Journal of King Saud University - Science 36 (2024) 103234

Contents lists available at ScienceDirect

Journal of King Saud University - Science

journal homepage: www.sciencedirect.com

Full Length Article

Certain properties of q-analogue of M-function

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ARTICLE INFO

MSC:
primary 26A33
33C05
33C45
secondary 33C47
33C90

Keywords:
M-series
q-beta functions
q-gamma functions
q-Laplace transform
q-Sumudu transform
Fractional q-derivative

ABSTRACT

The intent of this work is to create and study certain fundamental characteristics of new generalizations of the M-function using q-calculus. We establish its characteristics, like the convergence condition, recurrence relation, integral representations, q-derivative formulae, q-Laplace and q-Sumudu transformations, and image formulas for the Caputo fractional q-derivative and the Hilfer fractional q-derivative. Furthermore, we analyze some specific situations to demonstrate of our key findings.

1. Introduction

Sharma and Jain (2009) developed the new function known as the M-function in 2009. It is expressed in terms of power series for $z, \zeta, \vartheta \in \mathbb{C}$, $\Re(\zeta) > 0$ as:

$$ {}_{r}^{\zeta}M_{s}^{\vartheta}(z) = {}_{r}^{\zeta}M_{s}^{\vartheta}\left(a_{1,\ldots,}a_{r};\ c_{1,\ldots,}c_{s};\ z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(c_1)_n \cdots (c_s)_n} \frac{z^n}{\Gamma(\zeta n + \vartheta)}, \tag{1.1}$$

where the Pochhammer symbols are $(a_i)_n$, $(c_j)_n$; $i = 1, 2, \ldots, r; j = 1, 2, \ldots, s$, for more detail (see Kilbas et al., 2006). Special cases exist for the generalized M-function. For instance, the M-function reduces in the M-series described by Sharma (2008) if we give $\vartheta = 1$, for $\zeta = \vartheta = 1$, obtain well-known generalized hypergeometric function, when $\zeta = \vartheta = 1$, and $s = r = 0$, get Mittag–Leffler function (MLF) (Mittag-Leffler, 1903) with one parameter, put $s = r = 0$, obtain MLF with two parameter which is introduced by Wiman (1905), further, if we put $r = s = 1$, $c_1 = \gamma$, $d_1 = 1$, then, Eq. (1.1) becomes to Generalized MLF which is introduced by Prabhakar (1971). There are many characteristics of M-Series (1.1) and the unique situations have investigated by Miller (1993), Saxena et al. (2009), Saigo and Kilbas (1998), Purohit et al. (2010), and Mishra et al. (2017). In order to solve differential and integral equations of fractional order, the generalized M-function is important.

The article highlights the significance of gamma and beta functions, along with their diverse forms. Lahcene (2021) provides properties of the extended gamma and beta function together with a closed-form representation of more integral functions. Additionally, some of the extended function's relative behaviors, the special cases that arise from them when the parameters are fixed, the decomposition equation, the integrative representation of the suggested general formula, the frequency relationships, the correlations associated with the suggested formula, and the differentiation equation for these fundamental functions were examined. By investigating, integral representations, the author additionally looked into the basic decomposition equation, the differentiation formula, recurrence relations, convolutions, and the asymptotic behavior of a few particular examples. Applications of these functions to the infinite series and the complex of definite integrals of related fundamental functions have been described, as well as the assessment of various reversible Laplace transforms. Furthermore, by using the weights offered by extended gamma functions, Lahcene et al. (2022) provided rich theoretical and applied behaviors in models of special functions, particularly expansion-generalized gamma delta, and techniques to more thoroughly generalize integrals and derivatives. In order to establish a connection between all of the fundamental modifications that had previously been discovered, the researcher (Palsaniya et al., 2021) summarized the modifications that emerged on the most significant special functions pertaining to the extended generalized

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https://doi.org/10.1016/j.jksus.2024.103234
Received 8 December 2023; Received in revised form 21 March 2024; Accepted 29 April 2024
Available online 3 May 2024

gamma function that overlapped it in relationship to the fractional calculus (FC), and additional findings regarding the generalized gamma function that occurred in diffraction theory, as well as some special functions associated with fractional functions. The incomplete function of the beta and gamma functions, the H-function, and their generalization have been the focus of some researchers' recent work; for more information, see the article (Bhatter et al., 2023; Meena et al., 2020, 2022).

Inspired by the depiction of a series mentioned above, which is found in Eq. (1.1), as well as its significance and uses in applied math, we have determined the q-analogue of generalized M-series and obtained several fundamental properties. For these functions, a few q-integral formulations are developed. In the last section, unique cases of the main findings are presented.

In this work, We employ some definitions provided in the following section.

2. Preliminaries

Regarding the idea of q-series (see Gasper and Rahman, 2004), given $|q| < 1$ and $\lambda \in \mathbb{C}$, the q-shifted factorial is stated as follows:

$$(\eth; q)_h = \begin{cases} 1; & h = 0, \\ (1-\eth)(1-\eth q)\cdots(1-\eth q^{h-1}); & h \in \mathbb{N}, \end{cases} \tag{2.1}$$

and its natural expansion as

$$(\eth; q)_h = \frac{(\eth; q)_\infty}{(\eth q^h; q)_\infty}, \quad h \in \mathbb{C}, \tag{2.2}$$

where q^h's the fundamental value is determined.

For $\varepsilon \neq 0$, the power function $(\varepsilon - \tau)^m$ is stated as follows in terms of the q-analogue:

$$(\varepsilon-\tau)^m = \prod_{k=0}^{m-1}(\varepsilon - q^k\tau) = \varepsilon^m\left(\frac{\tau}{\varepsilon}; q\right)_m = \varepsilon^m \frac{\left(\frac{\tau}{\varepsilon}; q\right)_\infty}{\left(\frac{\tau}{\varepsilon}q^m; q\right)_\infty}; \ \varepsilon \neq 0. \tag{2.3}$$

Moreover, (see Ernst, 2003) the q-analogue of the power function $(\gamma \pm \tau)^n$ is

$$\begin{aligned}(\gamma \pm \tau)^{(n)} &= (\gamma \pm \tau)_n = \gamma^n (\mp\tau/\gamma; q)_n \\ &= \gamma^n \sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix}_q q^{t(t-1)/2} (\pm\tau/\gamma)^t, \ (n \in \mathbb{N}),\end{aligned} \tag{2.4}$$

wherein q-binomial coefficient is obtained from

$$\begin{aligned}\begin{bmatrix} n \\ t \end{bmatrix}_q &= \frac{(q^{-n}; q)_t}{(q; q)_t}(-q^n)^t q^{-(t-1)t/2} \\ &= \frac{\Gamma_q(n+1)}{\Gamma_q(t+1)\Gamma_q(1+n-t)}, \ (t \in \mathbb{N}, n \in \mathbb{C}).\end{aligned} \tag{2.5}$$

The q-gamma and q-beta function (cf. Gasper and Rahman, 2004), is defined as

$$\begin{aligned}\Gamma_q(\varepsilon) &= \frac{(q; q)_\infty}{(q^\varepsilon; q)_\infty}(1-q)^{1-\varepsilon} \\ &= (1-q)^{1-\varepsilon}(q; q)_{\varepsilon-1}; \ \varepsilon \in \mathbb{R}\setminus\{0, -1, -2, \ldots\}; |q| < 1,\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}B_q(\varepsilon, \eta) &= \frac{\Gamma_q(\varepsilon)\Gamma_q(\eta)}{\Gamma_q(\varepsilon+\eta)} = \int_0^1 t^{\varepsilon-1}\frac{(qt; q)_\infty}{(q^\eta t; q)_\infty} d_q t \\ &= \int_0^1 t^{\varepsilon-1}(tq; q)_{\eta-1} d_q t, \ (\Re(\varepsilon), \Re(\eta) > 0).\end{aligned} \tag{2.7}$$

Furthermore, Gasper and Rahman (2004) provides the q-derivative and q-difference operator of a function $f(t)$ defined on a subset of $\mathbb{C}$.

$$D_q f(t) = \frac{f(t) - f(tq)}{(1-q)t}, \quad (q \neq 1, \ t \neq 0), \tag{2.8}$$

$$D_q\left\{(t-\varepsilon)_q^\mu\right\} = [\mu]_q (t-\varepsilon)_q^{\mu-1}; \quad D_q\left\{(\eta-t)_q^\mu\right\} = -[\mu]_q(\eta - qt)_q^{\mu-1}, \tag{2.9}$$

and

$$\int_0^t f(z) d(z; q) = (1-q)t \sum_{k=0}^{\infty} q^k f(tq^k). \tag{2.10}$$

The fractional q-integral operator is given as

$$\left[I_{q,a+}^\delta (t-\varepsilon)_q^\mu\right](z) = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu+\delta+1)}(t-\varepsilon)_q^{\mu+\delta}; \ \delta \in (-1, \infty). \tag{2.11}$$

3. The q-analogue of M-function and its properties

Motivated by Eq. (1.1), we propose the q-analogue of M-function ${}_r\overset{\zeta,\vartheta}{M}_s(q,z)$ for $z, \zeta, \vartheta \in \mathbb{C}$; $|q| < 1$ and $\Re(\zeta) > 0$ as

$$\begin{aligned}{}_r\overset{\zeta,\vartheta}{M}_s(q,z) &= {}_r\overset{\zeta,\vartheta}{M}_s\left[\begin{matrix} q^{c_1}, \ldots, q^{c_r} \\ q^{d_1}, \ldots, q^{d_s} \end{matrix}; q, z\right] \\ &= \sum_{\xi=0}^{\infty} \frac{(q^{c_1}; q)_\xi \cdots (q^{c_r}; q)_\xi}{(q^{d_1}; q)_\xi \cdots (q^{d_s}; q)_\xi} \frac{z^\xi}{\Gamma_q(\zeta\xi + \vartheta)},\end{aligned} \tag{3.1}$$

where the q-analogue of Pochhammer symbol are as $(q^{c_i}, q)_\xi, (q^{d_j}, q)_\xi$; c_i, $d_j \neq 0, -1, -2, \ldots$ $(i = 1, 2, \ldots, r; j = 1, 2, \ldots, s)$, and $\Gamma_q(.)$ is the q-gamma function. If $0 < |q| < 1$, series (3.1) converges for all z if $r \le s$, and for $|z| < 1$ if $r = s + 1$. When $|q| > 1$, the seriesconverges for $|z| < |d_1 \cdots d_s q| / |c_1 \cdots c_r|$.

The connections are additionally noted as particular examples of ${}_r\overset{\zeta,\vartheta}{M}_s(q,z)$ with additional special functions as indicated below.

(1) For $s = r = 0$, the q-ML function, which Mansour (2009) defined, replaces the M-function.

$${}_0\overset{\zeta,\vartheta}{M}_0\left[\begin{matrix}-\\-\end{matrix}; q, z\right] = \sum_{\xi=0}^{\infty} \frac{z^\xi}{\Gamma_q(\zeta\,\xi + \vartheta)} = e_{\zeta,\vartheta}(z, q). \tag{3.2}$$

(2) The M-function reduces to the q-ML function, which is defined by Jain (2018) as follows for $\vartheta = 1$, $r = s = 0$.

$${}_0\overset{\zeta,1}{M}_0\left[\begin{matrix}-\\-\end{matrix}; q, z\right] = \sum_{\xi=0}^{\infty} \frac{z^\xi}{\Gamma_q(\zeta\,\xi + 1)} = E_\zeta(z, q), \ (\Re(\zeta) > 0). \tag{3.3}$$

(3) We get the generalized small q-ML function, first suggested by Purohit and Kalla (2011), as follows after setting $r = s = 1$, $c_1 = \delta$, $d_1 = 1$ in (3.1):

$${}_1\overset{\zeta,\vartheta}{M}_1\left[\begin{matrix} q^\delta; \\ q; \end{matrix}\ q, z\right] = \sum_{\xi=0}^{\infty} \frac{(q^\delta; q)_\xi}{(q; q)_\xi} \frac{z^\xi}{\Gamma_q(\zeta\xi + \vartheta)} = e^\delta_{\zeta,\vartheta}(q, z). \tag{3.4}$$

(4) The generalized q-ML function that Sharma and Jain (2016) proposed is obtained as follows when $r = s = 1$, $c_1 = \gamma\, d_1 = \delta$ is entered into (3.1).

$${}_1\overset{\zeta,\vartheta}{M}_1\left[\begin{matrix} q^\gamma; \\ q^\delta; \end{matrix}\ q, z\right] = \sum_{\xi=0}^{\infty} \frac{(q^\gamma; q)_\xi}{(q^\delta; q)_\xi} \frac{z^\xi}{\Gamma_q(\zeta\xi + \vartheta)} = E^{\gamma,\delta}_{\zeta,\vartheta}(z, q). \tag{3.5}$$

(5) Put $s = 0, r = 1$, $c_1 = \delta$ in (3.1), we get

$${}_1\overset{1,1}{M}_0\left[\begin{matrix} q^\delta; \\ -; \end{matrix}\ q, z\right] = \sum_{n=0}^{\infty} \frac{(q^\delta; q)_\xi\, z^\xi}{(q; q)_\xi} = \frac{(q^\delta z; q)_\infty}{(q; q)_\infty} = {}_1\phi_0(q^\delta; -; q, z), \tag{3.6}$$

where the function ${}_1\phi_0(q^\delta; -; q, z) = (1-z)^{-\delta}$ is also known as the q-binomial function.

(6) Lastly, considering the relationships

$$\lim_{q\to 1^-}\frac{\left(q^{\delta};q\right)_{\xi}}{(1-q)^{\xi}}=(\delta)_{\xi}, \tag{3.7}$$

and

$$\lim_{q\to 1^-}\Gamma_q(z)=\Gamma(z), \tag{3.8}$$

further note that

$$\lim_{q\to 1^-}{}_{r}\overset{\zeta,\vartheta}{M}_{s}(q,z)={}^{\zeta}_{r}M^{\vartheta}_{s}(z). \tag{3.9}$$

We begin our investigation from the convergence condition of q-analogue of the M-function.

4. Convergence of ${}_{r}\overset{\zeta,\vartheta}{M}_{s}(q,z)$

Theorem 4.1. *The q-analogue of the M-function given by the summation formula (3.1) converges absolutely for $|z|<(1-q)^{-\zeta}$ given that $0<q<1$, $\zeta>0$.*

Proof. By Eq. (3.1), we have

$$ {}_{r}\overset{\zeta,\vartheta}{M}_{s}(q,z)=\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_{\xi}\cdots(q^{c_r};q)_{\xi}}{\left(q^{d_1};q\right)_{\xi}\cdots\left(q^{d_s};q\right)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}=\sum_{\xi=0}^{\infty}u_{\xi}(say). \tag{4.1}$$

Now, apply the ratio formula $\lim_{\xi\to\infty}\left|\frac{u_{\xi+1}}{u_{\xi}}\right|$on Eq. (3.1). So

$$\lim_{\xi\to\infty}\left|\frac{u_{\xi+1}}{u_{\xi}}\right| = \lim_{\xi\to\infty}\left|\frac{(q^{c_1};q)_{\xi+1}\cdots(q^{c_r};q)_{\xi+1}}{\left(q^{d_1};q\right)_{\xi+1}\cdots\left(q^{d_s};q\right)_{\xi+1}}\frac{\left(q^{d_1};q\right)_{\xi}\cdots\left(q^{d_s};q\right)_{\xi}}{(q^{c_1};q)_{\xi}\cdots(q^{c_r};q)_{\xi}}\frac{\Gamma_q(\zeta\xi+\vartheta)\,z^{\xi+1}}{\Gamma_q(\zeta\xi+\zeta+\vartheta)\,z^{\xi}}\right|.$$

Using results $(q^{\zeta},q)_{\xi}=\frac{(q^{\zeta},q)_{\infty}}{(q^{\zeta+\xi},q)_{\infty}}$ *and* $\Gamma_q(\xi)=(1-q)^{1-\xi}\frac{(q,q)_{\infty}}{(q^{\xi},q)_{\infty}}$, we get

$$\lim_{\xi\to\infty}\left|\frac{u_{\xi+1}}{u_{\xi}}\right|=\lim_{\xi\to\infty}\left|\frac{\frac{(q^{d_1},q)_{\infty}}{(q^{d_1+\xi},q)_{\infty}}\cdots\frac{(q^{d_s},q)_{\infty}}{(q^{d_s+\xi},q)_{\infty}}\;\frac{(q^{c_1},q)_{\infty}}{(q^{c_1+\xi+1},q)_{\infty}}\cdots\frac{(q^{c_r},q)_{\infty}}{(q^{c_r+\xi+1},q)_{\infty}}}{\frac{(q^{d_1},q)_{\infty}}{(q^{d_1+\xi+1},q)_{\infty}}\cdots\frac{(q^{d_s},q)_{\infty}}{(q^{d_s+\xi+1},q)_{\infty}}\;\frac{(q^{c_1},q)_{\infty}}{(q^{c_1+\xi},q)_{\infty}}\cdots\frac{(q^{c_r},q)_{\infty}}{(q^{c_r+\xi},q)_{\infty}}}\right.$$
$$\left.\times\frac{(1-q)^{1-\zeta\xi-\vartheta}\frac{(q,q)_{\infty}}{(q^{(\zeta\xi+\vartheta)},q)_{\infty}}}{(1-q)^{1-\zeta\xi-\vartheta-\zeta}\frac{(q,q)_{\infty}}{(q^{(\zeta\xi+\zeta+\vartheta)},q)_{\infty}}}z\right|$$
$$=\lim_{\xi\to\infty}\left|\frac{(q^{d_1+\xi+1},q)_{\infty}\cdots(q^{d_s+\xi+1},q)_{\infty}}{(q^{d_1+\xi},q)_{\infty}\cdots(q^{d_s+\xi},q)_{\infty}}\frac{(q^{c_1+\xi},q)_{\infty}\cdots(q^{c_r+\xi},q)_{\infty}}{(q^{c_1+\xi+1},q)_{\infty}\cdots(q^{c_r+\xi+1},q)_{\infty}}\right.$$
$$\left.\times\frac{(q^{\zeta\xi+\zeta+\vartheta},q)_{\infty}(1-q)^{\zeta}}{(q^{\zeta\xi+\vartheta},q)_{\infty}}z\right|$$

Using q-analogue of power function which is define in (2.3), Then, we obtain

$$\lim_{\xi\to\infty}\left|\frac{u_{\xi+1}}{u_{\xi}}\right|\leq\lim_{\xi\to\infty}\left|\frac{(1-q^{c_1+\xi})\cdots(1-q^{c_r+\xi})}{(1-q^{d_1+\xi})\cdots(1-q^{d_s+\xi})}(1-q^{\zeta\xi+\vartheta})^{\zeta}(1-q)^{\zeta}\right||z|.$$

$$\Rightarrow\lim_{\xi\to\infty}\left|\frac{u_{\xi+1}}{u_{\xi}}\right|\leq\begin{cases}(1-q)^{\zeta}\,|z|; & 0<|q|<1\\ 0; & q=1\end{cases}\quad as\,\xi\to\infty.$$

Hence series ${}_{r}\overset{\zeta,\vartheta}{M}_{s}(q,z)$ is convergent for $0<|q|<1$ if $|z|<(1-q)^{-\zeta}$. Further, if $q\to 1$ then (3.1) coincide with (1.1), which was defined by Sharma and Jain (2009).

5. Recurrence relation

Theorem 5.1. *If $\zeta,\vartheta\in\mathbb{C}$, $\Re(\zeta)>0$, $\Re(\vartheta)>0$;then there hold the formula*

$$ {}_{r}\overset{\zeta,\vartheta}{M}_{s}(q,z)={}_{r+1}\overset{\zeta,\vartheta}{M}_{s}(q,z)-\frac{q^{c_i}}{(1-q^{d_j})}z\;{}_{r+1}\overset{\zeta,\zeta+\vartheta}{M}_{s+1}(q,z)-\frac{q^{c_i+1}}{(1-q^{d_j})}z\;{}_{r+1}\overset{\zeta,\zeta+\vartheta}{M}_{s+1}(q,qz), \tag{5.1}$$

where $i=1,2,\ldots,r$ and $j=1,2,\ldots,s$.

Proof. To show the outcomes (5.1), we use the left-hand side and we can write the definition (3.1) as

$$ {}_{r}\overset{\zeta,\vartheta}{M}_{s}(q,z)=\frac{1}{\Gamma_q(\vartheta)}+\sum_{\xi=1}^{\infty}\frac{(q^{c_1};q)_{\xi}\cdots(q^{c_r};q)_{\xi}}{\left(q^{d_1};q\right)_{\xi}\cdots\left(q^{d_s};q\right)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}. \tag{5.2}$$

Here, we can express the above term as

$$ {}_{r}\overset{\zeta,\vartheta}{M}_{s}(q,z)=\frac{1}{\Gamma_q(\vartheta)}+\sum_{\xi=1}^{\infty}\frac{(q^{c_i};q)_{\xi}}{\left(q^{d_j};q\right)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)};\quad i=1,2,\ldots,r;\,j=1,2,\ldots,s. \tag{5.3}$$

On using the q-identity, we get

$$ {}_{r}\overset{\zeta,\vartheta}{M}_{s}(q,z)=\frac{1}{\Gamma_q(\vartheta)}+\sum_{\xi=1}^{\infty}\frac{(1-q^{c_i})(q^{c_i+1};q)_{\xi-1}}{\left(q^{d_j};q\right)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}. \tag{5.4}$$

Since $(1-q^{c})=(1-q^{c+\xi})-q^{c}(1-q^{\xi})$, the above Eq. (5.4) reduces to

$$\begin{aligned}{}_{r}\overset{\zeta,\vartheta}{M}_{s}(q,z)&=\frac{1}{\Gamma_q(\vartheta)}+\sum_{\xi=1}^{\infty}\frac{\left\{(1-q^{c_i+\xi})-q^{c_i}(1-q^{\xi})\right\}(q^{c_i+1};q)_{\xi-1}}{\left(q^{d_j};q\right)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}\\&=\frac{1}{\Gamma_q(\vartheta)}+\sum_{\xi=1}^{\infty}\frac{\left\{(1-q^{c_i+\xi})(q^{c_i+1};q)_{\xi-1}-q^{c_i}(1-q^{\xi})(q^{c_i+1};q)_{\xi-1}\right\}}{\left(q^{d_j};q\right)_{\xi}}\\&\times\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}\\&=\frac{1}{\Gamma_q(\vartheta)}+\sum_{\xi=1}^{\infty}\frac{\left\{(1-q^{c_i+\xi})(q^{c_i+1};q)_{\xi-1}\right\}}{\left(q^{d_j};q\right)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}\\&-\sum_{\xi=1}^{\infty}\frac{\left\{q^{c_i}(1-q^{\xi})(q^{c_i+1};q)_{\xi-1}\right\}}{\left(q^{d_j};q\right)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}.\end{aligned} \tag{5.5}$$

Using the q-identity in Eq. (5.5) as in denominator of second term, we get

$$\begin{aligned}{}_{r}\overset{\zeta,\vartheta}{M}_{s}(q,z)&=\frac{1}{\Gamma_q(\vartheta)}+\sum_{\xi=1}^{\infty}\frac{(q^{c_i+1};q)_{\xi}}{\left(q^{d_j};q\right)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}\\&-\frac{q^{c_i}}{(1-q^{d_j})}\sum_{\xi=1}^{\infty}\frac{(1-q^{\xi})(q^{c_i+1};q)_{\xi-1}}{(q^{d_j+1};q)_{\xi-1}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}\\&=\sum_{\xi=0}^{\infty}\frac{(q^{c_i+1};q)_{\xi}}{\left(q^{d_j};q\right)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}-\frac{q^{c_i}}{(1-q^{d_j})}\sum_{\xi=1}^{\infty}\frac{(q^{c_i+1};q)_{\xi-1}}{(q^{d_j+1};q)_{\xi-1}}\frac{(1-q^{\xi})z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}\\&=\sum_{\xi=0}^{\infty}\frac{(q^{c_i+1};q)_{\xi}}{\left(q^{d_j};q\right)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}-\frac{q^{c_i}}{(1-q^{d_j})}\sum_{\xi=1}^{\infty}\frac{(q^{c_i+1};q)_{\xi-1}}{(q^{d_j+1};q)_{\xi-1}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}\\&-\frac{q^{c_i}}{(1-q^{d_j})}\sum_{\xi=1}^{\infty}\frac{(q^{c_i+1};q)_{\xi-1}}{(q^{d_j+1};q)_{\xi-1}}\frac{(qz)^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}.\end{aligned} \tag{5.6}$$

By changing ξ by $\xi+1$ in second and third summation, the RHS of Eq. (5.6) becomes

$$\begin{aligned}{}_{r}\overset{\zeta,\vartheta}{M}_{s}(q,z)&=\sum_{\xi=0}^{\infty}\frac{(q^{c_i+1};q)_{\xi}}{\left(q^{d_j};q\right)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}-\frac{q^{c_i}}{(1-q^{d_j})}\sum_{\xi=0}^{\infty}\frac{(q^{c_i+1};q)_{\xi}}{(q^{d_j+1};q)_{\xi}}\frac{z^{\xi+1}}{\Gamma_q(\zeta(\xi+1)+\vartheta)}\\&-\frac{q^{c_i}}{(1-q^{d_j})}\sum_{\xi=0}^{\infty}\frac{(q^{c_i+1};q)_{\xi}}{(q^{d_j+1};q)_{\xi}}\frac{(qz)^{\xi+1}}{\Gamma_q(\zeta(\xi+1)+\vartheta)}\\&=\sum_{\xi=0}^{\infty}\frac{(q^{c_i+1};q)_{\xi}}{\left(q^{d_j};q\right)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)}-\frac{q^{c_i}z}{(1-q^{d_j})}\sum_{\xi=0}^{\infty}\frac{(q^{c_i+1};q)_{\xi}}{(q^{d_j+1};q)_{\xi}}\frac{z^{\xi+1}}{\Gamma_q(\zeta\xi+\zeta+\vartheta)}\\&-\frac{q^{c_i+1}z}{(1-q^{d_j})}\sum_{\xi=0}^{\infty}\frac{(q^{c_i+1};q)_{\xi}}{(q^{d_j+1};q)_{\xi}}\frac{(qz)^{\xi}}{\Gamma_q(\zeta\xi+\zeta+\vartheta)}.\end{aligned} \tag{5.7}$$

In view of the definition (3.1), the above expression becomes

$$ {}_{r}\overset{\zeta,\vartheta}{M}_{s}(q,z)={}_{r+1}\overset{\zeta,\vartheta}{M}_{s}(q,z)-\frac{q^{c_i}}{(1-q^{d_j})}z\,{}_{r+1}\overset{\zeta,\zeta+\vartheta}{M}_{s+1}\times(q,z)-\frac{q^{c_i+1}}{(1-q^{d_j})}z\,{}_{r+1}\overset{\zeta,\zeta+\vartheta}{M}_{s+1}(q,qz).$$

This concludes the proof of the result (5.1).

Theorem 5.2. *If $\zeta,\vartheta \in \mathbb{C}$, $\Re(\zeta)>0, \Re(\vartheta)>0$; then there hold the formula*

$$a\,z^{\zeta} q^{c_i-1}\,{}_r\overset{\zeta,\vartheta}{M}_s(q,a\,z^{\zeta}) = {}_r\overset{\zeta,\vartheta-\zeta}{M}_s(q,a\,z^{\zeta}) - {}_{r-1}\overset{\zeta,\vartheta-\zeta}{M}_{s+1}(q,a\,z^{\zeta}) \tag{5.8}$$

where $i=1,2,\dots,r$ and $j=1,2,\dots,s$.

Proof. To show the outcomes (5.8), we use the left-hand side and apply definition (3.1) as

$$a\,z^{\zeta} q^{c_i-1}\,{}_r\overset{\zeta,\vartheta}{M}_s(q,a\,z^{\zeta}) = q^{c_i-1}\sum_{\xi=0}^{\infty}\frac{(q^{c_i};q)_{\xi}}{(q^{d_j};q)_{\xi}}\frac{\left(a\,z^{\zeta}\right)^{\xi+1}}{\Gamma_q(\zeta\xi+\vartheta)}$$

On using the q-identity, namely

$$q^{c-1}(1-q^{\xi+1})(q^c;q)_{\xi} = (q^c;q)_{\xi+1} - (q^{c-1};q)_{\xi+1}, \tag{5.9}$$

and the q-identity defined in [Mishra et al., 2017, p. 6, Eq. (1.2.33)], we have

$$a\,z^{\zeta} q^{c_i-1}\,{}_r\overset{\zeta,\vartheta}{M}_s(q,a\,z^{\zeta}) = \sum_{\xi=0}^{\infty}\frac{(q^{c_i};q)_{\xi+1}-(q^{c_i-1};q)_{\xi+1}}{(q^{d_j};q)_{\xi}(1-q^{\xi+1})}\frac{\left(a\,z^{\zeta}\right)^{\xi+1}}{\Gamma_q(\zeta\xi+\vartheta)}.$$

Once more, the aforementioned series exists for $\xi=-1$ and the associated value is zero, therefore for $\Re(\vartheta)>\Re(\zeta)>0$, we may write

$$a\,z^{\zeta} q^{c_i-1}\,{}_r\overset{\zeta,\vartheta}{M}_s(q,a\,z^{\zeta}) = \sum_{\xi=-1}^{\infty}\frac{(q^{c_i};q)_{\xi+1}-(q^{c_i-1};q)_{\xi+1}}{(q^{d_j};q)_{\xi}(1-q^{\xi+1})}\frac{\left(a\,z^{\zeta}\right)^{\xi+1}}{\Gamma_q(\zeta\xi+\vartheta)}.$$

We obtain the right-hand side of (5.8) by replacing ξ with $\xi-1$ and proving that (3.1) is used.

6. Elementary properties of ${}_r\overset{\zeta,\vartheta}{M}_s(q,z)$

We start from the following theorem, by illustrates the integral representation of the q-analogue of the M-function.

6.1. *q*-integral representations

Theorem 6.1. *If $z,\zeta,\vartheta\in\mathbb{C}$, $\Re(\zeta)>0$, and $\Re(\vartheta)>\Re(\zeta)>0$, then*

$$\begin{aligned}{}_r\overset{\zeta,\vartheta}{M}_s(q,\,z) &= \frac{z^{\zeta-\vartheta}}{(1-q^{1/m})}\int_0^{\infty} e_q(-t^m/z^m)\,t^{\vartheta-\zeta-1}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_{\xi}\cdots(q^{c_r};q)_{\xi}}{(q^{d_1};q)_{\xi}\cdots(q^{d_s};q)_{\xi}}\\ &\times\frac{q^{\delta(\delta-1)/2}t^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)(q;q)_{\delta-1}}d_qt,\end{aligned} \tag{6.1}$$

where $\delta=(\vartheta-\zeta+\xi)/m$ and any non-zero positive integer is denoted by m.

Proof. To show the outcomes (6.1), we use the R.H.S. (say-R) of (6.1).

$$\begin{aligned}R &= \frac{z^{\zeta-\vartheta}}{(1-q^{1/m})}\int_0^{\infty} e_q(-t^m/z^m)\,t^{\vartheta-\zeta-1}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_{\xi}\cdots(q^{c_r};q)_{\xi}}{(q^{d_1};q)_{\xi}\cdots(q^{d_s};q)_{\xi}}\\ &\times\frac{q^{\delta(\delta-1)/2}t^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)\ (q;q)_{\delta-1}}d_qt.\end{aligned} \tag{6.2}$$

Substituting $t^m/z^m=w$, in see of the q-difference operator (2.8), we have

$$d_qt = \frac{(1-q^{1/m})}{(1-q)}z\,w^{1/m-1}d_qw.$$

As a result, we can write

$$\begin{aligned}R &= \frac{1}{(1-q)}\int_0^{\infty} e_q(-w)\,w^{((\vartheta-\zeta)/m)-1}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_{\xi}\cdots(q^{c_r};q)_{\xi}}{(q^{d_1};q)_{\xi}\cdots(q^{d_s};q)_{\xi}}\\ &\times\frac{q^{\delta(\delta-1)/2}\left(zw^{1/m}\right)^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)\ (q;q)_{\delta-1}}d_qw.\end{aligned} \tag{6.3}$$

If we reverse the integration and summation order, we have (6.1), under which the condition is valid.

$$\begin{aligned}R &= \frac{1}{(1-q)}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_{\xi}\cdots(q^{c_r};q)_{\xi}}{(q^{d_1};q)_{\xi}\cdots(q^{d_s};q)_{\xi}}\frac{q^{\delta(\delta-1)/2}\ (z)^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)\ (q;q)_{\delta-1}}\int_0^{\infty}e_q(-w)\,w^{\delta-1}d_qw\\ &= \frac{1}{(1-q)}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_{\xi}\cdots(q^{c_r};q)_{\xi}}{(q^{d_1};q)_{\xi}\cdots(q^{d_s};q)_{\xi}}\frac{q^{\delta(\delta-1)/2}\ (z)^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)\ (q;q)_{\delta-1}}L_q\left\{u^{\delta-1};1\right\},\end{aligned} \tag{6.4}$$

In this case, Hahn (1949) established the q-Laplace transform of $f(w)$, which is shown by $L_q\{f(w);s\}$.

$$L_q\{f(w);s\} = \frac{1}{(1-q)}\int_0^{\infty}e_q(sw)\,f(w)d_qw. \tag{6.5}$$

Using Abdi's previously known result (Abdi, 1961), namely

$$L_q\left\{w^{\delta-1};s\right\} = \frac{(q;q)_{\delta-1}q^{-\delta(\delta-1)/2}}{s^{\delta}},\quad \Re(\delta)>0. \tag{6.6}$$

We have

$$L_q\left\{w^{\delta-1};1\right\} = (q;q)_{\delta-1}q^{-\delta(\delta-1)/2},$$

as result (6.4) leads to the left-hand side of (6.1).

This brings the proof of (6.1).

Theorem 6.2. *If $z,\zeta,\vartheta\in\mathbb{C}$, $\Re(\zeta)>0$ and $\Re(\vartheta)>\Re(\zeta)>0$, then*

$${}_r\overset{\zeta,\vartheta}{M}_s(q,\,z) = \frac{(1-q)}{(1-q^{\zeta})\Gamma_q(\vartheta-\zeta)}\int_0^1\left(qt^{1/\zeta};q\right)_{\vartheta-\zeta-1}\,{}_r\overset{\zeta,\zeta}{M}_s(q,\,zt)d_qt. \tag{6.7}$$

Proof. Applying the definition (3.1) in the right-hands of (6.7) we say (L), we obtain

$$\begin{aligned}L &= \frac{(1-q)}{(1-q^{\zeta})\Gamma_q(\vartheta-\zeta)}\int_0^1\left(qt^{1/\zeta};q\right)_{\vartheta-\zeta-1}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_{\xi}\cdots(q^{c_r};q)_{\xi}}{(q^{d_1};q)_{\xi}\cdots(q^{d_s};q)_{\xi}}\\ &\times\frac{z^{\xi}t^{\xi}}{\Gamma_q(\zeta\xi+\zeta)}d_qt\\ &= \frac{(1-q)}{(1-q^{\zeta})\Gamma_q(\vartheta-\zeta)}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_{\xi}\cdots(q^{c_r};q)_{\xi}}{(q^{d_1};q)_{\xi}\cdots(q^{d_s};q)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\zeta)}\\ &\times\int_0^1 t^{\xi}\left(qt^{1/\zeta};q\right)_{\vartheta-\zeta-1}d_qt.\end{aligned}$$

Let $t=y^{\zeta}$ then $d_qt=\left((1-q^{\zeta})/(1-q)\right)y^{\zeta-1}d_qy$ and using Eq. (2.7), we have

$$L = \frac{(1-q)}{(1-q^{\zeta})\Gamma_q(\vartheta-\zeta)}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_{\xi}\cdots(q^{c_r};q)_{\xi}}{(q^{d_1};q)_{\xi}\cdots(q^{d_s};q)_{\xi}}\frac{(1-q^{\zeta})\Gamma_q(\vartheta-\zeta)\Gamma_q(\zeta\xi+\zeta)\,z^{\xi}}{\Gamma_q(\zeta\xi+\zeta)(1-q)\Gamma_q(\zeta\xi+\vartheta)}. \tag{6.8}$$

After simplification, we obtain left hand side of Eq. (6.7).

Theorem 6.3. *If $z,\zeta,\vartheta\in\mathbb{C}$, $\Re(\zeta)>0$, and $\Re(\vartheta)>\Re(\zeta)>0$, then*

$${}_r\overset{\zeta,\vartheta}{M}_s(q,\,z) = \frac{1}{\Gamma_q(\zeta)}\int_0^1 t^{\zeta-1}(qt;q)_{\vartheta-\zeta-1}\,{}_r\overset{\zeta,\vartheta-\zeta}{M}_s(q,\,z(1-tq^{\vartheta-\zeta})^{(\zeta)})d_qt. \tag{6.9}$$

Proof. Applying the definition (3.1), we obtain

$${}_r\overset{\zeta,\vartheta}{M}_s(q,\,z) = \frac{1}{\Gamma_q(\zeta)}\int_0^1 t^{\zeta-1}(qt;q)_{\vartheta-\zeta-1}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_{\xi}\cdots(q^{c_r};q)_{\xi}}{(q^{d_1};q)_{\xi}\cdots(q^{d_s};q)_{\xi}}\frac{z^{\xi}(tq^{\vartheta-\zeta};q)_{\xi\zeta}}{\Gamma_q(\zeta\xi+\vartheta-\zeta)}d_qt.$$

The preceding equation results in when the integration and summation are performed in a different order and the q-identity is used [Gasper and Rahman, 1990, P. 234, 1.17].

$$\begin{aligned}{}_r\overset{\zeta,\vartheta}{M}_s(q,\,z) &= \frac{1}{\Gamma_q(\zeta)}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_{\xi}\cdots(q^{c_r};q)_{\xi}}{(q^{d_1};q)_{\xi}\cdots(q^{d_s};q)_{\xi}}\frac{z^{\xi}}{\Gamma_q(\zeta\xi+\vartheta-\zeta)}\\ &\times\int_0^1 t^{\zeta-1}(tq;q)_{\xi\zeta+\vartheta-\zeta-1}\,d_qt.\end{aligned} \tag{6.10}$$

From (2.7) and (3.1), Eq. (6.10) becomes to the left-hand side of (6.9).

6.2. *q*-derivative

Theorem 6.4. *For $z,\zeta,\vartheta\in\mathbb{C}$, $\Re(\zeta)>0$ and $\Re(\vartheta)>\Re(\zeta)>0$, then for any $m\in\mathbb{N}$, we have*

$$D_q^m\left[z^{\vartheta-1}\,{}_r\overset{\zeta,\vartheta}{M}_s(q,\,\gamma z^{\zeta})\right] = z^{\vartheta-m-1}\,{}_r\overset{\zeta,\vartheta-m}{M}_s(q,\,\gamma z^{\zeta}). \tag{6.11}$$

Proof. By using the function $f(z) = z^{\vartheta-1}{}_r\overset{\zeta,\vartheta}{M}_s(q,\gamma z^{\zeta})$ in (2.8) and using the definition (3.1), we obtain

$$D_q\left[z^{\vartheta-1}{}_r\overset{\zeta,\vartheta}{M}_s(q,\gamma z^{\zeta})\right] = \sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi}\frac{(1-q^{\zeta\xi+\vartheta-1})\gamma^\xi z^{\zeta\xi+\vartheta-2}}{(1-q)\Gamma_q(\zeta\xi+\vartheta)}. \quad (6.12)$$

The functional relation as

$$\Gamma_q(\varepsilon+1) = \frac{1-q^\varepsilon}{1-q}\Gamma_q(\varepsilon),$$

the right-hand side of Eq. (6.12) expression becomes

$$\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi}\frac{\gamma^\xi z^{\zeta\xi+\vartheta-2}}{\Gamma_q(\zeta\xi+\vartheta-1)} = z^{\vartheta-2}{}_r\overset{\zeta,\vartheta-1}{M}_s(q,\gamma z^{\zeta}).$$

Iterating this result, up to $m-1$ times, we have the required outcome (6.11).

6.3. q-Laplace transform

Theorem 6.5. *For $z,s,\zeta,\vartheta\in\mathbb{C}$, $\Re(\zeta)>0$, $\Re(s)>0$, and $\Re(\vartheta)>\Re(\zeta)>0$, then*

$${}_qL_s\left[{}_r\overset{\zeta,\vartheta}{M}_s(q,\gamma z^{\delta})\right] = \frac{1}{s}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi}\frac{\Gamma_q(1+\delta\xi)}{\Gamma_q(\zeta\xi+\vartheta)}\left(\frac{\gamma(1-q)^\delta}{s^\delta}\right)^\xi. \quad (6.13)$$

Proof. The following q- integral defines the q-Laplace transform of an appropriate function.

$${}_qL_s[f(z)] = \frac{1}{(1-q)}\int_0^{s^{-1}}E_q(qsz)f(z)d_qz, \quad (6.14)$$

since the q-exponential series is given by

$$E_q(z) = \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n}z^n = (-z;q)_\infty.$$

By applying the q-integral equation (2.10) and the above q-exponential series, Eq. (6.14) can be write as

$${}_qL_s[f(z)] = \frac{(q;q)_\infty}{s}\sum_{l=0}^{\infty}\frac{q^l f(s^{-1}q^l)}{(q;q)_l}. \quad (6.15)$$

Now, by applying the q-Laplace transform definition and definition (3.1), we get

$${}_qL_s\left[{}_r\overset{\zeta,\vartheta}{M}_s(q,\gamma z^{\delta})\right] = \frac{(q;q)_\infty}{s}\sum_{l=0}^{\infty}\frac{q^l}{(q;q)_l}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi}\frac{\{\gamma(s^{-1}q^l)^\delta\}^\xi}{\Gamma_q(\zeta\xi+\vartheta)}.$$

On switching the order of summations and summing the resulting inner ${}_0\phi_0(.)$ series using the result of Gasper and Rahman (Miller, 1993), namely,

$${}_0\phi_0(-;-;q,z) = \frac{1}{(z;q)_\infty}, \quad (6.16)$$

we obtain

$${}_qL_s\left[{}_r\overset{\zeta,\vartheta}{M}_s(q,\gamma z^{\delta})\right] = \frac{(q;q)_\infty}{s}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi(q^{1+\delta\xi};q)_\infty}\frac{\{\gamma\}^\xi}{\Gamma_q(\zeta\xi+\vartheta)s^{\delta\xi}}.$$

This, after certain simplifications reduces to the right-hand side of (6.13).

$$\begin{aligned}{}_qL_s\left[{}_r\overset{\zeta,\vartheta}{M}_s(q,\gamma z^{\delta})\right] &= \frac{1}{s}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi(1-q)^{-\delta\xi}}\frac{\Gamma_q(1+\delta\xi)\{\gamma\}^\xi}{\Gamma_q(\zeta\xi+\vartheta)s^{\delta\xi}}\\ &= \frac{1}{s}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi}\frac{\Gamma_q(1+\delta\xi)}{\Gamma_q(\zeta\xi+\vartheta)}\left(\frac{\gamma(1-q)^\delta}{s^\delta}\right)^\xi.\end{aligned}$$

The completes the proof.

6.4. q-Sumudu transform

Theorem 6.6. *For $z,s,\zeta,\vartheta\in\mathbb{C}$, $\Re(\zeta)>0$, $\Re(s)>0$, and $\Re(\vartheta)>\Re(\zeta)>0$, then*

$${}_tS_q\left[{}_r\overset{\zeta,\vartheta}{M}_s(q,\gamma z^{\delta})\right] = \sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi}\frac{\Gamma_q(1+\delta\xi)}{\Gamma_q(\zeta\xi+\vartheta)}\left(\gamma t^\delta(1-q)^\delta\right)^\xi. \quad (6.17)$$

Proof. By the definition of (3.1), we have

$${}_r\overset{\zeta,\vartheta}{M}_s(q,\gamma z^{\delta}) = \sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi}\frac{\gamma^\xi z^{\delta\xi}}{\Gamma_q(\zeta\xi+\vartheta)} = f(z)(say).$$

and

$$f(tq^m) = \sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi}\frac{\gamma^\xi t^{\delta\xi}q^{\delta m\xi}}{\Gamma_q(\zeta\xi+\vartheta)}. \quad (6.18)$$

The q-Sumudu transform (Albayrak et al., 2013) is defined as

$${}_tS_q[f(\eta)] = \frac{1}{(1-q)t}\int_0^t E_q\{(q/t)\eta\}f(\eta)d_q\eta = (q;q)_\infty\sum_{m=0}^{\infty}\frac{q^m f(tq^m)}{(q;q)_m}, t>0, \quad (6.19)$$

where $E_q(\eta) = \sum_{m=0}^{\infty}\frac{(-1)^m q^{\{m(m-1)/2\}}\eta^m}{(q;q)_m} = (\eta;q)_\infty$, $(\eta\in\mathbb{C})$ is classical exponential.

Using the definition of Sumudu transform (6.19) and (6.16), We obtain

$${}_tS_q\left[{}_r\overset{\zeta,\vartheta}{M}_s(q,\gamma z^{\delta})\right] = (q;q)_\infty\sum_{m=0}^{\infty}\frac{q^m}{(q;q)_m}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi}\frac{\gamma^\xi t^{\delta\xi}q^{\delta m\xi}}{\Gamma_q(\zeta\xi+\vartheta)}.$$

Switching the order of summation, we obtain

$$\begin{aligned}{}_tS_q\left[{}_r\overset{\zeta,\vartheta}{M}_s(q,\gamma z^{\delta})\right] &= \sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi}\frac{\gamma^\xi t^{\delta\xi}}{\Gamma_q(\zeta\xi+\vartheta)}\sum_{m=0}^{\infty}\frac{(q;q)_\infty q^{m(1+\delta\xi)}}{(q;q)_m}\\ &= \sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi}\frac{\gamma^\xi t^{\delta\xi}}{\Gamma_q(\zeta\xi+\vartheta)}\sum_{m=0}^{\infty}(q^{1+m};q)_\infty q^{m(1+\delta\xi)}.\end{aligned}$$

Using the result $\sum_{m=0}^{\infty}(q^{1+m};q)_\infty q^{(1+\delta\xi)m} = \Gamma_q(1+\delta\xi)(1-q)^{\delta\xi}$, we obtain

$${}_tS_q\left[{}_r\overset{\zeta,\vartheta}{M}_s(q,\gamma z^{\delta})\right] = \sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi}\frac{\Gamma_q(1+\delta\xi)}{\Gamma_q(\zeta\xi+\vartheta)}\left(\gamma t^\delta(1-q)^\delta\right)^\xi.$$

This is the result (6.17).

7. Fractional q-derivative of ${}_r\overset{\zeta,\vartheta}{M}_s(q,z)$

Theorem 7.1 (*Caputo Fractional q-Derivative*). *If $z,\zeta,\vartheta\in\mathbb{C}$, $\Re(\zeta)>0$, then for $\mu\in\mathbb{N}$*

$$\begin{aligned}&\left[{}_cD^\mu_{q,a+}\left\{{}_r\overset{\zeta,\vartheta}{M}_s(q,(t-qa)^\mu_q)\right\}\right](z) = (z-qa)_q^{-\mu}\\ &\times\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi\,\Gamma_q(1+\mu\xi)\;(z-qa)_q^{\mu\,\xi}}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi\,\Gamma_q(\zeta\xi+\vartheta)\,\Gamma_q(1+\mu\xi-\mu)}. \quad (7.1)\end{aligned}$$

Proof. The Caputo fractional q-derivative (Rajkovic et al., 2009) is given by

$$\left[{}_cD^\mu_{q,a+}f\right](z) = \left[I^{1-\mu}_{q,a+}D_qf\right](z) = \frac{1}{\Gamma_q(1-\mu)}\int_a^z(z-qt)_q^{-\mu}f(t)d_qt. \quad (7.2)$$

Now, by using (3.1) and (7.2), we have

$$\begin{aligned}&\left[{}_cD^\mu_{q,a+}\left\{{}_r\overset{\zeta,\vartheta}{M}_s(q,(t-qa)^\mu_q)\right\}\right](z)\\ &= \left[{}_cD^\mu_{q,a+}\sum_{\xi=0}^{\infty}\frac{(q^{c_1};q)_\xi\cdots(q^{c_r};q)_\xi}{(q^{d_1};q)_\xi\cdots(q^{d_s};q)_\xi}\frac{(t-qa)_q^{\mu\,\xi}}{\Gamma_q(\zeta\xi+\vartheta)}\right](z)\end{aligned}$$

$$= \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi \, \Gamma_q(\zeta\xi+\vartheta)} \left[{}_cD^{\mu}_{q,a+} \, (t-qa)_q^{\mu\,\xi} \right](z)$$
$$= \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi \, \Gamma_q(\zeta\xi+\vartheta)} \left[I^{1-\mu}_{q,a+} \, D_q \, (t-qa)_q^{\mu\,\xi} \right](z).$$

Using (2.9), we get

$$\left[{}_cD^{\mu}_{q,a+} \left\{ {}_r\overset{\zeta,\vartheta}{M}_s(q,\,(t-qa)_q^{\mu}) \right\} \right](z) = \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi \, \Gamma_q(\zeta\xi+\vartheta)} \times \left[I^{1-\mu}_{q,a+} \, [\mu\xi]_q \, (t-qa)_q^{\mu\,\xi-1} \right](z)$$
$$= \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi \, \Gamma_q(\zeta\xi+\vartheta)} [\mu\xi]_q \times \left[I^{1-\mu}_{q,a+} \, (t-qa)_q^{\mu\,\xi-1} \right](z).$$

Using (2.11), we obtain

$$\left[{}_cD^{\mu}_{q,a+} \left\{ {}_r\overset{\zeta,\vartheta}{M}_s(q,\,(t-qa)_q^{\mu}) \right\} \right](z) = \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi \, \Gamma_q(\mu\xi)}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi \, \Gamma_q(\zeta\xi+\vartheta)\, \Gamma_q(1+\mu\xi-\mu)} \times [\mu\xi]_q \left[(t-qa)_q^{\mu\,\xi-\mu} \right]$$
$$= \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi \, \Gamma_q(1+\mu\xi)\,(t-qa)_q^{\mu\,\xi-\mu}}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi \, \Gamma_q(\zeta\xi+\vartheta)\, \Gamma_q(1+\mu\xi-\mu)}$$
$$\Rightarrow \left[{}_cD^{\mu}_{q,a+} \left\{ {}_r\overset{\zeta,\vartheta}{M}_s(q,\,(t-qa)_q^{\mu}) \right\} \right](z)$$
$$= (z-qa)_q^{-\mu} \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi \, \Gamma_q(1+\mu\xi) \;\; (z-qa)_q^{\mu\,\xi}}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi \, \Gamma_q(\zeta\xi+\vartheta)\, \Gamma_q(1+\mu\xi-\mu)}.$$

This is the desired result (7.1).

Theorem 7.2 (*Hilfer Fractional q-Derivative*). *If* $z, \zeta, \vartheta \in \mathbb{C}$, $\Re(\zeta) > 0$, *then for* $\mu \in \mathbb{N}$

$$\left[{}_cD^{\mu,\omega}_{q,a+} \left\{ {}_r\overset{\zeta,\vartheta}{M}_s(q,\,(t-qa)_q^{\mu}) \right\} \right](z) = (z-qa)_q^{-\mu} \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi} \times \frac{\Gamma_q(1+\mu\xi)\,(z-qa)_q^{\mu\,\xi}}{\Gamma_q(\zeta\xi+\vartheta)\, \Gamma_q(1+\mu\xi-\mu)} \tag{7.3}$$

Proof. The Hilfer fractional q-derivative (Hilfer, 2000, 2002) of $0 < \mu < 1$ and $0 \le \omega \le 1$ is defined as

$$\left[{}_cD^{\mu,\omega}_{q,a+} f \right](z) = \left[I^{(1-\mu)\omega}_{q,a+} \, D_q \left(I^{(1-\mu)(1-\omega)}_{q,a+} \right) \right](z). \tag{7.4}$$

Using (7.4) and (3.1), we have

$$\left[D^{\mu,\omega}_{q,a+} \left\{ {}_r\overset{\zeta,\vartheta}{M}_s(q,\,(t-qa)_q^{\mu}) \right\} \right](z)$$
$$= D^{\mu,\omega}_{q,a+} \left[\sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi} \frac{(t-qa)_q^{\mu\,\xi}}{\Gamma_q(\zeta\xi+\vartheta)} \right](z)$$
$$= \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi} \frac{1}{\Gamma_q(\zeta\xi+\vartheta)} \left[D^{\mu,\omega}_{q,a+} \, (t-qa)_q^{\mu\,\xi} \right](z)$$
$$= \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi} \frac{1}{\Gamma_q(\zeta\xi+\vartheta)} \times \left[I^{(1-\mu)\omega}_{q,a+} \, D_q \left(I^{(1-\mu)(1-\omega)}_{q,a+} \right) (t-qa)_q^{\mu\,\xi} \right](z).$$

Using (2.11), we get

$$\left[D^{\mu,\omega}_{q,a+} \left\{ {}_r\overset{\zeta,\vartheta}{M}_s(q,\,(t-qa)_q^{\mu}) \right\} \right](z)$$
$$= \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi} \frac{1}{\Gamma_q(\zeta\xi+\vartheta)} \left[\frac{\left(I^{(1-\mu)\omega}_{q,a+} \, D_q \, (t-qa)_q^{1-\mu-\omega+\mu\omega+\mu\xi} \right)(z)}{\left(\frac{\Gamma_q(2-\mu-\omega+\mu\omega+\mu\xi)}{\Gamma_q(1+\mu\xi)} \right)} \right].$$

Using (2.9), we get

$$\left[D^{\mu,\omega}_{q,a+} \left\{ {}_r\overset{\zeta,\vartheta}{M}_s(q,\,(t-qa)_q^{\mu}) \right\} \right](z)$$
$$= \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi} \frac{1}{\Gamma_q(\zeta\xi+\vartheta)} \left[\frac{\left(I^{(1-\mu)\omega}_{q,a+} \, D_q \, (t-qa)_q^{-\mu-\omega+\mu\omega+\mu\xi} \right)(z)}{\left(\frac{\Gamma_q(2-\mu-\omega+\mu\omega+\mu\xi)}{(1-\mu-\omega+\mu\omega+\mu\xi)_q \, \Gamma_q(1+\mu\xi)} \right)} \right].$$

Again, using (2.11), we get

$$\left[D^{\mu,\omega}_{q,a+} \left\{ {}_r\overset{\zeta,\vartheta}{M}_s(q,\,(t-qa)_q^{\mu}) \right\} \right](z)$$
$$= \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi \, \Gamma_q(\zeta\xi+\vartheta)} \times \left[\frac{\left(\frac{\Gamma_q(1-\mu-\omega+\mu\omega+\mu\xi)}{\Gamma_q(1-\mu-\omega+\mu\omega+\mu\xi+\omega-\mu\omega)} \right)(z-qa)_q^{\omega-\mu\omega-\mu-\omega+\mu\omega+\mu\xi}}{\left(\frac{\Gamma_q(1-\mu-\omega+\mu\omega+\mu\xi)}{\Gamma_q(1+\mu\xi)} \right)} \right].$$
$$= \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi \, \Gamma_q(1+\mu\xi)}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi \, \Gamma_q(\zeta\xi+\vartheta)\, \Gamma_q(1-\mu+\mu\xi)} (z-qa)_q^{-\mu+\mu\xi}$$
$$\Rightarrow \left[D^{\mu,\omega}_{q,a+} \left\{ {}_r\overset{\zeta,\vartheta}{M}_s(q,\,(t-qa)_q^{\mu}) \right\} \right](z)$$
$$= (z-qa)_q^{-\mu} \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi \, \Gamma_q(1+\mu\xi)}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi \, \Gamma_q(\zeta\xi+\vartheta)\, \Gamma_q(1-\mu+\mu\xi)} (z-qa)_q^{\mu\xi}.$$

This is the required result (7.3).

8. Concluding observations

A few of the ramifications of the findings discussed in the previous sections are briefly discussed below. For example, the following is the outcome of Theorem 6.1 when $m = 1$, is used:

Corollary 8.1. *If* $z, \zeta, \vartheta \in \mathbb{C}$, $\Re(\zeta) > 0$, *and* $\Re(\vartheta) > \Re(\zeta) > 0$, *then*

$${}_r\overset{\zeta,\vartheta}{M}_s(q,\,z) = \frac{z^{\zeta-\vartheta}}{(1-q)} \int_0^{\infty} e_q(-t/z)\, t^{\vartheta-\zeta-1} \times \sum_{\xi=0}^{\infty} \frac{(q^{c_1};q)_\xi \cdots (q^{c_r};q)_\xi}{(q^{d_1};q)_\xi \cdots (q^{d_s};q)_\xi} \frac{q^{(\vartheta-\zeta+\xi)(\vartheta-\zeta+\xi-1)/2} t^{\xi}}{\Gamma_q(\zeta\xi+\vartheta)\,(q;q)_{\vartheta-\zeta+\xi-1}} d_q t. \tag{8.1}$$

For $q \to 1^-$, Corollary 8.1 and Theorem 6.2–6.3 yields the following outcomes involving integral representation for the M-series (1.1).

Corollary 8.2. *If* $z, \zeta, \vartheta \in \mathbb{C}$, $\Re(\zeta) > 0$, *and* $\Re(\vartheta) > \Re(\zeta) > 0$, *then*

$${}_r\overset{\zeta,\vartheta}{M}_s(z) = z^{\zeta-\vartheta} \int_0^{\infty} \exp(-t/z)\, t^{\vartheta-\zeta-1} \times \sum_{\xi=0}^{\infty} \frac{(c_1)_\xi \, \ldots \, (c_r)_\xi}{(d_1)_\xi \, \ldots \, (d_s)_\xi} \frac{t^{\xi}}{\Gamma(\zeta\xi+\vartheta)\,\Gamma(\vartheta-\zeta+\xi)} dt. \tag{8.2}$$

Corollary 8.3. *If* $z, \zeta, \vartheta \in \mathbb{C}$, $\Re(\zeta) > 0$ *and* $\Re(\vartheta) > \Re(\zeta) > 0$, *then*

$${}_r\overset{\zeta,\vartheta}{M}_s(q,\,z) = \frac{1}{\Gamma(\vartheta-\zeta)} \int_0^1 \left(1-t^{1/\zeta}\right)^{\vartheta-\zeta-1} {}_r\overset{\zeta,\zeta}{M}_s(zt)dt. \tag{8.3}$$

Corollary 8.4. *If* $z, \zeta, \vartheta \in \mathbb{C}$, $\Re(\zeta) > 0$, *and* $\Re(\vartheta) > \Re(\zeta) > 0$, *then*

$${}_r\overset{\zeta,\vartheta}{M}_s(q,\,z) = \frac{1}{\Gamma_q(\zeta)} \int_0^1 t^{\zeta-1}(1-t)^{\vartheta-\zeta-1} \, {}_r\overset{\zeta,\vartheta-\zeta}{M}_s(z(1-t)^{\zeta})dt. \tag{8.4}$$

Further, If we set $r = s = 1, c_1 = \gamma, d_1 = \delta$ and use Eq. (3.5), the outcomes of Theorem 5.1 and Theorem 6.4 gives, respectively, the known results due to Sharma and Jain [Sharma and Jain, 2016, p.793, Eq. (2).1] and Sharma and Jain [Sharma and Jain, 2016, p.794, Eq. (2).2]. Similar in way, by setting $r = s = 1, c_1 = \delta, d_1 = 1$ and special case defined in (3.4), the results of Theorem 5.2 gives, the known results due to Purohit and Kalla [Purohit and Kalla, 2011, p.19, Eq. (3).9].

The q-analogue of M-functions and their properties discussed in this article can be used to obtain results with extensions of the q-exp functions, the q-ML functions, and the q-hypergeometric function. They can also be applied to investigate solutions to a large number of fractional q-integral and q-difference equations. Finally, we conclude by noting that these properties hold true.

CRediT authorship contribution statement

Biniyam Shimelis: Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing – original draft. **D.L. Suthar:** Conceptualization, Formal analysis, Methodology, Validation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

We have not used any data for this study.

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