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Original article

Wiener polarity and Wiener index of double generalized Petersen graph

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A R T I C L E I N F O

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A B S T R A C T

A topological index is a numerical parameter of a graph which characterizes some of the topological properties of the graph. The concepts of Wiener polarity index and Wiener index were established in chemical graph theory by means of the distances. The double generalized Petersen graph denoted by $DP(n,k)$ is obtained by attaching the vertices of outer pendent vertices to inner pendent vertices lying at distance k. The length of the outer and inner cycle is n, thus the number of vertices are $4n$ and the number of edges in the $DP(n,k)$ are $6n$. In this paper, the Wiener polarity index of $DP(n,k$ for $3 \leq n \leq 6$ and for $n \geq 6k+1$ is computed. Further, the Wiener index of $DP(n,k)$, for $k = \{1,2\}$ is determined.

1. Introduction

Let G be a simple, connected, undirected graph with a vertex set $V(G)$ and an edge set $E(G)$. The *distance* between two vertices v_i and v_j, denoted by $d(v_i, v_j)$ is the length of the shortest path between the vertices v_i and v_j in G. The diameter $diam(G)$ of a connected graph G is the length of any longest geodesic. The *degree* of a vertex v_i in G is the number of edges incident to v_i and is denoted by $d_i = deg(v_i)$ (Adnan et al., 2021; Buckley and Harary, 1990).

In the modern age, network structures have great significance in the field of chemistry, information technology, communication, and physical structures. A *topological index* of graph G is a numerical quantity that describes the topology of the graph. It reflects the theoretical properties of chemical compounds when applied to the molecular structure of the chemical compounds. Several topological indices have been proposed so far by various researchers.The Wiener index was the first and most studied topological index (see for details in (Wiener, 1947)). Wiener demonstrates that the Wiener index number is strongly related to the boiling points of alkane molecules. In chemistry, it was the first molecular topological index used. Since then, many indices that relate topological indices to different physical properties have been introduced in (Bokhary et al., 2021; Bokhary and Adnan, 2021; Ul Haq Bokhary et al., 2021).

Generalized Petersen Graphs and Double Generalized Petersen graphs are extensively studied graph networks. Many graph properties like metric dimension, partition dimension, different kinds of graph labelings, etc of this family are already been explored. In (Liu et al., 2017), Hamiltonian cycles is studied.

The *Wiener polarity index* of G is denoted by $W_p(G)$ and defined as

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$W_p(G) = |\{\{u, v\}|d(u, v) = 3, u, v \in V(G)\}|$.

The name "Wiener polarity index" is introduced by Harold Wiener in 1947 (Ul Haq Bokhary et al., 2021).

The *Wiener index* of a graph G is denoted by $W(G)$ and is defined as:

$$W(G) = \sum_{u,v \in V(G)} d(x, y) = \sum_{k \geq 1} k\gamma(G, k).$$

The popularity of these indices is due to numerous of their chemical applications and mathematical properties reported in (Bokhary et al., 2021; Bokhary and Adnan, 2021; Du et al., 2008; Graovac and Pisanski, 1991; Harary, n.d.; Ul Haq Bokhary et al., 2021)

Consider a graph consisting of two cycles; one is called an outer and the other is an inner cycle. The vertex set of outer and inner cycles is denoted by X and Y, respectively. Let, each of the vertices of the cycles are attached to a pendant vertex. The pendant vertices attached to the outer cycle are called outer pendent vertices whereas the inner pendent vertices are the vertices are attached to the inner cycle. The vertex set of outer and inner pendent vertices is denoted by U and V, respectively. The double generalized Petersen graph denoted by $DP(n, k)$ is a graph obtained by attaching outer pendent vertices to inner pendent vertices lying at distance k. The vertices of $DP(n, k)$ are defined as follows:

$V(DP(n, k)) = \{X \cup Y \cup U \cup V\}$

$X = \{x_i\}$, $Y = \{y_i\}$, $U = \{u_i\}$, $V = \{V_i\}$ for $1 \leq i \leq n$.

The edge set is defined as follows:

$E(DP(n, k)) = \{x_i x_{i+1}, x_i u_i, u_i v_{i+k}, y_i y_{i+1}, y_i u_i : 1 \leq i \leq n\}$

The construction follows that the order of the graph $DP(n, k)$ is $6n$. The graph of $DP(6, 1)$ is depicted in Fig. 1. (See Fig 2 Fig 3)

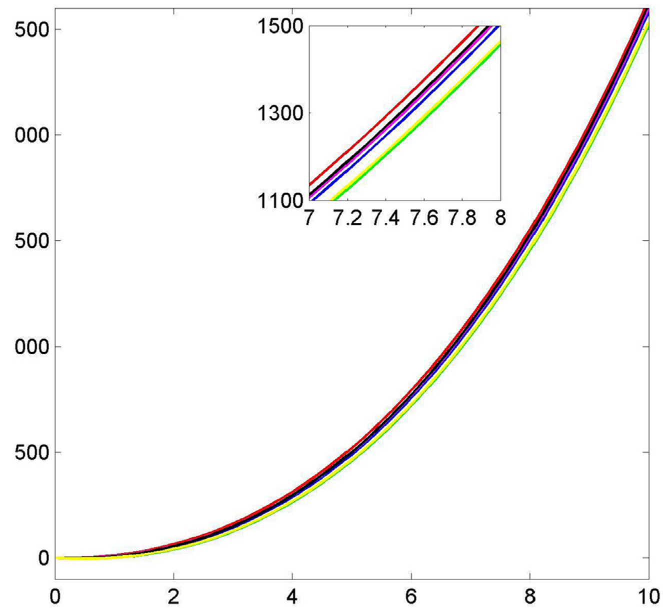

Fig. 2. Graphical representation of $W(q, 1)$ for different values of q

In this paper, the Wiener polarity and the Wiener index of dthe graphs $DP(n, k)$, where n and k be the nonnegative integers are investigated. Throughout this paper, the Wiener polarity index and the Wiener index of double generalized Petersen graph $DP(n, k)$ will be denoted by $W_p(q, k)$ and $W(q, k)$, respectively. All the indices that follow henceforth are taken under modulo n.

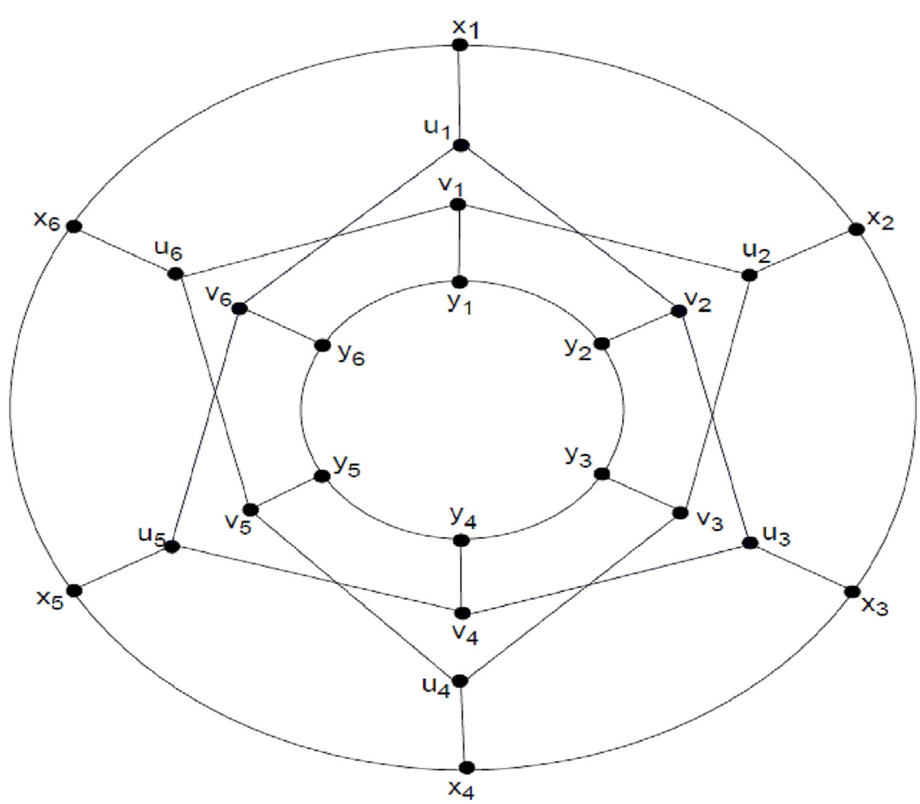

Fig. 1. The double generalized Petersen graph $DP(6, 1)$

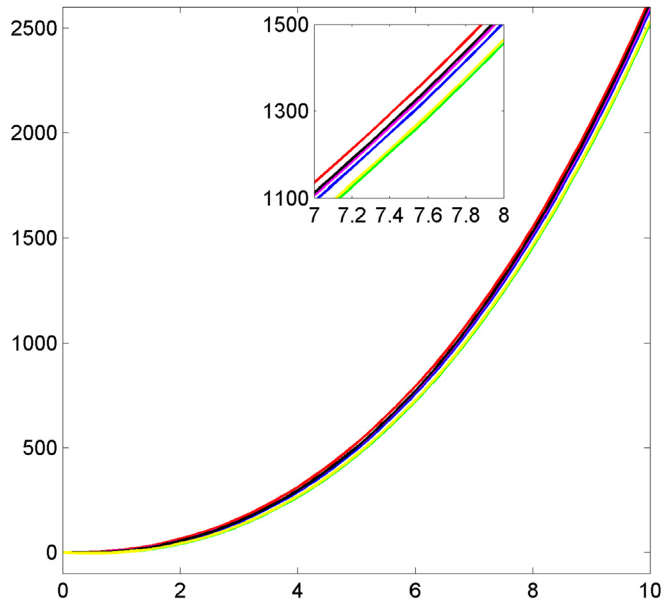

Fig. 3. Graphical representation of $W(q,2)$ for different values of q

2. The Wiener polarity index of Double Generalized Petersen graphs

In this section, the $W_p(n,1)$ of $DP(n,k))$ is computed for different values of n and k.

Theorem 2.1. *For* $n \geq 3$, *the Wiener polarity index* $W_p(n,1)$ *is.*

$$W_p(n,1) = \begin{cases} 15 & for n = 3 \\ 40 & for n = 4 \\ 80 & for n = 5 \\ 96 & for n = 6 \\ 18n & for n \geq 7 \end{cases}$$

Proof. For $3 \leq n \leq 6$, it is easy to verify the results. For $n \geq 7$ and $1 \leq l \leq n$, let a be an arbitrary vertex of $DP(n,k)$. Then, the vertex a can belong to either of the set X, Y, U or V:

Case 1. *Let* $a \in X$, *then* a *be one of the outer vertex* x_l. *The vertices* $x_{l+3}, u_{l+2}, u_{l-2}, v_{l+2}, v_{l-2}, v_l$, y_{l+1}, y_{l-1} *are the vertices which have a distance of three from* x_l. *There are* $8n$ *vertices that have a distance of three from the vertices of the set* X.

Case 2. *Let* $a \in U$. *In this case,* a *be one of the inner vertex* u_l. *The vertices* $x_{l+2}, x_{l-2}, u_{l+1}, v_{l+3}, v_{l-3}, y_{l+2}, y_{l-2}, y_l$ *are the vertices at distance three from each* u_l. *So,* $6n$ *vertices from* $8n$ *that are yet to be counted.*

Case 3. *Let* $a \in V$, *then* a *is one of the inner vertex* v_l. *The vertices* $x_{l+2}, x_{l-2}, x_l, u_{l+3}, u_{l-3}, v_{l+1}, y_{l+2}, y_{l-2}$ *are the vertices at distance of three from each vertex of* V. *So,* $3\ n$ *vertices from* $8n$ *which are yet to be counted.*

Case 4. *Let* $a \in Y$, *then* a *be one of the inner vertex* y_i. *The vertices* $x_{l+1}, x_{l-1}, u_{l+2}, u_{l-2}, v_{l+2}, u_l$, v_{l-2}, y_{l+3} *are the vertices at distance of three. So,* n *vertices from* $8n$ *which are yet to be counted.The above cases imply that.*

$W_p(n,1) = 8n + 6n + 3n + n = 18n.$

Theorem 2.2. *For* $n \geq 6k+1$, $W_p(n,k) = 24n$, *where* $k \in \mathbb{N}$.

Proof. For $1 \leq l \leq n$, let a be an arbitrary vertex of $DP(n,k)$. Then, the vertex a can belong to either of the set X, Y, U or V:

If $a \in X$, then a is one of the outer vertex x_l. The vertices x_{l+3}, $u_{l+2}, u_{l-2}, u_{l+2k}, u_{l-2k}, v_{l+1+k}, v_{l+1-k}, v_{l+1}, v_{l-k-1} y_{l+k}$, and y_{l-k} are the vertices at distance three from each x_l. It is clear that u_{l+2k} and u_{l-2k} are the vertices that have the largest gap from the vertex x_l. Thus, the maximum value of n for which the vertices u_{l+2k} and u_{l-2k} can be equal is $n = 4k+1$ but $n \geq 6k+1$. Therefore, u_{l+2k} and u_{l-2k} are distinct vertices and hence, all the vertices mentioned above are distinct. Since, $1 \leq i \leq n$, therefore there are total $11n$ vertices that have a distance of three from the set X.

If the vertex a belongs to set U or V. Without loss of generality, one can suppose that $a \in U$, then a is one of the inner vertex u_l. The vertices $x_{l+2}, x_{l-2}, u_{l+1}, v_{l+3k}, v_{l-3k}, y_{l+1+k}, y_{l+1-k}, y_{l+3}$,y_{l-3} are the vertices at distance three. The vertices v_{i+3k} and v_{i-3k} have the largest gap from the vertex u_l. Thus, the maximum value of n for which the vertex v_{l+3k} and v_{l-3k} are equal is $n = 6k$ but $n \geq 6k+1$. Therefore, v_{l+3k} and v_{l-3k} are distinct vertices and hence all the vertices mentioned above are distinct. Since 4 of these vertices belong to a set X and are already counted, therefore there are total $7n$ vertices that are yet to be counted and have a distance of three from the vertices of the set U. Similarly, for each $a \in V$, there are 11 vertices with distance 3 and out of them 4 belong to set U and 2 belong to a set X. Therefore there are $5n$ vertices that are yet to be counted and have a distance of three from V.

Finally, if $a \in Y$ then for any arbitrary vertex y_l, the vertex y_{l+3} is the vertex with distance three and is not counted before. Thus, there are n new vertices having a distance of three from a set Y. Therefore,

$W_p(n,k) = 11n + 7n + 5n + n = 24n.$

2.1. The Wiener index of Double Generalized Petersen Graphs $\boldsymbol{DP(n,k)}$

In this section, the $W(q,k)$ for $k = 1,2$ is computed, where q and k are positive integers. Let $d^*(x, y)$ and $d^{**}(x,y)$ be the minimum and maximum distances between any pair of vertices x and y.

Theorem 2.3. *For* $q \geq 3$,

$$W(q,1) = \begin{cases} 2q^3 + 8q^2 + 8q & ifqiseven, \\ 2q^3 + 8q^2 + 4q & ifqisodd, \end{cases}$$

Proof. The equation for calculating the Wiener index is

$$W(G) = \sum_{p_1,p_2 \in V(G)} d(p_1,p_2) \quad (1)$$

The proof is divided into two cases.

Case 1. *q is even.*

- If $p_1, p_2 \in XorY$, then $d^{**}(p_1,p_2) = \frac{q}{2}$. There are q and $\frac{q}{2}$ pairs of vertices when $1 \leq d(x,y) \leq \frac{q-2}{2}$ and $d(p_1,p_2) = \frac{q}{2}$ respectively. Therefore,

$$\sum_{p_1,p_2 \in X} d(p_1,p_2) = q(1+2+3+\cdots+\frac{q-2}{2}) + \frac{q}{2}(\frac{q}{2}) = \frac{q^3}{8} \quad (2)$$

- If $p_1 \in X$ and $p_2 \in U$ or $p \in Y$ and $y \in V$, then $d^{**}(p_1,p_2) = \frac{q+2}{2}$. There are $2q$ and q pairs of vertices when $2 \leq d(p_1,p_2) \leq \frac{q}{2}$ and $d(p_1,p_2) = 1 or \frac{q+2}{2}$ respectively. Thus,

$$\sum_{p_1\in X,p_2\in U} d(p_1,p_2) = 2q(2+\cdots+\frac{q}{2}) + q(1+\frac{q+2}{2}) = \frac{q^3+4q^2}{4} \quad (3)$$

- If $p_1 \in X$ and $p_2 \in V$ or $p_1 \in Y$ and $p_2 \in U$. In this case, $d^*(p_1, p_2) = 2$ and $d^{**}(p_1, p_2) = \frac{q+2}{2}$. There are $2q$ and q pairs of vertices when $2 \le d(p_1,p_2) \le \frac{q}{2}$ and $d(p_1,p_2) = 3 or \frac{q+2}{2}$ respectively. Thus,

$$\sum_{p_1\in X,p_2\in V} d(p_1,p_2) = 2q(2+\cdots+\frac{q}{2}) + q(3+\frac{q+2}{2}) = \frac{q^3+4q^2+8q}{4} \quad (4)$$

- If $p_1 \in X$ and $p_2 \in Y$, then $d^*(p_1, p_2) = 3$ and $d^{**}(p_1, p_2) = \frac{q+4}{2}$. There are $2q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+2}{2}$ and q if $d(p_1,p_2) = \frac{q}{2}$ or $\frac{q+4}{2}$. Thus,

$$\sum_{p_1\in Y,p_2\in X} d(p_1,p_2) = 2q(3+4+\cdots+\frac{q+2}{2}) + q((\frac{q}{2}) + (\frac{q+4}{2})) = \frac{q^3+8q^2+8q}{4} \quad (5)$$

- If $p_1,p_2 \in U$ or $p_1,p_2 \in V$. In both these cases, $d^*(p_1, p_2) = 2$ and $d^{**}(p_1, p_2) = \frac{q+2}{2}$. There are q pairs of vertices when $2 \le d(p_1,p_2) \le \frac{q+2}{2}$ and $\frac{q}{2}$ if $d(p_1,p_2) = \frac{q}{2}$. Therefore,

$$\sum_{p_1,p_2\in U} d(p_1,p_2) = q(2+3+\cdots+\frac{q-2}{2}+\frac{q+2}{2}) + \frac{q}{2}(\frac{q}{2}) = \frac{q^3+4q^2}{8} \quad (6)$$

- If $p_1 \in U$ and $p_2 \in V$, then $d^*(p_1, p_2) = 1$ and $d^{**}(p_1, p_2) = \frac{q+4}{2}$. It is important to note that there is no pair which have a distance of three. There are $2q$ pairs of vertices when $d(p_1,p_2) = 1$ or $3 \le d(p_1,p_2) \le \frac{q}{2}$ and q if the $d(p_1,p_2) = \frac{q}{2} or \frac{q+4}{2}$. So,

$$\sum_{p_1\in V,p_2\in U} d(p_1,p_2) = 2q(1+3+\cdots+\frac{q}{2}) + q(\frac{q}{2}+\frac{q+4}{2}) = \frac{q^3+4q^2+8q}{4} \quad (7)$$

By adding Eqs. (2), (3), (4), (5), (6) and (7), we get
$W(q,1) = 2q^3 + 8q^2 + 8q$.

Case 2. *q is odd.*

- If $p_1, p_2 \in X or Y$. In both these cases, $d^*(p_1, p_2) = 1$ and $d^{**}(p_1, p_2) = \frac{q-1}{2}$. There are q pairs of vertices when $1 \le d(p_1,p_2) \le \frac{q-1}{2}$. Thus,

$$\sum_{p_1,p_2\in X} d(p_1,p_2) = q(1+2+3+\cdots+\frac{q-1}{2}) = \frac{q^3-q}{8} \quad (8)$$

- If $p_1 \in X$ and $p_2 \in Y$ or $p_1 \in Y$ and $p_2 \in V$. In both these cases, $d^*(p_1, p_2) = 1$ and $d^{**}(p_1, p_2) = \frac{q+1}{2}$. There are q pairs of vertices when $d(p_1,p_2) = 1$ and $2q$ if the $2 \le d(p_1,p_2) \le \frac{q+1}{2}$. Therefore,

$$\sum_{p_1\in X,p_2\in U} d(p_1,p_2) = 2q(2+3+\cdots+\frac{q+1}{2}) + 1(q) = \frac{q^3+4q^2-q}{4} \quad (9)$$

- If $p_1 \in X$ and $p_2 \in V$ or $p_1 \in Y$ and $p_2 \in U$. Then $d^*(p_1, p_2) = 2$ and $d^{**}(p_1, p_2) = \frac{q+1}{2}$. There are $2q$ pairs of vertices when $2 \le d(p_1,p_2) \le \frac{q+1}{2}$ and q if the $d(p_1,p_2) = 3$. Thus,

$$\sum_{p_1\in X,p_2\in V} d(p_1,p_2) == 2q(2+3+\cdots+\frac{q+1}{2}) + 3q = \frac{q^3+4q^2+7q}{4} \quad (10)$$

- If $p_1 \in X$ and $p_2 \in Y$, then $d^*(p_1, p_2) = 3$ and $d^{**}(p_1, p_2) = \frac{q+3}{2}$. There are $2q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+3}{2}$ and q if the $d(p_1,p_2) = 4$. Therefore,

$$\sum_{p_1\in X,p_2\in Y} d(p_1,p_2) = 2q(3+4+\cdots+\frac{q+3}{2}) + 4q = \frac{q^3+8q^2+7q}{4} \quad (11)$$

- If $p_1,p_2 \in U or V$, then $d^*(p_1, p_2) = 2$ and $d^{**}(p_1, p_2) = \frac{q+1}{2}$. There are q pairs of vertices when $2 \le d(p_1,p_2) \le \frac{q+1}{2}$. Thus,

$$\sum_{p_1,p_2\in U} d(p_1,p_2) = q(2+3+\ldots+\frac{q+1}{2}) = \frac{q^3+4q^2-5q}{8} \quad (12)$$

- If $p_1 \in U$ and $p_2 \in V$, then $d^*(p_1, p_2) = 1$ and $d^{**}(p_1, p_2) = \frac{q+1}{2}$. There are $2q$ pairs of vertices when $1 or 3 \le d(p_1,p_2) \le \frac{q+1}{2}$ and q if the $d(p_1,p_2) = \frac{q-1}{2}$. So,

$$\sum_{p_1\in U,p_2\in V} d(p_1,p_2) = 2q(1+3+4\ldots+\frac{q+1}{2}) + q(\frac{q-1}{2}) = \frac{q^3+4q^2+3q}{4} \quad (13)$$

By adding Eqs. (8), (9), (10), (11), (12) and (13), we get
$W(q,1) = 2q^3 + 8q^2 + 4q$.

Theorem 2.4. *For* $7 \le q \le 11$,

$$W((q,2)) = \begin{cases} 2q^3+12q^2-30q & if q is even,\\ 2q^3+11q^2-37q & if q is odd,\end{cases}$$

Proof. The proof is divided into two cases.

Case 1. *q is even.*

- If $p_1,p_2 \in Y$, then $d^*(p_1, p_2) = 1$ and $d^{**}(p_1, p_2) = \frac{q}{2}$. There are q pairs of vertices when $1 \le d(p_1,p_2) \le \frac{q-2}{2}$ and $\frac{q}{2}$ if the $d(p_1,p_2) = \frac{q}{2}$. Therefore,

$$\sum_{p_1,p_2\in Y} d(p_1,p_2) = q(1+2+3+\cdots+\frac{q-2}{2}) + \frac{q}{2}(\frac{q}{2}) = \frac{q^3}{8} \quad (14)$$

- If $p_1 \in Y$ and $p_2 \in V$, then $d^*(p_1, p_2) = 1$ and $d^{**}(p_1, p_2) = \frac{q}{2}$. There are $2q$ pairs of vertices when $2 \le d(p_1,p_2) \le \frac{q}{2}$ and q if the $d(p_1,p_2) = 1\ 0r \frac{q-2}{2}$. Thus,

$$\sum_{p_1\in Y,p_2\in V} d(p_1,p_2) = 2q(2+\ldots+\frac{q}{2}) + q(1+\frac{q-2}{2}) = \frac{q^3+4q^2-8q}{4} \quad (15)$$

- If $p_1 \in Y$ and $p_2 \in U$, then $d^*(p_1, p_2) = 2$ and $d^{**}(p_1, p_2) = \frac{q}{2}$. There are $2q$ pairs of vertices when $2 \le d(p_1,p_2) \le \frac{q}{2}$ and $2q$ if the $d(p_1,p_2) = \frac{q-2}{2}$. Therefore,

$$\sum_{p_1\in y,p_2\in U} d(p_1,p_2) = 2q(2+3+\ldots+\frac{q}{2}) + 2q(\frac{q-2}{2}) = \frac{q^3+6q^2-16q}{4} \quad (16)$$

- If $p_1 \in Y$ and $p_2 \in X$, then $d^*(p_1, p_2) = 3$ and $d^{**}(p_1, p_2) = \frac{q+2}{2}$. There are $2q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+2}{2}$ and $2q$ if the $d(p_1,p_2) = \frac{q}{2}$. Therefore,

$$\sum_{p_1\in Y,p_2\in X} d(p_1,p_2) = 2q(3+4+\ldots+\frac{q+2}{2}) + 2q(\frac{q}{2}) = \frac{q^3+10q^2-16q}{4} \quad (17)$$

- If $p_1, p_2 \in V$, then $d^*(p_1, p_2) = 3$ and $d^{**}(p_1, p_2) = \frac{q+2}{2}$. There are q pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+2}{2}$ and $\frac{q}{2}$ if the $d(p_1,p_2) = \frac{q-4}{2}$. Thus,

$$\sum_{p_1,p_2\in V} d(p_1,p_2) = q(3+4+\ldots+\frac{q+2}{2}) + \frac{q}{2}(\frac{q-4}{2}) = \frac{q^3+8q^2-24q}{8} \quad (18)$$

- If $p_1 \in V$ and $p_2 \in U$, then then $d^*(p_1, p_2) = 1$ and $d^{**}(p_1, p_2) = \frac{q+2}{2}$. There are $2q$ pairs of vertices when $1 \le d(p_1,p_2) \le \frac{q+2}{2}$ and $2q$ if the $d(p_1,p_2) = \frac{q}{2}$. Thus,

$$\sum_{p_1\in V,p_2\in U} d(p_1,p_2) = 2q(1+\ldots+\frac{q+2}{2}) + 2q(\frac{q}{2}) = \frac{q^3+10q^2-32q}{4} \quad (19)$$

By adding Eqs. (14), (15), (16), (17), (18) and (19), we get

$W(DP(q,2)) = 2q^3 + 12q^2 - 30q$.

Case 2. *q is odd.*

- If $p_1, p_2 \in Y$, then $d^*(p_1, p_2) = 1$ and $d^{**}(p_1, p_2) = \frac{q-1}{2}$. There are q pairs of vertices when $1 \le d(p_1,p_2) \le \frac{q-1}{2}$. Therefore,

$$\sum_{p_1,p_2\in Y} d(p_1,p_2) = q(1+2+3+\cdots+\frac{q-1}{2}) = \frac{q^3-q}{8} \quad (20)$$

- If $p_1 \in Y$ and $p_2 \in V$, then $d^*(p_1, p_2) = 1$ and $d^{**}(p_1, p_2) = \frac{q-1}{2}$. There are $2q$ pairs of vertices when $2 \le d(p_1,p_2) \le \frac{q-1}{2}$ and q if the $d(p_1,p_2) = 1$ and $2q$ if the $\frac{q-3}{2}$. Therefore,

$$\sum_{p_1\in Y,p_2\in V} d(p_1,p_2) = q(1) + 2q(2+\ldots+\frac{q-1}{2}) + 2q(\frac{q-3}{2}) = \frac{q^3+4q^2-17q}{4} \quad (21)$$

- If $p_1 \in Y$ and $p_2 \in U$, then $d^*(p_1, p_2) = 2$ and $d^{**}(p_1, p_2) = \frac{q-1}{2}$. There are $2q$ pairs of vertices when $2 \le d(p_1,p_2) \le \frac{p_1-1}{2}$ and q if the $d(p_1,p_2) = \frac{q-1}{2}$ and $2q$ if the $\frac{q-3}{2}$. Thus,

$$\sum_{p_1\in Y,p_2\in U} d(p_1,p_2) = 2q(2+3+\ldots+\frac{q-1}{2}) + q(\frac{q-1}{2}) + 2q(\frac{q-3}{2}) = \frac{q^3+6q^2-23q}{4} \quad (22)$$

- If $p_1 \in Y$ and $p_2 \in X$, then $d^*(p_1, p_2) = 3$ and $d^{**}(p_1, p_2) = \frac{q+1}{2}$. There are $2q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+1}{2}$ and q if the $d(p_1,p_2) = \frac{q+1}{2}$ and $2q$ if the $\frac{q-1}{2}$. Therefore,

$$\sum_{p_1\in Y,p_2\in X} d(p_1,p_2) = 2q(3+4+\ldots+\frac{q+1}{2}) + 2q(\frac{q-1}{2}) + q(\frac{q+1}{2}) = \frac{q^3+10q^2-23q}{4} \quad (23)$$

- If $p_1, p_2 \in V$, then then $d^*(p_1, p_2) = 2$ and $d^{**}(p_1, p_2) = \frac{q+1}{2}$. There are q pairs of vertices when $2 \le d(p_1,p_2) \le \frac{q+1}{2}$. Therefore,

$$\sum_{p_1,p_2\in V} d(p_1,p_2) = q(2+3+\ldots+\frac{q+1}{2}) = \frac{q^3+4q^2-5q}{8} \quad (24)$$

- If $p_1 \in V$ and $p_2 \in U$, then $d^*(p_1, p_2) = 1$ and $d^{**}(p_1, p_2) = \frac{q+1}{2}$. There are $2q$ pairs of vertices when $1 or 4 \le d(p_1,p_2) \le \frac{q+1}{2}$ and q if the $d(p_1,p_2) = \frac{q+1}{2}$ and $2q$ if the $d(p_1,p_2) = \frac{q-1}{2}$. Thus,

$$\sum_{p_1\in V,p_2\in U} d(p_1,p_2) = q(1+4\ldots+\frac{q+1}{2}) + 2q(\frac{q-1}{2}) + q(\frac{q+1}{2}) = \frac{q^3+10q^2-39q}{4} \quad (25)$$

By adding Eqs. (20), (21), (22), (23), (24) and (25), we get q $W(DP(q,2)) = 2q^3 + 11q^2 - 37q$.

Theorem 2.5. *For $q \ge 12$,*

$$W((q,2)) = \begin{cases} q^3+17q^2-6q & if q \equiv 0(mod 8), \\ q^3+17q^2-12q & if q \equiv 1(mod 8), \\ q^3+17q^2-10q & if q \equiv 2,5,6(mod 8), \\ q^3+17q^2-18q & if q \equiv 3(mod 8), \\ \frac{4q^3+69q^2-44q}{4} & if q \equiv 4(mod 8), \\ q^3+17q^2-17q & if q \equiv 7(mod 8), \end{cases}$$

Proof. The proof is divided into two cases.

Case 1.

$q \equiv 0(mod 8)$

- If $p_1, p_2 \in X$, then $d^*(p_1, p_2) = 1$ and $d^{**}(p_1, p_2) = \lfloor\frac{q}{4}\rfloor + 3$. There are q pairs of vertices when $1 \le d(p_1,p_2) \le 5$ and $\frac{q+12}{4}$ and $2q$ pairs of vertices if the $\frac{q}{4} \le d(p_1,p_2) \le \frac{q+4}{4}$ and also $\frac{3q}{2}$ pairs of vertices when $d(p_1,p_2) = \frac{q+8}{4}$. Thus,

$\sum_{p_1,p_2\in X} d(p_1,p_2) = q(1+2+3+\cdots+5) + q(\frac{q+8}{4}+\frac{q+12}{4}) + \frac{q}{2}(\frac{q+8}{4})$
$+2q(6+7+\cdots+\frac{q+4}{4})$

$$= \frac{q^3+22q^2-112q}{16} \quad (26)$$

- If $p_1 \in X$ and $p_2 \in U$, then $d^*(p_1,\ p_2)=1$ and $d^{**}(p_1,\ p_2)=\frac{q+8}{4}$. There are $4q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q}{4}$ and $3q$ if the $d(p_1,p_2)=\frac{q+4}{4}$ and $2q$ if the $d(p_1,p_2)=2\,0r\,\frac{q+8}{4}$ and also q if the $d(p_1,p_2)=1$. Therefore,

$$\sum_{p_1\in X,p_2\in U} d(p_1,p_2) = q\left(1+\frac{q+4}{4}\right) + 2q\left(2+\frac{q+4}{4}+\frac{q+8}{4}\right) + 4q\left(3+4+\cdots+\frac{q}{4}\right) = \frac{q^3+14q^2}{8} \quad (27)$$

- If $p_1 \in X$ and $p_2 \in V$, then $d^*(p_1,\ p_2)=2$ and $d^{**}(p_1,\ p_2)=\frac{q+8}{4}$. There are $4q$ pairs of vertices when $3or5 \le d(p_1,p_2) \le \frac{q+4}{4}$ and $5q$ if the $d(p_1,p_2)=4$ and $2q$ if the $d(p_1,p_2)=2$ and q if the $d(p_1,p_2)=\frac{q+8}{4}$. Thus,

$$\sum_{p_1\in X,p_2\in V} d(p_1,p_2) = q(4+\frac{q+8}{4}) + 2q(2) + 4q(3+4+\cdots+\frac{q+4}{4}) = \frac{q^3+14q^2+16q}{8} \quad (28)$$

- If $p_1 \in X$ and $p_2 \in Y$, then $d^*(p_1,\ p_2)=3$ and $d^{**}(p_1,\ p_2)=\frac{q+12}{4}$. There are $4q$ pairs of vertices when $4or6 \le d(p_1,p_2) \le \frac{q+8}{4}$ and $5q$ if the $d(p_1,p_2)=5$ and $2q$ if the $d(p_1,p_2)=2$ and q if the $d(p_1,p_2)=\frac{q+12}{4}$. Therefore,

$$\sum_{p_1\in X,p_2\in Y} d(p_1,p_2) = q\left(5+\frac{q+12}{4}\right) + 2q(3) + 4q\left(4+5+\cdots+\frac{q+8}{4}\right) = \frac{q^3+22q^2+16q}{8} \quad (29)$$

- If $p_1,p_2 \in U$, then $d^*(p_1,\ p_2)=2$ and $d^{**}(p_1,\ p_2)=\frac{q+12}{4}$. There are $2q$ pairs of vertices when $4 \le d(p_1,p_2) \le 5or\frac{q+4}{4}$ and q if the $d(p_1,p_2)=2,\ 3,\ \frac{q+8}{4},\ \frac{q+12}{4}$ and $\frac{3q}{2}$ if the $d(p_1,p_2)=\frac{q}{4}$. So,

$$\sum_{p_1,\ p_2\in U} d(p_1,p_2) = q\left(2+3+\frac{q}{4}+\frac{q+8}{4}+\frac{q+12}{4}\right) + \frac{q}{2}\left(\frac{q}{4}\right) + 2q\left(\frac{q+4}{4}\right) + 2q\left(4+5+\cdots+\frac{q-4}{4}\right) = \frac{q^3+18q^2}{16} \quad (30)$$

- If $p_1 \in U$ and $p_2 \in V$, then $d^*(p_1,\ p_2)=1$ and $d^{**}(p_1,\ p_2)=\frac{q+12}{4}$. There are $5q$ if the $d(p_1,p_2)=5$ and $4q$ if the $d(p_1,p_2)=4or\frac{q}{4}or\frac{q+8}{4}$ and $2q$ if the $d(p_1,p_2)=1or3\,or\frac{q+4}{4}$ and q if the $d(p_1,p_2)=\frac{q+12}{4}$. Thus,

$$\sum_{p_1\in U,p_2\in V} d(p_1,p_2) = q(5+\frac{q+12}{4}) + 2q(1+3+\frac{q+4}{4}) + 4q(\frac{q+8}{4}) + 4q(4+5+\cdots+\frac{q}{4}) \times \frac{q^3+18q^2+16q}{8} \quad (31)$$

By adding Eqs. (26), (27), (28), (29), (30) and (31), we get $W(DP(q,2)) = q^3+17q^2-6q$.

Case 2.

$q \equiv 1(mod8)$

- If $p_1,p_2 \in X$, then $d^*(p_1,\ p_2)=1$ and $d^{**}(p_1,\ p_2)=\frac{q+11}{4}$. There are q pairs of vertices when $1 \le d(p_1,p_2) \le 5or\frac{q+11}{4}$ and $2q$ if the $6 \le d(p_1,p_2) \le \frac{q+7}{4}$. Thus,

$$\sum_{p_1,p_2\in X} d(p_1,p_2) = q(1+2+3+\cdots+5) + q\left(\frac{q+11}{4}\right) + 2q\left(6+\cdots+\frac{q+7}{4}\right)$$

$$= \frac{q^3+22q^2-119q}{16} \quad (32)$$

- If $p_1 \in X$ and $p_2 \in U$, then $d^*(p_1,\ p_2)=1$ and $d^{**}(p_1,\ p_2)=\frac{q+7}{4}$. There are $4q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+3}{4}$ and $2q$ if the $d(p_1,p_2)=2or\frac{q+7}{4}$ and also q if the $d(p_1,p_2)=1$. Therefore,

$$\sum_{p_1\in X,p_2\in U} d(p_1,p_2) = q(1) + 2q(2+\frac{q+7}{4}) + 4q(3+4+\cdots+\frac{q+3}{4}) = \frac{q^3+14q^2-7q}{8} \quad (33)$$

- If $p_1 \in X$ and $p_2 \in V$, then $d^*(p_1,\ p_2)=2$ and $d^{**}(p_1,\ p_2)=\frac{q+7}{4}$. There are $4q$ pairs of vertices when $3or5 \le d(p_1,p_2) \le \frac{q+3}{4}$ and $5q$ if the $d(p_1,p_2)=4$ and $2q$ if the $d(p_1,p_2)=2or\frac{q+7}{4}$. Thus,

$$\sum_{p_1\in X,p_2\in V} d(p_1,p_2) = 2q(2+\frac{q+7}{4}) + q(4) + 4q(3+4+\cdots+\frac{q+3}{4}) = \frac{q^3+14q^2+17q}{8} \quad (34)$$

- If $p_1 \in X$ and $p_2 \in Y$, then $d^*(p_1,\ p_2)=3$ and $d^{**}(p_1,\ p_2)=\frac{q+11}{4}$. There are $4q$ pairs of vertices when $4or6 \le d(p_1,p_2) \le \frac{q+7}{4}$ and $5q$ if the $d(p_1,p_2)=5$ and $2q$ if the $d(p_1,p_2)=3or\frac{q+11}{4}$. Thus,

$$\sum_{p_1\in X,p_2\in Y} d(p_1,p_2) = q(5) + 2q(3+\frac{q+11}{4}) + 4q(4+5+\cdots+\frac{q+7}{4}) = \frac{q^3+22q^2+17q}{8} \quad (35)$$

- If $p_1,p_2 \in U$, then $d^*(p_1,\ p_2)=2$ and $d^{**}(p_1,\ p_2)=\frac{q+11}{4}$. There are $2q$ pairs of vertices when $4 \le d(p_1,p_2)\frac{q+3}{4}$ and q if the $d(p_1,p_2)=2,\ 3,\ \frac{q+7}{4},\ \frac{q+11}{4}$. Therefore,

$$\sum_{p_1,p_2\in U} d(p_1,p_2) = q(2+3+\frac{q+7}{4}+\frac{q+11}{4}) + 2q(4+5+\cdots+\frac{q+3}{4}) = \frac{q^3+18q^2-19q}{16} \quad (36)$$

- If $p_1 \in U$ and $p_2 \in V$, then $d^*(p_1,\ p_2)=1$ and $d^{**}(p_1,\ p_2)=\frac{q+11}{4}$. There are $5q$ if the $d(p_1,p_2)=5$ and $4q$ if the

$4 \le d(p_1,p_2) \le \frac{q+3}{4}$ and $2q$ if the $d(p_1,p_2) = 1 or 3 or \frac{q+7}{4} or \frac{q+11}{4}$. Thus,

$$\sum_{p_1 \in U, p_2 \in V} d(p_1,p_2) = 2q(1+3+\frac{q+7}{4}+\frac{q+11}{4}) + q(5) + 4q(4 + \cdots + \frac{q+3}{4}) = \frac{q^3+18q^2+5q}{8} \quad (37)$$

By adding Eqs. (32), (33), (34), (35), (36) and (37), we get

$W(DP(q,2)) = q^3 + 17q^2 - 12q.$

Case 3.

$q \equiv 2(mod 8)$

- If $p_1, p_2 \in X$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+10}{4}$. There are q pairs of vertices when $1 \le d(p_1,p_2) \le 5$ and $2q$ if the $6 \le d(p_1,p_2) \le \frac{q+6}{4}$ and $\frac{3q}{2}$ if the $d(p_1,p_2) = \frac{q+10}{4}$. Thus,

$$\sum_{p_1, p_2 \in X} d(p_1,p_2) = q(1+2+3+\cdots+5) + 2q\left(6+\cdots+\frac{q+6}{4}\right) + \frac{3q}{2}\left(\frac{q+10}{4}\right) = \frac{q^3+22q^2-120q}{16} \quad (38)$$

- If $p_1 \in X$ and $p_2 \in U$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+6}{4}$. There are $4q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+2}{4}$ and $3q$ if the $d(p_1,p_2) = \frac{q+6}{4}$ and $2q$ if the $d(p_1,p_2) = 2$ and also q if the $d(p_1,p_2) = 1$. Thus,

$$\sum_{p_1 \in X, p_2 \in U} d(p_1,p_2) = q(1+\frac{q+6}{4}) + 2q(2+\frac{q+6}{4}) + 4q(3+4+\cdots+\frac{q+2}{4}) = \frac{q^3+14q^2-8q}{8} \quad (39)$$

- If $p_1 \in X$ and $p_2 \in V$, then $d^*(p_1,\ p_2) = 2$ and $d^{**}(p_1,\ p_2) = \frac{q+10}{4}$. There are $4q$ pairs of vertices when $3 or 5 \le d(p_1,p_2) \le \frac{q+2}{4}$ and $5q$ if the $d(p_1,p_2) = 4$ and $2q$ if the $d(p_1,p_2) = 2 or \frac{q+6}{4}$ and n if the $d(p_1,p_2) = \frac{q+10}{4}$. Therefore,

$$\sum_{p_1 \in X, p_2 \in V} d(p_1,p_2) = q(4+\frac{q+10}{4}) + 2q(2+\frac{q+6}{4}) + 4q(3+4+\cdots+\frac{q+2}{4}) = \frac{q^3+14q^2+24q}{8} \quad (40)$$

- If $p_1 \in X$ and $p_2 \in Y$, then $d^*(p_1,\ p_2) = 3$ and $d^{**}(p_1,\ p_2) = \frac{q+14}{4}$. There are $4q$ pairs of vertices when $4 or 6 \le d(p_1,p_2) \le \frac{q+6}{4}$ and $5q$ if the $d(p_1,p_2) = 5$ and $2q$ if the $d(p_1,p_2) = 3 or \frac{q+10}{4}$ and also q if the $d(p_1,p_2) = \frac{q+14}{4}$. So,

$$\sum_{p_1 \in X, p_2 \in Y} d(p_1,p_2) = q(5+\frac{q+14}{4}) + 2q(3+\frac{q+10}{4}) + 4q(4+5+\cdots+\frac{q+6}{4}) = \frac{q^3+22q^2+24q}{8} \quad (41)$$

- If $p_1, p_2 \in U$, then $d^*(p_1,\ p_2) = 2$ and $d^{**}(p_1,\ p_2) = \frac{q+10}{4}$. There are $2q$ pairs of vertices when $4 \le d(p_1,p_2) \le \frac{q+2}{4}$ and q if the $d(p_1,p_2) = 2,\ 3,\ \frac{q+6}{4}$ and also $\frac{3q}{2}$ if the $d(p_1,p_2) = \frac{q+10}{4}$. Thus,

$$\sum_{p_1, p_2 \in U} d(p_1,p_2) = q(2+3+\frac{q+6}{4}+\frac{q+10}{4}) + 2q(4+5+\cdots+\frac{q+2}{4}) + \frac{q}{2}(\frac{q+10}{4}) = \frac{q^3+18q^2-16q}{16} \quad (42)$$

- If $p_1 \in U$ and $p_2 \in V$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+14}{4}$. There are $5q$ if the $d(p_1,p_2) = 5$ and $4q$ if the $4 \le d(p_1,p_2) \le \frac{q+6}{4}$ and $2q$ if the $d(p_1,p_2) = 1 or 3$ and also q if the $d(p_1,p_2) = \frac{q+14}{4}$. Therefore,

$$\sum_{p_1 \in U, p_2 \in V} d(p_1,p_2) = 2q(1+3) + q(5+\frac{q+14}{4}) + 4q(4+\cdots+\frac{q+6}{4}) = \frac{q^3+18q^2}{8} \quad (43)$$

By adding Eqs. (38), (39), (40), (41), (42) and (43), we get

$W(DP(q,2)) = q^3 + 17q^2 - 10q.$

Case 4.

$q \equiv 3(mod 8)$

- If $p_1, p_2 \in X$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+9}{4}$. There are q pairs of vertices when $1 \le d(p_1,p_2) \le 5$ and $2q$ if the $6 \le d(p_1,p_2) \le \frac{q+9}{4}$. So,

$$\sum_{p_1, p_2 \in X} d(p_1,p_2) = q(1+2+3+\cdots+5) + 2q(6+\cdots+\frac{q+9}{4})) = \frac{q^3+22q^2-123q}{16} \quad (44)$$

- If $p_1 \in X$ and $p_2 \in U$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+5}{4}$. There are $4q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+5}{4}$ and $2q$ if the $d(p_1,p_2) = 2$ and also q if the $d(p_1,p_2) = 1$. Therefore,

$$\sum_{p_1 \in X, p_2 \in U} d(p_1,p_2) = q(1) + 2q(2) + 4q(3+4+\cdots+\frac{q+5}{4}) = \frac{q^3+14q^2-11q}{8} \quad (45)$$

- If $p_1 \in X$ and $p_2 \in V$, then $d^*(p_1,\ p_2) = 2$ and $d^{**}(p_1,\ p_2) = \frac{q+5}{4}$. There are $4q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+5}{4}$ and $5q$ if the $d(p_1,p_2) = 4$ and $2q$ if the $d(p_1,p_2) = 2$. Therefore,

$$\sum_{p_1 \in X, p_2 \in V} d(p_1,p_2) = 2q(2) + 4(q) + 4q(3+4+\cdots+\frac{q+5}{4}) = \frac{q^3+14q^2+13q}{8} \quad (46)$$

- If $p_1 \in X$ and $p_2 \in Y$, then $d^*(p_1,\ p_2) = 3$ and $d^{**}(p_1,\ p_2) = \frac{q+9}{4}$. There are $4q$ pairs of vertices when $4 or 6 \le d(p_1,p_2) \le \frac{q+9}{4}$ and $5q$ if the $d(p_1,p_2) = 5$ and $2q$ if the $d(p_1,p_2) = 3$. Therefore,

$$\sum_{p_1\in X,p_2\in Y} d(p_1,p_2) = q(5) + 2q(3) + 4q(4+5+\cdots+\frac{q+9}{4}) = \frac{q^3+22q^2+13q}{8} \quad (47)$$

- If $p_1,p_2 \in U$, then $d^*(p_1,\ p_2) = 2$ and $d^{**}(p_1,\ p_2) = \frac{q+9}{4}$. There are $2q$ pairs of vertices when $4 \le d(p_1,p_2) \le \frac{q+5}{4}$ and q if the $d(p_1,p_2) = 2,\ 3,\ \frac{q+9}{4}$. Therefore,

$$\sum_{p_1,p_2\in U} d(p_1,p_2) = q(2+3+\frac{q+9}{4}) + 2q(4+5+\cdots+\frac{q+5}{4}) = \frac{q^3+18q^2-31q}{16} \quad (48)$$

- If $p_1 \in U$ and $p_2 \in V$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+9}{4}$. There are $5q$ if the $d(p_1,p_2) = 5$ and $4q$ if the $4 \le d(p_1,p_2) \le \frac{q+5}{4}$ and $2q$ if the $d(p_1,p_2) = 1 or 3 or \frac{q+9}{4}$. So, we have

$$\sum_{p_1\in U,p_2\in V} d(p_1,p_2) = 2q(1+3+\frac{q+9}{4}) + q(5) + 4q(4+\cdots+\frac{q+5}{4}) = \frac{q^3+18q^2-7q}{8} \quad (49)$$

By adding Eqs. (44), (45), (46), (47), (48) and (49) we get

$$W(DP(q,2)) = q^3 + 17q^2 - 18q.$$

Case 5.

$q \equiv 4(mod 8)$

- If $p_1,\ p_2 \in X$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+12}{4}$. There are q pairs of vertices when $1 \le d(p_1,p_2) \le 5$ and $2q$ if the $6 \le d(p_1,p_2) \le \frac{q+8}{4}$ and $\frac{q}{2}$ if the $d(p_1,p_2) = \frac{q+12}{8}$. Thus,

$$\sum_{p_1,p_2\in X} d(p_1,p_2) = q(1+2+3+\cdots+5) + 2q\left(6+\cdots+\frac{q+8}{4}\right) + \frac{q}{2}\left(\frac{q+12}{4}\right)$$

$$= \frac{q^3+22q^2-120q}{16} \quad (50)$$

- If $p_1 \in X$ and $p_2 \in U$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+8}{4}$. There are $4q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+4}{4}$ and $2q$ if the $d(p_1,p_2) = 2$ and also q if the $d(p_1,p_2) = 1 or \frac{q+8}{4}$. Therefore,

$$\sum_{p_1\in X,p_2\in U} d(p_1,p_2) = q(1+\frac{q+8}{4}) + 2q(2) + 4q(3+4+\cdots+\frac{q+4}{4}) = \frac{q^3+14q^2-8q}{8} \quad (51)$$

- If $p_1 \in X$ and $p_2 \in V$, then $d^*(p_1,\ p_2) = 2$ and $d^{**}(p_1,\ p_2) = \frac{q+8}{4}$. There are $4q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q}{4}$ and $5q$ if the $d(p_1,p_2) = 4$ and $2q$ if the $d(p_1,p_2) = 2 or \frac{q+8}{4}$ and $3q$ if the $d(p_1,p_2) = \frac{q+4}{4}$. Therefore,

$$\sum_{p_1\in X,p_2\in V} d(p_1,p_2) = 2q(2+\frac{q+4}{4}+\frac{q+8}{4}) + q(4+\frac{q+4}{4}) + 4q(3+4+\cdots+\frac{q}{4}) = \frac{q^3+14q^2+24q}{8} \quad (52)$$

- If $p_1 \in X$ and $p_2 \in Y$, then $d^*(p_1,\ p_2) = 3$ and $d^{**}(p_1,\ p_2) = \frac{q+12}{4}$. There are $4q$ pairs of vertices when $4 \le d(p_1,p_2) \le \frac{q+4}{4}$ and $5q$ if the $d(p_1,p_2) = 5$ and $2q$ if the $d(p_1,p_2) = 3 or \frac{q+12}{4}$ and also $3q$ if the $d(p_1,p_2) = \frac{q+8}{4}$. So,

$$\sum_{p_1\in X,p_2\in Y} d(p_1,p_2) = q\left(5+\frac{q+8}{4}\right) + 2q\left(3+\frac{q+8}{4}+\frac{q+12}{4}\right) + 4q\left(4+5+\cdots+\frac{q+4}{4}\right) = \frac{+22q^2+24q}{8} \quad (53)$$

- If $p_1,p_2 \in U$, then $d^*(p_1,\ p_2) = 2$ and $d^{**}(p_1,\ p_2) = \frac{q+12}{4}$. There are $2q$ pairs of vertices when $4 \le d(p_1,p_2) \frac{q}{4} or \frac{q+8}{4}$ and q if the $d(p_1,p_2) = 2, 3, \frac{q+4}{4}$ and $\frac{q}{2}$ if the $d(p_1,p_2) = \frac{q+12}{4}$. Therefore,

$$\sum_{p_1,p_2\in U} d(p_1,p_2) = q(2+3+\frac{q+4}{4}) + 2q(4+5+\cdots+\frac{q}{4}) + 2q(\frac{q+8}{4}) + \frac{q}{2}(\frac{q+12}{4}) = \frac{q^3+18q^2-8q}{16} \quad (54)$$

- If $p_1 \in U$ and $p_2 \in V$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+12}{4}$. There are $4q$ if the $4 \le d(p_1,p_2) \le \frac{q+4}{4}$ and $2q$ if the $d(p_1,p_2) = 1 or 3 or \frac{q+8}{4} or \frac{q+12}{4}$. So,

$$\sum_{p_1\in U,p_2\in V} d(p_1,p_2) = 2q(1+3+\frac{q+8}{4}+\frac{q+12}{4}) + 4q(4+\cdots+\frac{q+4}{4}) = \frac{q^3+20q^2-16q}{8} \quad (55)$$

By adding Eqs. (50), (51), (52), (53), (54) and (55), we get

$$W(DP(q,2)) = \frac{4q^3+69q^2-44q}{4}.$$

Case 6.

$q \equiv 5(mod 8)$

- If $p_1,p_2 \in X$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+11}{4}$. There are q pairs of vertices when $1 \le d(p_1,p_2) \le 5\, 0r \frac{q+11}{4}$ and $2q$ if the $6 \le d(p_1,p_2) \le \frac{q+7}{4}$. Therefore,

$$\sum_{p_1,p_2\in X} d(p_1,p_2) = q(1+2+3+\cdots+5) + q\left(\frac{q+11}{4}\right) + 2q\left(6+7+\cdots+\frac{q+7}{4}\right) = \frac{q^3+22q^2-119q}{16} \quad (56)$$

- If $p_1 \in X$ and $p_2 \in U$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+7}{4}$. There are $4q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+3}{4}$ and $2q$ if the $d(p_1,p_2) = 2\, 0r \frac{q+7}{2}$ and also q if the $d(p_1,p_2) = 1$. Therefore,

$$\sum_{p_1\in X,p_2\in U} d(p_1,p_2) = q(1) + 2q(2+\frac{q+7}{4}) + 4q(3+4+\cdots+\frac{q+3}{4}) = \frac{q^3+14q^2-7q}{8} \quad (57)$$

- If $p_1 \in X$ and $p_2 \in V$, then $d^*(p_1,\ p_2) = 2$ and $d^{**}(p_1,\ p_2) = \frac{q+7}{4}$. There are $4q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+3}{4}$ and $5q$ if the $d(p_1,p_2) = 4$ and $2q$ if the $d(p_1,p_2) = 2 or \frac{q+7}{4}$. Thus,

$$\sum_{p_1\in X,p_2\in V} d(p_1,p_2) = q(4) + 2q(2+\frac{q+7}{4}) + 4q(3+4+\cdots+\frac{q+3}{4}) = \frac{q^3+14q^2+17q}{8} \quad (58)$$

- If $p_1 \in X$ and $p_2 \in Y$, then $d^*(p_1,\ p_2) = 3$ and $d^{**}(p_1,\ p_2) = \frac{q+11}{4}$. There are $4q$ pairs of vertices when $4 \le d(p_1,p_2) \le \frac{q+7}{4}$ and $5q$ if the $d(p_1,p_2) = 5$ and $2q$ if the $d(p_1,p_2) = 3 or \frac{q+11}{4}$. Therefore,

$$\sum_{p_1\in X,p_2\in Y} d(p_1,p_2) = q(5) + 2q(3+\frac{q+11}{4}) + 4q(4+5+\cdots+\frac{q+7}{4}) = \frac{q^3+22q^2+17q}{8} \quad (59)$$

- If $p_1,p_2 \in U$, then $d^*(p_1,\ p_2) = 2$ and $d^{**}(p_1,\ p_2) = \frac{q+11}{4}$. There are $2q$ pairs of vertices when $4 \le d(p_1,p_2) \le \frac{q-1}{4} or \frac{q+7}{4}$ and q if the $d(p_1,p_2) = 2, 3, \frac{q+3}{4}, \frac{q+11}{4}$. Therefore,

$$\sum_{p_1,p_2\in U} d(p_1,p_2) = q(2+3+\frac{q+3}{4}+\frac{q+11}{4}) + 2q(4+5+\cdots+\frac{q-1}{4}) + 2q(\frac{q+7}{4}) = \frac{q^3+18q^2-3q}{16} \quad (60)$$

- If $p_1 \in U$ and $p_2 \in V$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+11}{4}$. There are $4q$ pairs of vertices if the $4 \le d(p_1,p_2) \le \frac{q+3}{4}$ and $5q$ if the $d(p_1,p_2) = 5$ and $2q$ if the $d(p_1,p_2) = 1 or 3 or \frac{q+7}{4} or \frac{q+11}{4}$. Therefore,

$$\sum_{p_1\in U,p_2\in V} d(p_1,p_2) = 2q(1+3+\frac{q+7}{4}+\frac{q+11}{4}) + q(5) + 4q(4+\cdots+\frac{q+3}{4}) = \frac{q^3+18q^2+5q}{8} \quad (61)$$

By adding Eqs. (56), (57), (58), (59), (60) and (61), we get

$W(DP(q,2)) = q^3 + 17q^2 - 10q.$

Case 7.

$q \equiv 6(mod 8)$

- If $p_1,p_2 \in X$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+11}{4}$. There are q pairs of vertices when $1 \le d(p_1,p_2) \le 5\, 0r \frac{q+10}{4}$ and $2q$ if the $6 \le d(p_1,p_2) \le \frac{q+6}{4}$ and $\frac{q}{2}$ if the $d(p_1,p_2) = \frac{q+14}{4}$. Thus,

$$\sum_{p_1,p_2\in X} d(p_1,p_2) = q\left(1+2+3+\cdots+5+\frac{q+10}{4}\right) + \frac{q}{2}\left(\frac{q+14}{4}\right) + 2q\left(6+7+\cdots+\frac{q+6}{4}\right) = \frac{q^3+22q^2-112q}{16} \quad (62)$$

- If $p_1 \in X$ and $p_2 \in U$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+10}{4}$. There are $4q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+2}{4}$ and $2q$ if the $d(p_1,p_2) = 2\, 0r \frac{q+6}{2}$ and also q if the $d(p_1,p_2) = 1 0r \frac{q+10}{4}$. Thus,

$$\sum_{p_1\in X,p_2\in U} d(p_1,p_2) = q\left(1+\frac{q+10}{4}\right) + 2q\left(2+\frac{q+6}{4}\right) + 4q\left(3+4+\cdots+\frac{q+2}{4}\right) = \frac{q^3+14q^2}{8} \quad (63)$$

- If $p_1 \in X$ and $p_2 \in V$, then $d^*(p_1,\ p_2) = 2$ and $d^{**}(p_1,\ p_2) = \frac{q+6}{4}$. There are $4q$ pairs of vertices when $3 \le d(p_1,p_2) \le \frac{q+2}{4}$ and $5q$ if the $d(p_1,p_2) = 4$ and $2q$ if the $d(p_1,p_2) = 2$ and $3q$ if the $d(p_1,p_2) = \frac{q+6}{4}$. Therefore,

$$\sum_{p_1\in X,p_2\in V} d(p_1,p_2) = q(4+\frac{q+6}{4}) + 2q(2+\frac{q+6}{4}) + 4q(3+4+\cdots+\frac{q+2}{4}) = \frac{q^3+14q^2+16q}{8} \quad (64)$$

- If $p_1 \in X$ and $p_2 \in Y$, then $d^*(p_1,\ p_2) = 3$ and $d^{**}(p_1,\ p_2) = \frac{q+10}{4}$. There are $4q$ pairs of vertices when $4 \le d(p_1,p_2) \le \frac{q+6}{4}$ and $5q$ if the $d(p_1,p_2) = 5$ and $2q$ if the $d(p_1,p_2) = 3$ and $3q$ if the $d(p_1,p_2) = \frac{q+10}{4}$. Thus,

$$\sum_{p_1\in X,p_2\in Y} d(p_1,p_2) = q(5+\frac{q+10}{4}) + 2q(3+\frac{q+10}{4}) + 4q(4+5+\cdots+\frac{q+6}{4}) = \frac{q^3+22q^2+16q}{8} \quad (65)$$

- If $p_1,p_2 \in U$, then $d^*(p_1,\ p_2) = 2$ and $d^{**}(p_1,\ p_2) = \frac{q+14}{4}$. There are $2q$ pairs of vertices when $4 \le d(p_1,p_2) \le \frac{q+6}{4}$ and q if the $d(p_1,p_2) = 2,\ 3$ and $\frac{q}{2}$ if the $d(p_1,p_2) = \frac{q+14}{2}$. So,

$$\sum_{p_1,p_2\in U} d(p_1,p_2) = q(2+3) + \frac{q}{2}(\frac{q+14}{4}) + 2q(4+5+\cdots+\frac{q+6}{4}) = \frac{q^3+18q^2-24q}{16} \quad (66)$$

- If $p_1 \in U$ and $p_2 \in V$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+10}{4}$. There are $4q$ pairs of vertices if the $4 \le d(p_1,p_2) \le \frac{q+2}{4}$ and $5q$ if the $d(p_1,p_2) = 5$ and $2q$ if the $d(p_1,p_2) = 1 or 3 or \frac{q+6}{4}$ and $3q$ if the $d(p_1,p_2) = \frac{q+10}{4}$. Thus,

$$\sum_{p_1\in U,p_2\in V} d(p_1,p_2) = 2q\left(1+3+\frac{q+6}{4}+\frac{q+10}{4}\right) + q\left(5+\frac{q+10}{4}\right) + 4q\left(4+\cdots+\frac{q+2}{4}\right) = \frac{q^3+18q^2+8q}{8} \quad (67)$$

By adding Eqs. (62), (63), (64), (65), (66) and (67), we get

$W(DP(q,2)) = q^3 + 17q^2 - 10q.$

Case 8.

$q \equiv 7(mod 8)$

- If $p_1, p_2 \in X$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+9}{4}$. There are q pairs of vertices when $1 \le d(p_1, p_2) \le 5$ and $2q$ if the $6 \le d(p_1, p_2) \le \frac{q+9}{4}$. Thus,

$$\sum_{p_1, p_2 \in X} d(p_1, p_2) = q(1 + 2 + 3 + \cdots + 5) + 2q(6 + 7 + \cdots + \frac{q+9}{4}) = \frac{q^3 + 22q^2 - 123q}{16} \quad (68)$$

- If $p_1 \in X$ and $p_2 \in U$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+5}{4}$. There are $4q$ pairs of vertices when $3 \le d(p_1, p_2) \le \frac{q+5}{4}$ and $2q$ if the $d(p_1, p_2) = 2$ and also q if the $d(p_1, p_2) = 1$. Therefore,

$$\sum_{p_1 \in X, p_2 \in U} d(p_1, p_2) = q(1) + 2q(2) + 4q(3 + 4 + \cdots + \frac{q+5}{4}) = \frac{q^3 + 14q^2 - 11q}{8} \quad (69)$$

- If $p_1 \in X$ and $p_2 \in V$, then $d^*(p_1,\ p_2) = 2$ and $d^{**}(p_1,\ p_2) = \frac{q+5}{4}$. There are $4q$ pairs of vertices when $3 \le d(p_1, p_2) \le \frac{q+5}{4}$ and $5q$ if the $d(p_1, p_2) = 4$ and $2q$ if the $d(p_1, p_2) = 2$. Thus,

$$\sum_{p_1 \in X, p_2 \in V} d(p_1, p_2) = q(4) + 2q(2) + 4q(3 + 4 + \cdots + \frac{q+5}{4}) = \frac{q^3 + 14q^2 + 13q}{8} \quad (70)$$

- If $p_1 \in X$ and $p_2 \in Y$, then $d^*(p_1,\ p_2) = 3$ and $d^{**}(p_1,\ p_2) = \frac{q+9}{4}$. There are $4q$ pairs of vertices when $4 \le d(p_1, p_2) \le \frac{q+9}{4}$ and $5q$ if the $d(p_1, p_2) = 5$ and $2q$ if the $d(p_1, p_2) = 3$. So,

$$\sum_{p_1 \in X, p_2 \in Y} d(p_1, p_2) = q(5) + 2q(3) + 4q(4 + 5 + \cdots + \frac{q+9}{4}) = \frac{q^3 + 22q^2 + 13q}{8} \quad (71)$$

- If $p_1, p_2 \in U$, then $d^*(p_1,\ p_2) = 2$ and $d^{**}(p_1,\ p_2) = \frac{q+9}{4}$. There are $2q$ pairs of vertices when $4 \le d(p_1, p_2) \frac{q+5}{4}$ and q if the $d(p_1, p_2) = 2, 3, \frac{q+9}{4}$. Thus,

$$\sum_{p_1, p_2 \in U} d(p_1, p_2) = q(2 + 3 + \frac{q+9}{4}) + 2q(4 + 5 + \cdots + \frac{q+5}{4}) = \frac{q^3 + 18q^2 - 31q}{16} \quad (72)$$

- If $p_1 \in U$ and $p_2 \in V$, then $d^*(p_1,\ p_2) = 1$ and $d^{**}(p_1,\ p_2) = \frac{q+9}{4}$.. There are $4q$ pairs of vertices if the $4 \le d(p_1, p_2) \le \frac{q+1}{4}$ and $5q$ if the $d(p_1, p_2) = 5$ and $2q$ if the $d(p_1, p_2) = 1 or 3$ and $3q$ if the $d(p_1, p_2) = \frac{q+5}{4} or \frac{q+9}{4}$. Thus,

$$\sum_{p_1 \in U, p_2 \in V} d(p_1, p_2) = 2q\left(1 + 3 + \frac{q+5}{4} + \frac{q+9}{4}\right) + q\left(5 + \frac{q+5}{4} + \frac{q+9}{4}\right) + 4q\left(4 + \cdots + \frac{q+1}{4}\right) = \frac{q^3 + 18q^2 + q}{8} \quad (73)$$

By adding Eqs. (68), (69), (70), (71), (72) and (73), we get $W(DP(q, 2)) = q^3 + 17q^2 - 17q$.

3. Graphical representation of $W(q, 1)$ and $W(q, 2)$

In this section, the graphical representation of $W(q, 1)$ and $W(q, 2)$ of $DP(q, 1)$ and $DP(q, 2)$ for different valuers of q are determined. These compact formulas are easy to understand and draw and can be beneficial to the people working in the area.

4. Concluding remarks

The Generalized Petersen Graph and Double Generalized Petersen graphs are extensively studied families in graph theory. We have extended the study of topological indices by finding the Wiener index and the Wiener polarity indices of the Double Generalized Petersen Graph which is constructed from the Generalized Petersen Graph. We close this section by raising the following questions.

Open Problems

1. Determine the Wiener Index of Double Generalized Petersen Graph $DP(q, k)$ for $k \ge 3$.
2. Explore the other topological properties of $DP(q, k)$.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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