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Constrained backorders inventory system with varying order cost: Lead time demand uniformly distributed

Mona F. El-Wakeel *

Department of Statistics and Operation Research, Faculty of Science, King Saud University, Riyadh, Saudi Arabia

Abstract This paper discusses the probabilistic backorders inventory system when the order cost unit is a function of the order quantity. Our objective is to minimize the expected annual total cost under a restriction on the expected annual holding cost when the lead time demand follows the uniform distribution. Then some special cases are deduced and an illustrative numerical example with its graphs is added.

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1. Introduction

Many authors including Feldman (1978), Richards (1975) and Sahin (1979) have studied continuous review inventory models with constant units of cost and stationary distributions of inventory level. The inventory models under continuous review with stationary distribution of inventory level, or inventory position in the case of positive lead-time, have been derived using renewal theory as in Arrow et al. (1958). In addition, Ben-Daya and Abdul (1994) examined unconstrained inventory model with constant units of cost, demand follows a normal distribution and the lead-time is one of the decision variables.

Taha (1997) treated unconstrained probabilistic inventory problems with constant units of cost. Hadley and Whitin

(1963) discussed probabilistic continuous review inventory models with constant units of cost and the lead-time demand is a random variable. Their work gives heuristic approximate treatment for each of the backorders and the lost sales cases. Fabrycky and Banks (1967) studied the probabilistic single-item, single source (SISS) inventory system with zero lead-time, using the classical optimization. Abou-El-Ata et al. (2003) introduced a probabilistic multi-item inventory model with varying order cost; zero lead-time demand under two restrictions and no shortage are to be allowed. Fergany and El-Wakeel (2006a,b), applied several continuous distributions for constrained probabilistic lost sales inventory models with varying order cost using Lagrangian method. Recently, Kotb and Fergany (2011) deduced multi-item EOQ model with varying holding cost using geometric programming approach.

This paper considering the backorders inventory model with varying order cost, a restriction on the expected annual holding cost and the lead-time demand follows Uniform distribution. The policy variables of this model are the order quantity and the reorder point, which minimize the annual total cost. Finally, two special cases are deduced, which have been previously published and a numerical illustrative example is added with its graphs.

* Tel.: +966 501477097.
E-mail address: melwakeel@ksu.edu.sa.
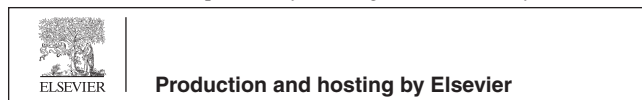
2. Assumptions and notations

The following assumptions are usually made in the simple treatments for developing the mathematical model:

1. The reorder point r is positive.
2. The demand is a random variable with known probability.
3. An order quantity of size Q per cycle is placed every time the stock level reaches a certain reorder point r.
4. Assume that the system repeats itself in the sense that the inventory position varies between r and $r + Q$ during each cycle.

The following notations are adopted for developing our model:

c_h = The holding cost per year
c_b = The backorder cost per unit backordered per cycle
c_o = The order cost per cycle
$C_o(Q) = c_o Q^\beta$ = The varying order cost per cycle, β is a real number
$\overline{D}$ = The average demand per year
$E(r - x) = ss$ = Safety stock = The expected net inventory
$\overline{H}$ = The average units on hand inventory = $\frac{\text{Max. on hand} + \text{Min. on hand}}{2}$ $= \frac{ss + Q + ss}{2} = \frac{Q}{2} + ss = \frac{Q}{2} + r - E(x)$
K = The limitation on the expected annual holding cost
L = The lead time between the placement of an order and its receipt
n = The number of cycles
N = The inventory cycle
Q = The order quantity per cycle
x = The continuous random variable represents the units demanded during L
$f(x)$ = The probability density function of the lead-time demand x
$r - x$ = The random variable represents the net inventory
$p(x > r)$ = The probability of shortage = $\int_r^\infty f(x)\,\mathrm{d}x = P(r)$ $= 1 - F(r)$ = The reliability function
$\overline{B}(r)$ = The expected number of backorders per cycle = $\int_r^\infty (x - r)f(x)\,\mathrm{d}x$

3. The mathematical model

Using the expression of the expected value of a random variable, it is possible to develop the expected annual total cost as follows:

$E(\text{Total Cost}) = E(\text{Order Cost}) + E(\text{Holding Cost}) + E(\text{Backorders Cost})$.

i.e., $\quad E(TC) = E(OC) + E(HC) + E(BC)$ (1)

where

$$E(OC) = C_o(Q) \cdot n = c_o Q^\beta \frac{\overline{D}}{Q} = c_o \overline{D} Q^{\beta-1} \quad (2)$$

$$E(HC) = c_h \overline{H} = c_h \left(\frac{Q}{2} + r - E(x) \right) \quad (3)$$

and

$$E(BC) = c_b \cdot n \cdot \overline{B}(r) = \frac{c_b \overline{D}}{Q} \int_r^\infty (x - r)f(x)\,\mathrm{d}x \quad (4)$$

Therefore

$$E[TC(Q, r)] = c_o \overline{D} Q^{\beta-1} + c_h \left(\frac{Q}{2} + r - E(x) \right) + \frac{c_b \overline{D}}{Q}$$
$$\times \int_r^\infty (x - r)f(x)\,\mathrm{d}x \quad (5)$$

Our objective is to minimize the expected annual total cost $E[TC(Q, r)]$ under the following constraint:

$$c_h \left(\frac{Q}{2} + r - E(x) \right) \leqslant K \quad (6)$$

To solve this primal function which is a convex programming problem, let us write it in the following form:

$$E[TC(Q, r)] = c_o \overline{D} Q^{\beta-1} + c_h \left(\frac{Q}{2} + r - E(x) \right) + \frac{c_b \overline{D}}{Q} \overline{B}(r) \quad (7)$$

subject to : $\quad c_h \left(\frac{Q}{2} + r - E(x) \right) \leqslant K \quad (8)$

To find the optimal values Q^* and r^* which minimize Eq. (7) under the constraint (8), we will use the Lagrangian multiplier technique as follows:

$$L(Q, r, \lambda) = c_o \overline{D} Q^{\beta-1} + c_h \left(\frac{Q}{2} + r - E(x) \right) + \frac{c_b \overline{D}}{Q} \overline{B}(r)$$
$$+ \lambda \left[c_h \left(\frac{Q}{2} + r - E(x) \right) - K \right] \quad (9)$$

where λ is the Lagrangian multiplier.

The optimal values Q^* and r^* can be found by setting each of the corresponding first partial derivatives of Eq. (9) equal to zero at $Q = Q^*$ and $r = r^*$ respectively, we obtain:

$$Q^{*2} + \frac{2(\beta - 1)c_o \overline{D}}{(1 + \lambda)c_h} Q^{*\beta} - \frac{2c_b \overline{D}}{(1 + \lambda)c_h} \overline{B}(r) = 0 \quad (10)$$

and

$$P(r^*) = \frac{(1 + \lambda)c_h}{c_b \overline{D}} Q^* \quad (11)$$

Clearly, it is difficult to find an exact solution of Q^* and r^* of Eqs. (10) and (11). So, we have to solve the two equations numerically, by the following algorithm that gives a closed approximate solution of these equations in a finite number of iterations:

- Step 1: Assume that $\overline{B} = 0$ and $r = E(x)$, then from Eq. (10) we have:

$$Q_1 = \left(\frac{2(1 - \beta)c_o \overline{D}}{(1 + \lambda)c_h} \right)^{\frac{1}{2-\beta}} \quad (12)$$

- Step 2: Substituting from Eq. (12) into Eq. (11) we get:

$$P(r_1) = \frac{(1 + \lambda)c_h}{c_b \overline{D}} \left(\frac{2(1 - \beta)c_o \overline{D}}{(1 + \lambda)c_h} \right)^{\frac{1}{2-\beta}} \quad (13)$$

- Step 3: Substituting by r_1 from Eq. (13) into Eq. (10) to find Q_2 as:

$$Q_2^2 + \frac{2(\beta - 1)c_o \overline{D}}{(1 + \lambda)c_h} Q_2^\beta - \frac{2c_b \overline{D}}{(1 + \lambda)c_h} \overline{B}(r_1) = 0 \quad (14)$$

The procedure is to vary λ in Eqs. (13) and (14) until the smallest value of $\lambda. > 0$ is found such that the constraint holds for the different values of β.

- Step 4: Repeating the steps 2 and 3 until obtaining successive values of Q and r, such that they are sufficiently close, which are the optimal values Q^* and r^*.

4. The uniform model

To get an exact solution, assume that the lead-time demand follows the Uniform distribution, as follows:

Let

$$f(x) = \frac{1}{b}; \quad 0 \leqslant x \leqslant b, \quad \text{with } E(x) = \frac{b}{2} \text{ and } P(r) = 1 - \frac{r}{b}$$

and

$$\overline{B}(r) = \frac{r^2}{2b} + \frac{b}{2} - r \tag{15}$$

Thus our simple model becomes:

$$E[TC(Q,r)] = c_o\overline{D}Q^{\beta-1} + c_h\left(\frac{Q}{2} + r - \frac{b}{2}\right) + \frac{c_b\overline{D}}{Q}\left(\frac{r^2}{2b} - r + \frac{b}{2}\right) \tag{16}$$

subject to: $\quad c_h\left(\dfrac{Q}{2} + r - \dfrac{b}{2}\right) \leqslant K \tag{17}$

To find the optimal values Q^* and r^* which minimize Eq. (16) under the constraint (17), we use the Lagrangian multiplier technique, thus the Lagrangian function is:

$$L(Q,r,\lambda) = c_o\overline{D}Q^{\beta-1} + c_h\left(\frac{Q}{2} + r - \frac{b}{2}\right) + \frac{c_b\overline{D}}{Q}\left(\frac{r^2}{2b} - r + \frac{b}{2}\right)$$
$$+ \lambda\left[c_h\left(\frac{Q}{2} + r - \frac{b}{2}\right) - K\right] \tag{18}$$

The optimal values Q^* and r^* are found by setting each of the corresponding first partial derivatives of Eq. (18) equal to zero, as follows:

$$(1 + \lambda)c_hbQ^{*2} + 2(\beta - 1)c_o\overline{D}bQ^{*\beta} - c_b\overline{D}(r - b)^2 = 0 \tag{19}$$

$$r^* = b\left(1 - \frac{(1 + \lambda)c_h}{c_b\overline{D}}Q^*\right) \tag{20}$$

Solving Eqs. (19) and (20) simultaneously, we get:

$$Q^* = \left(\frac{2(1 - \beta)c_bc_o\overline{D}^2}{(1 + \lambda)c_h[c_b\overline{D} - (1 + \lambda)bc_h]}\right)^{\frac{1}{2-\beta}} \tag{21}$$

$$r^* = b\left(1 - \left(\frac{2(1 - \beta)c_o[(1 + \lambda)c_h]^{1-\beta}\overline{D}^\beta}{c_b^{1-\beta}[c_b\overline{D} - (1 + \lambda)bc_h]}\right)^{\frac{1}{2-\beta}}\right) \tag{22}$$

Now Eqs. (21) and (22) gives the optimal values Q^* and r^* respectively. The procedure is to vary λ in Eqs. (21) and (22) until the smallest positive value of λ is found such that the constraint holds for the different values of β.

5. Special cases

We deduce two special cases of our model as follows:

Case 1: Let $\beta = 0$ and $K \to \infty \Rightarrow C_o(Q) = c_o$ and $\lambda = 0$. Thus Eqs. (10) and (11) become:

$$Q^* = \sqrt{\frac{2\overline{D}(c_o + c_b\overline{B}(r))}{c_h}} \quad \text{and} \quad P(r^*) = \frac{c_h}{c_b\overline{D}}Q^*$$

This is unconstrained probabilistic inventory model with backorders case and constant units of cost, which are the same results as in Hadley and Whitin (1963).

Case 2: Let $\beta = 0$ and $K \to \infty \Rightarrow C_o(Q) = c_o$ and $\lambda = 0$, when the lead-time demand has the Uniform distribution. In this case, we obtain an exact solution of the optimal inventory policy. So Eqs. (21) and (22) will be in the form:

$$Q^* = \sqrt{\frac{2c_oc_b\overline{D}^2}{c_h(c_b\overline{D} - bc_h)}} \quad \text{and} \quad r^* = b\left(1 - \frac{c_h}{c_b\overline{D}}Q^*\right)$$

This is unconstrained probabilistic inventory model under backorders case with uniformly lead-time demand and constant units of cost. Which are the same results as in Fabrycky and Banks (1967) when we consider that the unit cost c_P of the item is a constant independent of Q.

6. An illustrative example

Consider an item for the demand treated as a constant, the average demand being 100 units per year. The cost of placing an order is a function of Q and equal to $40.00. The item's holding cost is $4.00 per year. The procurement lead-time is not a constant and the lead-time demand is distributed uniformly in the interval 0–20 units. All demands occurring when the system is out of stock are backordered, and the cost is $7.00 per backorder. The inventory is controlled using a $\langle Q, r \rangle$ system under the constraint that the average holding cost is either less than or equal $120 per year. Determine Q^* and r^* for this model.

Table 1 The expected minimum total cost for Q^* and r^* at all different values of β.

| β	λ^*	Q^*	r^*	E(OC)	E(HC)	E(BC)	min(TC)
0	0	47.5191	14.2062	84.1767	111.143	13.1424	208.462
0.1	0.09	52.968	13.4017	112.318	119.543	14.3843	246.245
0.2	0.302	56.92	11.5303	157.706	119.961	22.0551	299.722
0.3	0.54	61.6328	9.1526	223.456	119.876	33.4099	376.742
0.4	0.789	67.405	6.2956	319.775	119.992	48.7603	488.528
0.5	1.025	74.4607	2.7677	463.55	119.992	69.791	653.333
0.6	1.246	82.1577	−1.0887	685.805	119.961	94.7305	900.496
0.7	1.414	89.2209	−4.6148	1039.73	119.983	118.84	1278.55
0.8	1.464	91.5611	−5.7836	1620.76	119.988	127.062	1867.81
0.9	1.167	79.2261	0.37909	2583.291	119.969	85.03	2788.3
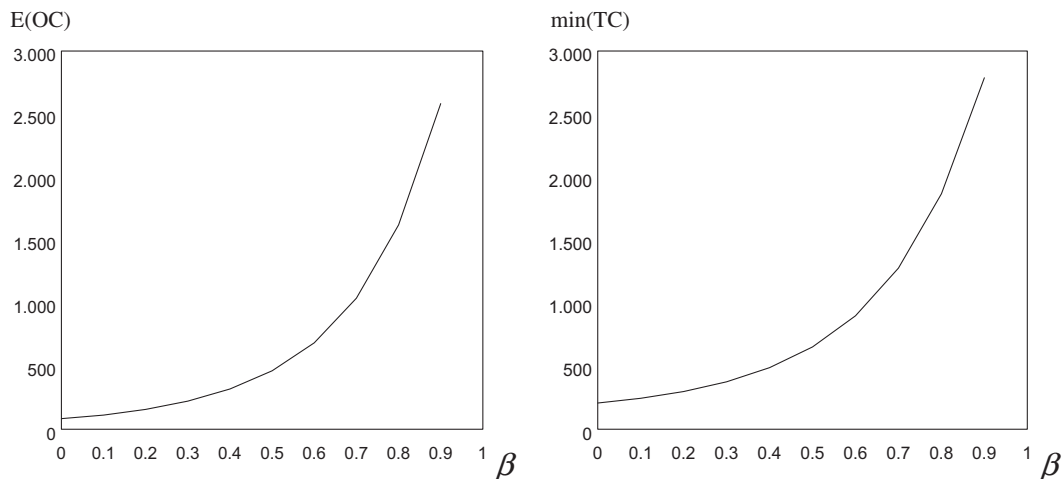
Figure 1   The expected order cost and the expected total cost via the values of β.

Using Eqs. (21) and (22), we can obtain the optimal values Q^* and r^* for different values of β at varying values of λ that yield the holding constraint as shown in Table 1.

From the above table, we can draw each of $E(OC)$ and $\min E(TC)$ against β as shown in Fig. 1.

7. Conclusion

This paper investigating probabilistic backorder inventory system with varying order cost and continuous lead time demand under the holding cost restriction. We have calculated Q^* and r^* numerically and obtain the min (TC) for different values of β by using Lagranian multiplier technique. In the Uniform model, we can evaluate the exact solution for each Q^* and r^* at different values of β, which minimize the expected total cost, mathematically.

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