concluding remark.

## 2. The Maximum Likelihood Estimation

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be an ordered sample of $n$ independent units obtained from a WED as defined in (1.1). We observe that hybrid censored observations may occur in one of the following types:

$$\begin{cases} I : \{X_{1:n}, X_{2:n}, \ldots, X_{r:n}\}, & \text{if} X_{r:n} < T, \\ II : \{X_{1:n}, X_{2:n}, \ldots, X_{m:n}\}, & \text{if } m < r, X_{m+1:n} > T. \end{cases}$$

In case II $m$ failures have recorded up to time $T$ while $(m+1)th$ failure obtain after $T$. The likelihood of Type-I hybrid censored sample can be written in this form

$$\begin{cases} I : L(\alpha,\beta,\theta) \propto \prod_{i=1}^{r} f(x_{i:n})[1-F(x_{r:n})]^{(n-r)}, \\ II : L(\alpha,\beta,\theta) \propto \prod_{i=1}^{m} f(x_{i:n})[1-F(T)]^{(n-m)}. \end{cases}$$

The likelihood of $\alpha, \beta$ and $\theta$ for the model (1.1) can be described as,

$$L(\alpha,\beta,\theta) \propto \alpha^d \beta^d \theta^d e^{\theta \sum_{i=1}^{d} x_{i:n}} e^{-\alpha \sum_{i=1}^{d} (e^{\theta x_{i:n}}-1)^{\beta}} e^{-\alpha(n-d)(e^{\theta c}-1)^{\beta}} \prod_{i=1}^{d} (e^{\theta x_{i:n}}-1)^{\beta-1}, \tag{2.1}$$

with $d$ and $c$ defined as

$$d = \begin{cases} r, & \text{for case } I, \\ m, & \text{for case } II, \end{cases} \quad c = \begin{cases} x_{r:n}, & \text{for case } I, \\ T, & \text{for case } II. \end{cases}$$

and corresponding log likelihood function is

$$L(\underline{x}) \propto d\log\alpha + d\log\beta + d\log\theta + \theta\sum_{i=1}^{d} x_i - \alpha\sum_{i=1}^{d}(e^{\theta x_i}-1)^{\beta} - \alpha(n-d)(e^{\theta c}-1)^{\beta} + (\beta-1)\sum_{i=1}^{d}\log(e^{\theta x_i}-1) \tag{2.2}$$

where $x_{i:n} = x_i$ and $\underline{x} = (x_1, x_2, \ldots, x_d)$. The corresponding likelihood equations are calculated as

$$\frac{\partial \log L}{\partial \alpha} = \frac{d}{\alpha} - \sum_{i=1}^{d}(e^{\theta x_i}-1)^{\beta} - (n-d)(e^{\theta c}-1)^{\beta} = 0 \tag{2.3}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{d}{\beta} + \sum_{i=1}^{d}\left(1-\alpha(e^{\theta x_i}-1)^{\beta}\right)\log(e^{\theta x_i}-1) - \alpha(n-d)(e^{\theta c}-1)^{\beta}\log(e^{\theta c}-1) = 0 \tag{2.4}$$

$$\frac{\partial \log L}{\partial \theta} = \frac{d}{\theta} + \sum_{i=1}^{d} x_i - \alpha\beta\sum_{i=1}^{d} x_i e^{\theta x_i}(e^{\theta x_i}-1)^{\beta-1} - \alpha\beta c(n-d)e^{\theta c}(e^{\theta c}-1)^{\beta-1} + (\beta-1)\sum_{i=1}^{d}\frac{x_i e^{\theta x_i}}{e^{\theta x_i}-1} = 0. \tag{2.5}$$

$$\pi(\alpha,\beta,\theta|\underline{x}) \propto \alpha^{d+a_1-1}\beta^{d+a_2-1}\theta^{d+a_3-1}e^{-b_2\beta} e^{-\theta(b_3-\sum_{i=1}^{d} x_i)} e^{(\beta-1)\sum_{i=1}^{d}\log(e^{\theta x_i}-1)} e^{-\alpha(b_1+\sum_{i=1}^{d}(e^{\theta x_i}-1)^{\beta}+(n-d)(e^{\theta c}-1)^{\beta})} \tag{3.1}$$

After solving the above likelihood Eqs. (2.3), (2.4) and (2.5) numerically, we derived the corresponding MLE's $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ of $\alpha, \beta$ and $\theta$. It is clear that the system cannot be solved in closed form. It has to be solved by using numerical method such as Newton Raphson method procedure using any programming software like Matlab or R.

The asymptotic variance–covariance matrix of $\hat{\gamma}$ can be calculated by $[I_X(\hat{\gamma})]^{-1}$. The corresponding $100(1-\xi)\%, 0 < \xi < 1$, asymptotic confidence intervals can be constructed using normality property of MLEs.

### 2.1. Asymptotic confidence interval

In this subsection, the asymptotic distribution of the MLE of $\gamma = (\alpha,\beta,\theta)$ are obtained as $((\hat{\gamma}-\gamma)) \to N(0, I^{-1}(\gamma))$, where $I^{-1}(\gamma)$ is the variance–covariance matrix and given below

$$I_0^{-1} = \begin{pmatrix} -\frac{\partial^2 \log l}{\partial\alpha^2} & -\frac{\partial^2 \log l}{\partial\alpha\partial\beta} & -\frac{\partial^2 \log l}{\partial\alpha\partial\theta} \\ -\frac{\partial^2 \log l}{\partial\beta\partial\alpha} & -\frac{\partial^2 \log l}{\partial\beta^2} & -\frac{\partial^2 \log l}{\partial\beta\partial\theta} \\ -\frac{\partial^2 \log l}{\partial\theta\partial\alpha} & -\frac{\partial^2 \log l}{\partial\theta\partial\beta} & -\frac{\partial^2 \log l}{\partial\theta^2} \end{pmatrix}_{(\hat{\alpha},\hat{\beta},\hat{\theta})} = \begin{pmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) & cov(\hat{\alpha},\hat{\theta}) \\ cov(\hat{\beta},\hat{\alpha}) & var(\hat{\beta}) & cov(\hat{\beta},\hat{\theta}) \\ cov(\hat{\theta},\hat{\alpha}) & cov(\hat{\theta},\hat{\beta}) & var(\hat{\theta}) \end{pmatrix}$$

The pivotal quantity $\frac{\hat{\gamma}-\gamma}{\sqrt{Var(\hat{\gamma})}}$ is approximately distributed as standard normal. The symmetric $100(1-\tau)\%$ approximate CI for the parameter $\gamma$ is calculated as $\hat{\gamma} \pm Z_{\tau/2}\sqrt{Var(\hat{\gamma})}$ where $Z_{\tau/2}$ is the $(\tau/2)^{th}$ denotes the upper $\frac{\xi}{2}$th percentile of the standard normal distribution.

## 3. Bayesian estimation

In this section, we obtain the Bayes estimates of the unknown parameters using squared error loss function (quadratic loss), which is symmetrical, and associates equal importance to the losses due to overestimation and underestimation of equal magnitude. $\tilde{\eta}$ is the estimator to estimate parameter $\eta$. The square error loss function is given as $L(\eta,\tilde{\eta}) = (\tilde{\eta}-\eta)^2$. The Bayes estimator under square error loss function is the posterior mean $\tilde{\eta}$ of $\eta$.

Suppose all the unknown parameters are stochastically independent. Assume that the prior densities for the parameter $\alpha$, is taken to be a $Gamma\,(a_1, b_1)$, the parameter $\beta$, the prior distribution is taken to be a $Gamma\,(a_2, b_2)$ and the parameter $\theta$, the prior distribution is taken to be a $Gamma\,(a_3, b_3)$. Hence, the joint prior distribution for $\alpha, \beta$ and $\theta$ is

$$g(\alpha,\beta,\theta) \propto \alpha^{a_1-1}\beta^{a_2-1}\theta^{a_3-1}e^{-\alpha b_1}e^{-\beta b_2}e^{-\theta b_3}, \quad 0<\alpha<\infty, 0<\beta<\infty, 0<\theta<\infty, a_1, b_1, a_2, b_2, a_3, b_3 > 0,$$

The joint posterior distribution is written as,

where $\underline{x} = (x_1, x_2, \ldots, x_d)$ and $k$ is the normalizing constant.

Under squared error loss function, the Bayes estimator of a function $h(\alpha,\beta,\theta)$ is the posterior mean of the function and is given by a ratio of three integrals as follows

$$\tilde{h}(\alpha,\beta,\theta) = E(h(\alpha,\beta,\theta)|\underline{x}) = \frac{1}{k}\int_0^\infty\int_0^\infty\int_0^\infty g(\alpha,\beta,\theta)\pi(\alpha,\beta,\theta|\underline{x})d\alpha d\beta d\theta,$$

Now for computing the Bayes estimate $\tilde{\alpha}$ of $\alpha$, take $h(\alpha,\beta,\theta)=\alpha$ and then the corresponding estimate is computed as

$$\tilde{\alpha} = \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{d+a_1}\beta^{d+a_2-1}\theta^{d+a_3-1}e^{-b_2\beta}e^{-\theta\left(b_3-\sum_{i=1}^{d}x_i\right)}e^{(\beta-1)\sum_{i=1}^{d}\log(e^{\theta x_i}-1)}e^{-\alpha\left(b_1+\sum_{i=1}^{d}(e^{\theta x_i}-1)^{\beta}+(n-d)(e^{\theta c}-1)^{\beta}\right)}d\alpha d\beta d\theta.$$

The Bayes estimators of $\beta$, and $\theta$ are calculated in a similar manner. But it is not possible to evaluate closed form expressions for these estimates. The Bayes estimates are obtained by Lindley (1980) approximation method and the M-H algorithm in next subsections.

### 3.1. Lindley approximation

It is not possible to calculate Bayes estimator in closed form expressions because it is the from of ratio of two integrals. Here we suggest an approaches to approximate Bayes estimates, namely Lindley's approximation method. Consider the ratio of integral $I(X)$, where

$$I(X) = \frac{\int_{(\vartheta_1,\vartheta_2,\vartheta_3)} h(\vartheta_1,\vartheta_2,\vartheta_3)e^{L(\vartheta_1,\vartheta_2,\vartheta_3)+\rho(\vartheta_1,\vartheta_2,\vartheta_3)}\partial(\vartheta_1,\vartheta_2,\vartheta_3)}{\int_{(\vartheta_1,\vartheta_2,\vartheta_3)} e^{L(\vartheta_1,\vartheta_2,\vartheta_3)+\rho(\vartheta_1,\vartheta_2,\vartheta_3)}\partial(\vartheta_1,\vartheta_2,\vartheta_3)} \quad (3.2)$$

where $h(\vartheta_1,\vartheta_2,\vartheta_3)$ is function of $\vartheta_1,\vartheta_2$ and $\vartheta_3$ only and $L(\vartheta_1,\vartheta_2,\vartheta_3)$ is the log-likelihood and $\rho(\vartheta_1,\vartheta_2,\vartheta_3)=\log\rho(\vartheta_1,\vartheta_2,\vartheta_3)$. Let $(\hat{\vartheta}_1,\hat{\vartheta}_2,\hat{\vartheta}_3)$ denote the MLE of $(\vartheta_1,\vartheta_2,\vartheta_3)$. Using the approach developed in Lindley (1980) for sufficiently large sample size $n$, the ratio of integral $I(X)$ as given in (3.2) can be written as

$$I(X) = u(\hat{\vartheta}_1,\hat{\vartheta}_2,\hat{\vartheta}_3) + (u_1\upsilon_1+u_2\upsilon_2+u_3\upsilon_3+\upsilon_4+\upsilon_5) + 0.5[A(u_1\sigma_{11}+u_2\sigma_{12}+u_3\sigma_{13})+B(u_1\sigma_{21}+u_2\sigma_{22}+u_3\sigma_{23}) + C(u_1\sigma_{31}+u_2\sigma_{32}+u_3\sigma_{33})]$$

$$\upsilon_i = \rho_1\sigma_{i1}+\rho_2\sigma_{i2}+\rho_3\sigma_{i3},\quad i=1,2,3\quad \upsilon_4 = u_{12}\sigma_{12}+u_{13}\sigma_{13}+u_{23}\sigma_{23}$$
$$\upsilon_5 = 0.5(u_{11}\sigma_{11}+u_{22}\sigma_{22}+u_{33}\sigma_{33})$$
$$A = \sigma_{11}L_{111}+2\sigma_{12}L_{121}+2\sigma_{13}L_{131}+2\sigma_{23}L_{231}+\sigma_{22}L_{221}+\sigma_{33}L_{331}$$
$$B = \sigma_{11}L_{112}+2\sigma_{12}L_{122}+2\sigma_{13}L_{132}+2\sigma_{23}L_{232}+\sigma_{22}L_{222}+\sigma_{33}L_{332}$$
$$C = \sigma_{11}L_{113}+2\sigma_{12}L_{123}+2\sigma_{13}L_{133}+2\sigma_{23}L_{233}+\sigma_{22}L_{223}+\sigma_{33}L_{333}$$

and subscripts 1,2,3 on the right-hand sides refer to $\vartheta_1,\vartheta_2,\vartheta_3$ respectively and

$$\rho_i = \frac{\partial\rho}{\partial\vartheta_i},\qquad u_i = \frac{\partial u(\vartheta_1,\vartheta_2,\vartheta_3)}{\partial\vartheta_i},\qquad u_{ij} = \frac{\partial^2 u(\vartheta_1,\vartheta_2,\vartheta_3)}{\partial\vartheta_i\partial\vartheta_j},$$
$$L_{ij} = \frac{\partial^2 L}{\partial\vartheta_i\partial\vartheta_j},\qquad L_{ijk} = \frac{\partial^3 L}{\partial\vartheta_i\partial\vartheta_j\partial\vartheta_k},$$

$i,j,k=1,2,3$ and also, $\sigma_{i,j}=(i,j)^{th}$ elements of the inverse of the matrix $\left[-\frac{\partial^2 l(\vartheta_1,\vartheta_2,\vartheta_3|\underline{x})}{\partial\vartheta_1\partial\vartheta_2\partial\vartheta_3}\right]^{-1}$ evaluated at $(\hat{\vartheta}_1,\hat{\vartheta}_2,\hat{\vartheta}_3)$. Other expressions are obtained as,

$$L_{11} = -\frac{d}{\alpha^2},\ L_{13} = -\beta\sum_{i=1}^{m}(1+r_i)x_ie^{\theta x_i}(e^{\theta x_i}-1)^{\beta-1} - \beta(n-d)ce^{\theta c}(e^{\theta c}-1)^{\beta-1},$$

$$L_{22} = -\frac{d}{\beta^2} - \alpha\sum_{i=1}^{d}(e^{\theta x_i}-1)^{\beta}(\log(e^{\theta x_i}-1))^2 - \alpha(n-d)(e^{\theta c}-1)^{\beta}(\log(e^{\theta c}-1))^2,$$

$$L_{33} = -\frac{d}{\theta^2} - \alpha\beta\sum_{i=1}^{d}x_i^2e^{\theta x_i}(e^{\theta x_i}-1)^{\beta-2}(\beta e^{\theta x_i}-1) - (\beta-1)\sum_{i=1}^{d}\frac{x_i^2e^{\theta x_i}}{(e^{\theta x_i}-1)^2} - \alpha\beta c^2(n-d)e^{\theta c}(e^{\theta c}-1)^{\beta-2}(\beta e^{\theta c}-1),$$

$$L_{12} = -\sum_{i=1}^{d}\log(e^{\theta x_i}-1)(e^{\theta x_i}-1)^{\beta-1} - (n-d)\log(e^{\theta c}-1)(e^{\theta c}-1)^{\beta-1}$$

$$L_{23} = L_{32} = -\alpha\sum_{i=1}^{d}x_ie^{\theta x_i}(e^{\theta x_i}-1)^{\beta-1}(1+\beta\log(e^{\theta x_i}-1)) - \alpha(n-d)ce^{\theta c}(e^{\theta c}-1)^{\beta-1}(1+\beta\log(e^{\theta c}-1)) + \sum_{i=1}^{d}\frac{x_ie^{\theta x_i}}{e^{\theta x_i}-1},$$

$$L_{111} = \frac{2d}{\alpha^3},\ L_{121} = L_{131} = 0,$$

$$L_{122} = -\sum_{i=1}^{d}(e^{\theta x_i}-1)^{\beta}(\log(e^{\theta x_i}-1))^2 - (n-d)(e^{\theta c}-1)^{\beta}(\log(e^{\theta c}-1))^2,$$

$$L_{222} = \frac{2d}{\beta^3} - \alpha\sum_{i=1}^{d}(e^{\theta x_i}-1)^{\beta}(\log(e^{\theta x_i}-1))^3 - \alpha(n-d)(e^{\theta c}-1)^{\beta}(\log(e^{\theta c}-1))^3,$$

$$L_{123} = -\sum_{i=1}^{d}x_ie^{\theta x_i}(e^{\theta x_i}-1)^{\beta-1}\{1+\beta\log(e^{\theta x_i}-1)\} - (n-d)ce^{\theta c}(e^{\theta c}-1)^{\beta-1}\{1+\beta\log(e^{\theta c}-1)\},$$

$$L_{133} = -\beta\sum_{i=1}^{d}x_i^2e^{\theta x_i}(e^{\theta x_i}-1)^{\beta-2}(\beta e^{\theta x_i}-1) - \beta c^2(n-d)e^{\theta c}(e^{\theta c}-1)^{\beta-2}(\beta e^{\theta c}-1),$$

$$L_{223} = -\alpha\sum_{i=1}^{d}x_ie^{\theta x_i}(e^{\theta x_i}-1)^{\beta-1}\log(e^{\theta x_i}-1)\{2+\beta\log(e^{\theta x_i}-1)\} - \alpha(n-d)ce^{\theta c}(e^{\theta c}-1)^{\beta-1}\log(e^{\theta c}-1)\{2+\beta\log(e^{\theta c}-1)\},$$

$$L_{233} = -\alpha\sum_{i=1}^{d}x_i^2e^{\theta x_i}(e^{\theta x_i}-1)^{\beta-2}\{2e^{\theta x_i}-1+\beta\log(e^{\theta x_i}-1)\} - \alpha c^2e^{\theta c}(e^{\theta c}-1)^{\beta-2}\{2e^{\theta c}-1+\beta\log(e^{\theta c}-1)\}$$

$$L_{333} = \frac{2d}{\theta^3} - \alpha\beta\sum_{i=1}^{d} x_i^3 e^{\theta x_i}\left(e^{\theta x_i}-1\right)^{\beta-3}\left(1+e^{\theta x_i}(\beta^2 e^{\theta x_i}-3\beta+1)\right)$$
$$-\alpha\beta c^3(n-d)e^{\theta c}(e^{\theta c}-1)^{\beta-3}\left(1+e^{\theta c}(\beta^2 e^{\theta c}-3\beta+1)\right),$$
$$\rho_1 = \frac{a_1-1}{\alpha}-b_1,\ \rho_2=\frac{a_2-1}{\beta}-b_2,\ \rho_3=\frac{a_3-1}{\theta}-b_3$$

The Bayes estimate of unknown parameter $\alpha$ based on SEL function,

$$\tilde{\alpha}_{LI} = \hat{\alpha} + \vartheta_1 + 0.5(\sigma_{11}A + \sigma_{21}B + \sigma_{31}C),$$

Similarly, we can calculate Bayes estimate of $\beta$ and $\theta$. But it is not possible to construct HPD intervals using Lindley's method. Therefore, we propose the M-H algorithm to generate samples from the respective posterior distributions and compute some more estimates of unknown parameters in next subsection. See Metropolis et al. (1953) and Hastings (1970) for more applications of this method.

### 3.2. The Metropolis-Hastings M-H algorithm

The M-H algorithm is a good method for parameter estimation. Where, it used to simulate samples from probability distribution using full joint density function and (independent) proposal distribution. Furthermore, the random samples are generated from prescribed posterior distribution according to the following algorithm:

Step 1: Specify an initial guess of $(\alpha,\beta,\theta)$ and say it $(\alpha_0,\beta_0,\theta_0)$
Step 2: Generate $\beta'$ using the proposal $N(\beta_{n-1},\sigma^2)$ distribution and $\theta'$ using the proposal $N(\theta_{n-1},\sigma^2)$ distribution and then $\alpha'$ from $G_{\alpha|(\beta,\theta)}\left(d+a_1, b_1+\sum_{i=1}^{d}(e^{\theta x_i}-1)^{\beta}+(n-d)(e^{\theta c}-1)^{\beta}\right)$
Step 3: Compute $h=\frac{\pi(\alpha',\beta',\theta'|x)}{\pi(\alpha_{n-1},\beta_{n-1},\theta_{n-1}|x)}$
Step 4: Generate a sample $u$ from the uniform $U(0,1)$ distribution
Step 5: Then if $u \leqslant h$ then set
$\alpha_n \leftarrow \alpha'$, $\beta_n \leftarrow \beta'$, $\theta_n \leftarrow \theta'$; otherwise
$\alpha_n \leftarrow \alpha_{n-1}$; $\beta_n \leftarrow \beta_{n-1}$, $\theta_n \leftarrow \theta_{n-1}$;
Step 6: Replicate steps (2–5) $Q$ times.

Finally, the associated Bayes estimates of $\alpha,\beta$ and $\theta$ are respectively given by

$$\tilde{\alpha}_{mh} = \frac{1}{Q-Q_0}\sum_{i=Q_0+1}^{Q}\alpha_i,\quad \tilde{\beta}_{mh} = \frac{1}{Q-Q_0}\sum_{i=Q_0+1}^{Q}\beta_i,$$
$$\tilde{\theta}_{mh} = \frac{1}{Q-Q_0}\sum_{i=Q_0+1}^{Q}\theta_i,$$

where $Q$ denotes the total number of generated samples and $Q_0$ denotes the initial burn-in samples. We must notice that the $100(1-\mu)\%, 0<\mu<1$, HPD intervals for the all unknown parameters can easily be structure using the M-H samples. The following section discussed the performance of all the above-mentioned estimators using simulations.

## 4. Numerical comparisons

The aim of the simulation is to compare the effect of the point and interval estimation. We arbitrarily specify true values of the three parameters as $\alpha=0.5, \beta=0.4$ and $\theta=0.5$ then generate Type-I hybrid censored samples from WED for different choices of $n, r$ and $T$. The average values (AV) and mean square error (MSE) values of all estimators are calculated using 10000 repetitions. The study is done to applied the MLEs and the associated 95% asymptotic CI estimates for each sampling situations. We specify informative prior set up hyperparasite as $a_1=4; b_1=8; a_2=2; b_2=5; a_3=4; b_3=8$. Tables 1,2 includes the average values and MSEs of all estimators. From the tabulated values, we realized the following observations.

**Table 1**
MSEs and average values of all estimates of $\alpha,\beta$ and $\theta$.

| $T$ | r | $\hat{\alpha}$ | $\tilde{\alpha}_{LI}$ | $\tilde{\alpha}_{MH}$ | $\hat{\beta}$ | $\tilde{\beta}_{LI}$ | $\tilde{\beta}_{MH}$ | $\hat{\theta}$ | $\tilde{\theta}_{LI}$ | $\tilde{\theta}_{MH}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 30 | 0.611813 | 0.65632 | 0.377178 | 0.42306 | 0.455501 | 0.389726 | 0.56902 | 0.477774 | 0.471545 |
| | | 0.137158 | 0.065186 | 0.021264 | 0.018831 | 0.015123 | 0.003151 | 0.113863 | 0.121423 | 0.007832 |
| | 35 | 0.575701 | 0.583291 | 0.43204 | 0.424961 | 0.427378 | 0.390552 | 0.548203 | 0.480493 | 0.475382 |
| | | 0.094922 | 0.039545 | 0.010703 | 0.018664 | 0.016782 | 0.00313 | 0.076226 | 0.054593 | 0.036902 |
| | 40 | 0.569694 | 0.547376 | 0.490556 | 0.424462 | 0.412593 | 0.400355 | 0.524493 | 0.505187 | 0.492077 |
| | | 0.071873 | 0.028558 | 0.009152 | 0.013788 | 0.010245 | 0.003506 | 0.053222 | 0.018819 | 0.011694 |
| 12 | 30 | 0.602685 | 0.653479 | 0.375968 | 0.423384 | 0.456354 | 0.390998 | 0.572621 | 0.474439 | 0.473121 |
| | | 0.131148 | 0.063255 | 0.041633 | 0.016762 | 0.014298 | 0.003219 | 0.112883 | 0.120247 | 0.097929 |
| | 35 | 0.580825 | 0.58348 | 0.434221 | 0.424547 | 0.422466 | 0.389656 | 0.542043 | 0.485641 | 0.475281 |
| | | 0.089835 | 0.040031 | 0.020299 | 0.014494 | 0.011714 | 0.003189 | 0.073319 | 0.052627 | 0.027161 |
| | 40 | 0.556267 | 0.544192 | 0.499443 | 0.42496 | 0.413408 | 0.407883 | 0.529824 | 0.504134 | 0.503321 |
| | | 0.061362 | 0.026985 | 0.010843 | 0.01373 | 0.00922 | 0.00351 | 0.052238 | 0.013994 | 0.009635 |

**Table 2**
MSEs and average values of all estimates of $\alpha,\beta$ and $\theta$.

| $T$ | r | $\hat{\alpha}$ | $\tilde{\alpha}_{LI}$ | $\tilde{\alpha}_{MH}$ | $\hat{\beta}$ | $\tilde{\beta}_{LI}$ | $\tilde{\beta}_{MH}$ | $\hat{\theta}$ | $\tilde{\theta}_{LI}$ | $\tilde{\theta}_{MH}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 50 | 0.55554 | 0.557353 | 0.39525 | 0.415645 | 0.414303 | 0.39143 | 0.536703 | 0.486758 | 0.479575 |
| | | 0.065924 | 0.027595 | 0.016268 | 0.009171 | 0.008186 | 0.002457 | 0.053216 | 0.018932 | 0.096401 |
| | 55 | 0.548301 | 0.536374 | 0.450788 | 0.416435 | 0.409861 | 0.392215 | 0.52536 | 0.504339 | 0.477371 |
| | | 0.049623 | 0.020493 | 0.012989 | 0.008971 | 0.005392 | 0.002647 | 0.042846 | 0.007009 | 0.006237 |
| | 60 | 0.548899 | 0.52696 | 0.489434 | 0.418195 | 0.407974 | 0.399974 | 0.511239 | 0.509336 | 0.487533 |
| | | 0.045313 | 0.018562 | 0.01022 | 0.008747 | 0.004432 | 0.002746 | 0.03289 | 0.00666 | 0.005149 |
| 12 | 50 | 0.549887 | 0.556505 | 0.394296 | 0.413567 | 0.413667 | 0.39115 | 0.540589 | 0.488729 | 0.48043 |
| | | 0.063011 | 0.029183 | 0.016252 | 0.008707 | 0.00813 | 0.002546 | 0.053769 | 0.019264 | 0.006099 |
| | 55 | 0.543468 | 0.536428 | 0.448303 | 0.414088 | 0.409976 | 0.391615 | 0.530188 | 0.504587 | 0.479578 |
| | | 0.049885 | 0.021081 | 0.014985 | 0.008491 | 0.005101 | 0.002627 | 0.042977 | 0.007135 | 0.006265 |
| | 60 | 0.532158 | 0.523156 | 0.500064 | 0.41527 | 0.411279 | 0.405214 | 0.519464 | 0.507791 | 0.501875 |
| | | 0.033897 | 0.012203 | 0.00976 | 0.008096 | 0.003682 | 0.002814 | 0.030635 | 0.003757 | 0.002942 |

We observed from Tables 1 and 2 that in most of the cases, for all three estimators, that for fixed $n$ as $r$ increases or $T$ increases, the performances of the Bayes estimates are better than the corresponding MLEs in estimating $\alpha, \beta$ as well as $\theta$. Furthermore, we observe from both tables that the Bayes estimates obtained from the MH method show superior behavior among Lindley and MH estimates.

Furthermore, Table 3 introduces different 95% asymptotic confidence and HPD intervals for all unknown parameters for different values of $(n, r)$ and $T$. In general, we observe that the HPD intervals compete better when compared with asymptotic intervals. This gathers for the unknown parameters and all given values of $r$ and $T$.

Overall, it may be noted that for fixed $n$ and $r$, by increasing $T$, the MSE values of all estimates become a little bit small. Also, similar trend is observed when $T, n$ are kept fixed and $r$ is allowed to increase. Finally in general, as the sample size $n$ increases, the MSEs and biases of different methods decrease, except for few.

## 5. Data analysis

In this section, we have analyzed two real data sets which have been recently considered by Oguntunde et al. (2015) for illustrative purposes. They fitted these real data sets to WED and found WED fits the both real data sets reasonably good.

**Example 1.** This data set demonstrates the breaking stress of carbon fibres of 50 mm length(GPa). The data has been previously utilized by Nichols and Padgett (2006). The data is as follows:

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61,1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Based on the above Type-I hybrid censored samples, we obtained the estimate of the parameters using classically and Bayesian procedures. Lindley and M-H estimates are derived using the SEL under a noninformative prior. In Table 4, all the estimates of $\alpha, \beta$ and $\theta$ are presented for different combinations of $r$ and $T$. In Table 5, we have also computed approximate 95% confidence intervals and noninformative HPD intervals of the unknown parameters.

**Table 3**
95% asymptotic confidence and HPD intervals estimates of $\alpha, \beta$ and $\theta$.

| | | T = 10 | | T = 12 | |
|---|---|---|---|---|---|
| n | r | Appr. Con. Int. | HPD Int. | Appr. Con. Int. | HPD Int. |
| 40 | 30 | (0, 1.11804) | (0.263862, 0.514021) | (0, 1.11039) | (0.261584, 0.514625) |
| | | (0.20171, 0.644421) | (0.302551, 0.480021) | (0.20087, 0.645898) | (0.303418, 0.481277) |
| | | (0, 1.19006) | (0.36431, 0.584473) | (0, 1.19931) | (0.365498, 0.587659) |
| | 35 | (0.024733, 1.02339) | (0.324614, 0.548846) | (0.02234, 1.04673) | (0.326944, 0.54966) |
| | | (0.210874, 0.639047) | (0.307251, 0.477454) | (0.21197, 0.636887) | (0.306703, 0.476437) |
| | | (0.025376, 1.05331) | (0.372941, 0.582588) | (0.022034, 1.04701) | (0.371927, 0.58352) |
| | 40 | (0.097942, 0.978024) | (0.383321, 0.60049) | (0.112166, 0.955331) | (0.391589, 0.609859) |
| | | (0.219097, 2.58598) | (0.318949, 0.485198) | (0.22047, 0.629521) | (0.326309, 0.493173) |
| | | (0.654194, 2.58832) | (0.390452, 0.596258) | (0.10704, 0.941467) | (0.401946, 0.607506) |
| 60 | 50 | (0.096334, 0.95227) | (0.289718, 0.51933) | (0.094752, 0.952104) | (0.289365, 0.517833) |
| | | (0.241265, 0.590025) | (0.313745, 0.471942) | (0.239349, 0.587785) | (0.313902, 0.471726) |
| | | (0.083953, 0.987755) | (0.378393, 0.586811) | (0.083722, 0.993138) | (0.379339, 0.587363) |
| | 55 | (0.149071, 0.913909) | (0.348563, 0.559434) | (0.148146, 0.905059) | (0.347119, 0.557422) |
| | | (0.247216, 0.585654) | (0.317061, 0.471532) | (0.245143, 0.583033) | (0.317087, 0.470203) |
| | | (0.13518, 0.911668) | (0.379437, 0.580334) | (0.13726, 0.920513) | (0.381906, 0.582127) |
| | 60 | (0.181033, 0.891345) | (0.386335, 0.595257) | (0.197139, 0.851851) | (0.397552, 0.605462) |
| | | (0.251525, 0.584865) | (0.325573, 0.479014) | (0.250465, 0.579839) | (0.330943, 0.483571) |
| | | (0.164013, 0.856293) | (0.390845, 0.588615) | (0.181122, 0.856061) | (0.404667, 0.602548) |

**Table 4**
Point estimates of $\alpha, \beta$ and $\theta$ for data set 1.

| $r$ | $T$ | $\hat{\alpha}$ | $\tilde{\alpha}_{LI}$ | $\tilde{\alpha}_{MH}$ | $\hat{\beta}$ | $\tilde{\beta}_{LI}$ | $\tilde{\beta}_{MH}$ | $\hat{\theta}$ | $\tilde{\theta}_{LI}$ | $\tilde{\theta}_{MH}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 60 | 3.5 | 5.09564 | 5.21406 | 5.17384 | 3.053 | 3.47442 | 2.74276 | 0.151757 | 0.14746 | 0.13145 |
| | 4 | 5.96719 | 5.02238 | 5.17794 | 3.14514 | 3.56083 | 2.82614 | 0.148072 | 0.12344 | 0.135906 |
| 66 | 3.8 | 5.96872 | 5.12238 | 5.45904 | 3.12018 | 3.55468 | 2.73751 | 0.147359 | 0.15276 | 0.133378 |
| | 4 | 5.5727 | 5.78224 | 5.06376 | 2.96234 | 3.88888 | 2.77513 | 0.145313 | 0.155313 | 0.135918 |

**Table 5**
95% intervals estimates of $\alpha, \beta$ and $\theta$ for data set 1.

| | | $\alpha$ | | $\beta$ | | $\theta$ | |
|---|---|---|---|---|---|---|---|
| $r$ | $t$ | Appr. con. Int. | HPD Int. | Appr. con. Int. | HPD Int. | Appr. con. Int. | HPD Int. |
| 60 | 3.5 | (5.03366, 5.15762) | (5.07933, 5.25331) | (2.99102, 3.11498) | (2.65505, 2.87629) | (0.089776, 0.213738) | (0.12153, 0.141436) |
| | 4 | (5.90521, 6.02917) | (5.09654, 5.27524) | (3.08316, 3.20712) | (2.68778, 2.9812) | (0.086091, 0.210052) | (0.125198, 0.145513) |
| 66 | 3.8 | (5.90674, 6.03076) | (5.27323, 5.62839) | (3.0582, 3.18216) | (2.65189, 2.84955) | (0.085378, 0.209341) | (0.123035, 0.143629) |
| | 4 | (5.51072, 5.63468) | (4.97349, 5.20393) | (2.90036, 3.02432) | (2.66716, 2.85787) | (0.083332, 0.207294) | (0.126549, 0.144892) |

**Table 6**
Point estimates of $\alpha, \beta$ and $\theta$ for data set 2.

| $r$ | $T$ | $\hat{\alpha}$ | $\tilde{\alpha}_{LI}$ | $\tilde{\alpha}_{MH}$ | $\hat{\beta}$ | $\tilde{\beta}_{LI}$ | $\tilde{\beta}_{MH}$ | $\hat{\theta}$ | $\tilde{\theta}_{LI}$ | $\tilde{\theta}_{MH}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 60 | 1.8 | 0.016031 | 0.017375 | 0.011408 | 3.23072 | 3.5553 | 3.00937 | 0.935313 | 0.821945 | 1.00585 |
| | 2 | 0.015967 | 0.011356 | 0.012131 | 3.24816 | 3.94743 | 3.05139 | 0.935374 | 0.644358 | 0.893187 |
| 63 | 2 | 0.010404 | 0.089768 | 0.014748 | 2.97893 | 3.18815 | 3.0264 | 1.05552 | 0.903603 | 0.969879 |
| | 2.5 | 0.01647 | 0.022785 | 0.029056 | 2.93347 | 3.09661 | 2.9637 | 0.986311 | 1.11555 | 0.892012 |

**Table 7**
95% intervals estimates of $\alpha, \beta$ and $\theta$ for data set 2.

| | | $\alpha$ | | $\beta$ | | $\theta$ | |
|---|---|---|---|---|---|---|---|
| $r$ | $t$ | Appr. con. Int. | HPD Int. | Appr. con. Int. | HPD Int. | Appr. con. Int. | HPD Int. |
| 60 | 1.8 | (0, 0.115809) | (0.006018, 0.014978) | (0.846664, 5.61477) | (2.94067, 3.05528) | (0, 2.27255) | (0.938418, 1.04609) |
| | 2 | (0, 0.097924) | (0.010585, 0.031746) | (1.233, 5.26333) | (2.99963, 3.10249) | (0, 2.03133) | (0.847588, 0.937197) |
| 63 | 2 | (0, 0.070845) | (0.007162, 0.024959) | (0.163934, 5.79392) | (2.98804, 3.06966) | (0, 2.7213) | (0.900791, 1.05592) |
| | 2.5 | (0, 0.147977) | (0.018238, 0.042189) | (0, 6.82763) | (2.85819, 3.04172) | (0, 3.19214) | (0.845884, 0.934036) |

**Example 2.** The second data set was originally discussed by workers at the UK National Physical Laboratory and it has been used by Smith and Naylor (1987). It is on the strengths of 1.5 cm glass fibres. The data are as follows:

0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24

The MLEs and Bayes estimates of unknown parameters are presented in Table 6. The approximate confidence intervals and non informative HPD intervals for the unknown parameters are also presented in this Table 7.

## 6. Conclusion

In this paper based on Type-I hybrid censored sample, both the classical and Bayesian inference procedures for the parameters of WED have been successfully discussed. We have considered MLE and Bayes estimates under square error loss function. Since the Bayes estimates cannot be obtained in explicit forms, so Lindley method and M-H algorithm are considered. The CIs and HPD interval of unknown parameters are calculated. We compared all classical and Bayesian estimations numerically and appropriate comments have been provided. Based on the results of the simulation study, we see obviously that, the Bayesian estimation exhibit reasonably good behavior compared to the respective MLEs. The computational results show that the performances of all estimators are improved when the effective sample size increases. HPD intervals is found to be preferable then asymptotic confidence intervals. Same method can be extended for other distribution and censoring schemes also. We believe, more work is needed along these directions.

## References


Balakrishnan, N., Kundu, D., 2013. Hybrid censoring: models, inferential results and applications. Comp. Stat. Data Anal. 57 (1), 166–209.

Epstein, B., 1954. Truncated life tests in the exponential case. Ann. of Math. Stat. 25, 555–564.

Hastings, W.K., 1970. Monte Carlo sampling methods using Markov chains and their applications. Biometrika. 57, 97–109.

Lawless, J.F., 1982. Statistical Models and Methods for Lifetime Data. Wiley, New York.

Lindley, D.V., 1980. Approximate Bayesian Methods. Trabajos Estadistica 31, 223–237.

Metropolis, N., Rosenbluth, A.W., Rosenbluth, M.N., Teller, A.H., Teller, E., 1953. Equations of state calculations by fast computing machines. J. Chem. Phys. 21, 1087–1092.

Nichols, M.D., Padgett, W.J., 2006. A bootstrap control chart for Weibull percentiles. Quality Reliabil. Eng. Int. 22, 141–151.

Oguntunde, P.E., Balogun, O.S., Okagbue, H.I., Bishop, S.A., 2015. The Weibull-Exponential distribution: its properties and applications. J. Appl. Sci. 54, 374–380.

Rastogi, M.K., 2017. Estimation Based on Progressively Censored Data from Weibull-Exponential Distribution. J. Indian Soc. Probab. Stat. 18, 237–265.

Rastogi, M.K., Tripathi, Y.M., 2013. Inference on unknown parameters of a Burr distribution under hybrid censoring. Stat. Papers 54, 619–643.

Rastogi, M.K., Tripathi, Y.M., 2013. Estimation using hybrid censored data from a two-parameter distribution with bathtub shape. Comp. Stat. Data Anal. 67, 268–281.

Smith, R.L., Naylor, J.C., 1987. A comparison of maximum likelihood and bayesian estimators for the three-parameter weibull distribution. Appl. Stat. 15, 1305–1311.