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The HK-Sobolev space and applications to one-dimensional boundary value problems

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ABSTRACT

In this paper, we introduce the HK-Sobolev space and establish a fundamental theorem of calculus and an integration by parts formula, then we give sufficient conditions for the existence and uniqueness of a solution to a variational problem associated with a Sturm–Liouville type equation involving Henstock-Kurzweil integrable functions as source terms.

1. Introduction

By way of introduction, let us begin with the motivation for this work.

A classical variational problem. The Sobolev space is given by

$$H^1([0,1]) = \left\{u \in L^2([0,1]) : \exists g \in L^2([0,1]) \text{ such that } \int_0^1 u\varphi' = -\int_0^1 g\varphi, \forall \varphi \in C_c^1(0,1)\right\}, \quad (1)$$

where $C_c^1(0,1)$ is the space of continuously differentiable functions defined on $[0,1]$ with support in $(0,1)$. The function g in (1) is named the weak derivative of u and is denoted by $\dot{u}$. We set

$$H_0^1([0,1]) = \{u \in H^1([0,1]) : u(0) = u(1) = 0\}.$$

Consider the following variational problem:

Given $f \in L^2([0,1])$, find $u \in H_0^1([0,1])$such that

$$\int_0^1 \dot{u}\dot{\varphi} + \int_0^1 u\varphi = \int_0^1 f\varphi \quad (2)$$

for all $\varphi \in H_0^1([0,1])$. This variational problem arises from considering the following boundary problem (Brezis, 2011):

$$\begin{aligned} &-\ddot{u} + u = f \text{ a.e. on } (0,1); \\ &u(0) = 0, \ u(1) = 0. \end{aligned} \quad (3)$$

The space $H_0^1([0,1])$ has certain properties that are useful in order to find a solution to variational problems (2): it is a Hilbert, separable and reflexive space.

What happens if f is not of square Lebesgue integrable? In several physical phenomena, highly oscillating or singular functions appear (Condon et al., 2009; Hamed and Cummins, 1991; Hong and Xu, 2001; Samoilov et al., 2005). The Lebesgue integral is not enough for some highly oscillating functions leading to the possibility that the integral on the right side of the Eq. (2) does not exist for this type of functions and so the variational problem (2) would not be well defined. One way to solve this problem is to change the type of integral to be considered, in this work we will use the Henstock–Kurzweil integral. Different authors have studied differential equations involving Henstock–Kurzweil integrable functions. In León-Velasco et al. (2019) the authors use the Finite Element Method (FEM) for finding numerical solutions of elliptic problems with Henstock–Kurzweil integrable functions. They use open quadratures and Lobatto quadratures to approximate numerically the integrals that appear in the FEM. In Liu et al. (2018) are given conditions to establish the existence of a solution to nonlinear second-order differential equations of type $-D^2x = f(t,x) + g(t,x)Du$ subject to the boundary conditions $x(0) = \beta Dx(0)$, $Dx(1) + Dx(\eta) = 0$, where the derivatives are in the distributional sense, x, u are regulated functions and g is of bounded variation.

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In that paper the Henstock–Kurzweil–Stieltjes integral is used to transform the distributional differential equation into an integral equation, then the Leray–Schauder nonlinear alternative theorem is applied for finding a solution. In Sánchez-Perales and Mendoza-Torres (2020) the existence and uniqueness of the Shrödinger equation, $-y'' + qy = f$ a.e. on $[a,b]$ subject to arbitrary boundary values, is guaranteed for functions f, q Henstock–Kurzweil integrable. Properties of the inverse of the Shrödinger operator are established, then the authors give conditions so that the solution of the differential equation can be expressed as a Fourier type series.

Henstock–Kurzweil–Sobolev space. Around the 1960s, R. Henstock and J. Kurzweil, independently, define a Riemann-type integral, known as Henstock–Kurzweil integral, which is equivalent to Denjoy and Perron integrals. This integral is more general than the Lebesgue integral. In this work we introduce, using the Henstock–Kurzweil integral instead of the Lebesgue integral, a space analogous to $H^1([0,1])$, which we will call the Henstock–Kurz weil–Sobolev space and denote it by W_{HK}. Since the product of two Henstock–Kurzweil integrable functions is not necessarily an integrable function, the W_{HK}space is not provided with a natural internal product. Thus we cannot apply classical theorems, such as Lax–Milgram's, to guarantee the existence and uniqueness of the solution to variational problems such as (2). In this paper, we will use Fredholm's alternative for compact operators and the properties of the Henstock–Kurzweil–Sobolev space to solve such problems.

2. Preliminaries

The symbol $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ stands for the complex numbers and $[a,b] \subset \mathbb{R}$ is a closed finite interval. A tagged partition $\{([t_{i-1}, t_i], \xi_i) : i = 1, \ldots, n\}$ of $[a,b]$ is a finite collection of non-overlapping intervals $[t_{i-1}, t_i]$ such that $[a,b] = \cup_{i=1}^n [t_{i-1}, t_i]$, and $\xi_i \in [t_{i-1}, t_i]$ for all $i = 1, \ldots, n$. A function $\delta : [a,b] \to \mathbb{R}$ is a gauge on $[a,b]$ if $\delta(t) > 0$ for every $t \in [a,b]$. Given a gauge δ on $[a,b]$ and a tagged partition $P = \{([t_{i-1}, t_i], \xi_i) : i = 1, \ldots, n\}$ of $[a,b], P$ is δ-fine if

$$[t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)), \quad i = 1, \ldots, n.$$

A function $f : [a,b] \to \mathbb{C}$ is Henstock–Kurzweil integrable (HK-integrable) on $[a,b]$ if there exists a number I such that for every $\epsilon > 0$ there exists a gauge δ on $[a,b]$ such that for each δ-fine tagged partition $\{([t_{i-1}, t_i], \xi_i) : i = 1, \ldots, n\}$ of $[a,b]$,

$$\left| \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) - I \right| < \epsilon.$$

The number I is called the integral of f over $[a,b]$ and it is denoted by $\int_a^b f$. The space of Henstock–Kurzweil integrable functions is denoted by $HK([a,b])$. The Alexiewicz semi-norm of a function $f \in HK([a,b])$ is defined by

$$\|f\|_A = \sup_{t\in[a,b]} \left\{ \left| \int_a^t f \right| \right\}.$$

A function $\varphi : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$ ($\varphi \in BV([a,b])$) if

$$V_{[a,b]}\varphi := \sup \left\{ \sum_{i=1}^{n} |\varphi(d_i) - \varphi(c_i)| \right\} < \infty,$$

where the supremum is taken over all finite collections $\{[c_i, d_i] : i = 1, \ldots, n\}$ of non-overlapping intervals of $[a,b]$.

Theorem 2.1. (Talvila, 1999, Lemma 24) *If $f \in HK([a,b])$ and $g \in BV([a,b])$, then $fg \in HK([a,b])$ and*

$$\left| \int_a^b fg \right| \leqslant \inf_{t\in[a,b]} |g(t)| \left| \int_a^b f(t)dt \right| + \|f\|_A V_{[a,b]} g.$$

Theorem 2.2. (Sargent, 1948, Theorem D) *Let f, φ be functions such that f is of real values and $f \in HK([a,b])$. Then, $f\varphi \in HK([a,b])$ if and only if there exists $\varphi_1 \in BV([a,b])$ such that $\varphi = \varphi_1$ a.e. on $[a,b]$.*

The next Fubini's Theorem is a direct consequence of Talvila (2002, Lemma 25).

Theorem 2.3. *If $f \in HK([a,b])$ and $h \in BV([a,b])$, then for any subintervals A, B of $[a,b]$, we have*

$$\int_A \int_B f(t)h(x)dtdx = \int_B \int_A f(t)h(x)dxdt.$$

A function $F : [a,b] \to \mathbb{C}$ is absolutely continuous (respectively, absolutely continuous in the restricted sense) on a set $E \subseteq [a,b]$, if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^s |F(d_i) - F(c_i)| < \epsilon$ (respectively, $\sum_{i=1}^s \sup\{|F(x) - F(y)| : x, y \in [c_i, d_i]\} < \epsilon$) whenever $\{[c_i, d_i]\}_{i=1}^s$ is a collection of non-overlapping intervals with endpoints in E and such that $\sum_{i=1}^s (d_i - c_i) < \delta$. The space of absolutely continuous functions on E is denoted by $AC(E)$, and the space of absolutely continuous functions in the restricted sense on E is denoted by $AC_*(E)$.

The function F is generalized absolutely continuous in the restricted sense on $[a,b]$ ($F \in ACG_*([a,b])$), if F is continuous on $[a,b]$ and there exists a countable collection $(E_n)_{n=1}^\infty$ of subsets of $[a,b]$ such that $[a,b] = \cup_{i=1}^\infty E_n$ and $F \in AC_*(E_n)$ for all $n \in \mathbb{N}$. This concept leads to a very strong version of the fundamental theorem of calculus:

Theorem 2.4. [Fundamental theorem of calculus] (Gordon, 1994) *Let $f, F : [a,b] \to \mathbb{C}$ be functions and let $c \in [a,b]$.*

1. If $f \in HK([a,b])$ and $F(x) = \int_c^x f$ for all $x \in [a,b]$, then $F \in ACG_*([a,b])$ and $F' = f$ almost everywhere on $[a,b]$. In particular, if f is continuous at $x \in [a,b]$, then $F'(x) = f(x)$.
2. $F \in ACG_*([a,b])$ if and only if F' exists almost everywhere on $[a,b]$ and $\int_c^x F' = F(x) - F(c)$ for all $x \in [a,b]$.

Theorem 2.5. [Integration by parts formula] (Sánchez-Perales and Mendoza-Torres, 2020, Corollary 2.4) *If $u \in ACG_*([a,b])$ and $v \in AC([a,b])$, then $u'v \in HK([a,b]), uv' \in L([a,b])$ and*

$$\int_a^b u'(t)v(t)dt = u(b)v(b) - u(a)v(a) - \int_a^b u(t)v'(t)dt.$$

3. The HK-Sobolev space

In what follows from this document, we have presented the results on the interval $[0,1]$ without loss of generality, since they can be generalized to any compact interval. Let $C_P^2([0,1])$ the space of all functions $\varphi \in C([0,1])$ for which there exists $\{[t_{i-1}, t_i]\}_{i=1}^n$ a partition of $[0,1]$ such that $\varphi \in C^2((t_{i-1}, t_i))$ for all $i = 1, \cdots, n$; and $\varphi^{(k)}(t_0+), \varphi^{(k)}(t_1-), \varphi^{(k)}(t_1+), \cdots, \varphi^{(k)}(t_{n-1}-), \varphi^{(k)}(t_{n-1}+), \varphi^{(k)}(t_n-)$ exist for all $k = 1, 2$. We set

$$V = \{\varphi \in C_P^2([0,1]) : \varphi(0) = \varphi(1) = 0\}.$$

It is clear that if $\varphi \in C^2_P([0,1])$, then φ and φ' belong to $AC([0,1]) (\subseteq BV([0,1]))$. The next theorem is proved in a similar way to Hestenes (1966, Lemma 15.2, p. 51) with some modifications.

Lemma 3.1. [Fundamental lemma of calculus of variations] *Let $f, g \in HK([0,1])$ with g continuous on the right at $t = 0$. Then*

$$\int_0^1 [f(t)\varphi(t) + g(t)\varphi'(t)]dt = 0$$

for every $\varphi \in V$, if and only if,

$$g(t) = \int_0^t f(s)ds + g(0),$$

for almost all $t \in [0,1]$.

Proof. We assume that f and g are HK-integrable on $[0,1]$, then the function $G(t) := \int_0^t g$ is differentiable except for a set K of measure zero. Let $\hat{t} \in (0,1) \setminus K$, choose ϵ such that $0 < \epsilon < \hat{t}/2$, and define the function

$$\varphi(t) = \begin{cases} \frac{t}{\epsilon}, & \text{if } 0 \leqslant t \leqslant \epsilon, \\ 1, & \text{if } \epsilon \leqslant t \leqslant \hat{t} - \epsilon, \\ \frac{\hat{t}-t}{\epsilon}, & \text{if } \hat{t} - \epsilon \leqslant t \leqslant \hat{t}, \\ 0, & \text{if } \hat{t} \leqslant t \leqslant 1. \end{cases}$$

Then $\varphi \in V$ and so

$$0 = \frac{1}{\epsilon}\int_0^{\epsilon} f(t)tdt + \frac{1}{\epsilon}\int_0^{\epsilon} g + \int_{\epsilon}^{\hat{t}-\epsilon} f + \frac{1}{\epsilon}\int_{\hat{t}-\epsilon}^{\hat{t}} f(t)(\hat{t} - t)dt - \frac{1}{\epsilon}\int_{\hat{t}-\epsilon}^{\hat{t}} g. \tag{4}$$

Since g is continuous on the right at 0, it follows that

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon}\int_0^{\epsilon} g = g(0).$$

From Bartle (2001, Theorem 12.5),

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\int_0^{\epsilon} f(t)tdt = 0 = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\int_{\hat{t}-\epsilon}^{\hat{t}} f(t)(\hat{t} - t)dt.$$

Thus the right side of (4) tends to $g(0) + \int_0^{\hat{t}} f - G'(\hat{t})$ as $\epsilon \to 0$. Therefore, $g(0) + \int_0^{\hat{t}} f - G'(\hat{t}) = 0$, i.e.

$$g(\hat{t}) = \int_0^{\hat{t}} f + g(0).$$

Conversely, suppose that

$$g(t) = \int_0^t f(s)ds + g(0),$$

for almost all $t \in [0,1]$. We set $H(t) = \int_0^t f(s)ds + g(0)$, then $H = g$ and $H' = f$ a.e. on $[0,1]$, hence by Theorem 2.5,

$$\int_0^1 [f\varphi + g\varphi'] = \int_0^1 [H'\varphi + H\varphi'] = H(1)\varphi(1) - H(0)\varphi(0) = 0. \quad \square$$

Corollary 3.2. *Let $f \in HK[0,1]$. If*

$$\int_0^1 f\varphi = 0,$$

for every $\varphi \in V$, then $f = 0$ a.e on $[0,1]$.

Proof. Taking $g = 0$ in Lemma 3.1, we obtain that

$$\int_0^t f = 0,$$

for almost all $t \in [0,1]$. We set $F(t) = \int_0^t f$, by continuity of F and the zero function, we have that $f = 0$ a.e. on $[0,1]$.

Corollary 3.3. *Let $f \in HK[0,1]$. If*

$$\int_0^1 f\varphi' = 0,$$

for every $\varphi \in V$, then there exists a constant C such that $f = C$ a.e on $[0,1]$.

Proof. Consider the function

$$\psi(t) = \begin{cases} 4t, & \text{if } 0 \leqslant t \leqslant \frac{1}{2}; \\ -4t + 4, & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Let $\varphi \in V$ and define

$$z(t) = \int_0^t \left[\varphi - \left(\int_0^1 \varphi\right)\psi\right].$$

Then $z \in C^2_P([0,1])$ and $z(0) = z(1) = 0$, hence $z \in V$. Thus by hypothesis and Fubini's Theorem, we have that

$$\begin{aligned} 0 = \int_0^1 fz' &= \int_0^1 f(t)\left[\varphi(t) - \left(\int_0^1 \varphi(x)dx\right)\psi(t)\right]dt \\ &= \int_0^1 f(t)\varphi(t)dt - \int_0^1\int_0^1 f(t)\varphi(x)\psi(t)dxdt \\ &= \int_0^1 f(x)\varphi(x)dx - \int_0^1\int_0^1 f(t)\varphi(x)\psi(t)dtdx \\ &= \int_0^1 \left(f(x) - \int_0^1 f(t)\psi(t)dt\right)\varphi(x)dx. \end{aligned}$$

Therefore, by Corollary 3.2, $f - \int_0^1 f\psi = 0$ a.e. on $[0,1]$.

Theorem 3.4. *Let $g \in HK([0,1])$. For a fixed $y_0 \in [0,1]$ define*

$$v(x) = \int_{y_0}^x g(t)dt, x \in [0,1].$$

Then $v \in C[0,1]$ and

$$\int_0^1 v\varphi' = -\int_0^1 g\varphi,$$

for all $\varphi \in V$.

Proof. It is clear that v is continuous. By Fubini's Theorem it follows that

$$\begin{aligned} \int_0^1 v\varphi' &= \int_0^1\int_{y_0}^x g(t)\varphi'(x)dtdx \\ &= -\int_0^{y_0}\int_x^{y_0} g(t)\varphi'(x)dtdx + \int_{y_0}^1\int_{y_0}^x g(t)\varphi'(x)dtdx \\ &= -\int_0^{y_0}\int_0^t g(t)\varphi'(x)dxdt + \int_{y_0}^1\int_t^1 g(t)\varphi'(x)dxdt \\ &= -\int_0^{y_0} g(t)(\varphi(t) - \varphi(0))dt + \int_{y_0}^1 g(t)(\varphi(1) - \varphi(t))dt \\ &= -\int_0^1 g(t)\varphi(t)dt. \end{aligned}$$

Definition 1. The HK-Sobolev space W_{HK} is defined to be

$$W_{HK} = \{u \in HK([0,1]) : \exists g \in HK([0,1]) \text{ such that } \int_0^1 u\varphi' = -\int_0^1 g\varphi,\ \forall \varphi \in V\}.$$

For $u \in W_{HK}$ we define the weak derivative of u, denoted by $\dot{u}$, as

$$\dot{u} = g,$$

where g is the function given in Definition 1. Observe, by Corollary 3.2, $\dot{u}$ is well defined. Also, from Theorem 3.4 we have that for every $f \in HK([0,1])$, there exists a continuous function v defined on $[0,1]$ such that $\dot{v} = f$, that is, each HK-integrable function is the weak derivative of a some continuous function.

Proposition 3.5. *If $u = u_1$ a.e on $[0,1]$ and u_1 belongs to $ACG_*([0,1])$, then $u \in W_{HK}$ and $\dot{u} = u_1'$.*[1]

Proof. Let u, u_1 be functions such that $u_1 \in ACG_*([0,1])$ and $u = u_1$ a.e. on $[0,1]$. Then u_1' exists a.e on $[0,1]$, and by Theorem 2.5,

$$\int_0^1 u_1'\varphi = u_1\varphi|_0^1 - \int_0^1 u_1\varphi'$$

for all $\varphi \in V$. Therefore

$$\int_0^1 u\varphi' = \int_0^1 u_1\varphi' = -\int_0^1 u_1'\varphi$$

for all $\varphi \in V$. Consequently, $u \in W_{HK}$ and $\dot{u} = u_1'$. □

Remark 1. As an immediate consequence of the previous proposition, we have that if u is a continuous function on I such that u' exists except on a countable set, then $u \in W_{HK}$ and $\dot{u} = u'$.

Theorem 3.6. [Fundamental theorem of calculus] *Let $u \in W_{HK}$. Then there exists a function $\tilde{u} \in C([0,1])$ such that*

$$u = \tilde{u} \text{ a.e. on } [0,1]$$

and

$$\tilde{u}(d) - \tilde{u}(c) = \int_c^d \dot{u}, \text{ for all } c, d \in [0,1].$$

Proof. Write $\hat{u}(x) = \int_0^x \dot{u}$. By Theorem 3.4 we have

$$\int_0^1 \hat{u}\varphi' = -\int_0^1 \dot{u}\varphi,$$

for all $\varphi \in V$. On the other hand, as $u \in W_{HK}$, then

$$-\int_0^1 \dot{u}\varphi$$

Therefore

$$\int_0^1 (u - \hat{u})\varphi' = 0,$$

for all $\varphi \in V$. Consequently by Corollary 3.3, there exists a constant C such that $u - \hat{u} = C$ a.e. on $[0,1]$. The function $\tilde{u} = \hat{u} + C$ is the desired and satisfies the second part of the theorem. □

Remark 2.

1. Every function $u \in W_{HK}$ admits one (and only one) continuous representative $\tilde{u}$ on $[0,1]$. Therefore $u \in L^2([0,1])$. Moreover, $\tilde{u} \in ACG_*([0,1])$ and by Proposition 3.5, $\dot{u} = \tilde{u}'$.
2. If $v \in W_{HK}$ and there exists $w \in C([0,1])$ such that $w = \dot{v}$ a.e on $[0,1]$, then by Theorem 3.6, $\tilde{v} \in C^1([0,1])$, i.e. v has a continuously differentiable representative on $[0,1]$. Therefore by Theorem 2.2, $gv \in HK([0,1])$ for all real valued function $g \in HK([0,1])$.

Corollary 3.7. [Integration by parts formula] *If $u, v \in W_{HK}$, then $uv \in W_{HK}$ and*

$$(uv)^\cdot = \dot{u}v + u\dot{v}.$$

Also, if $\dot{u}v \in HK([0,1])$ and $u(0+) = u(0)$, $v(0+) = v(0)$, $u(1-) = u(1)$ and $v(1-) = v(1)$, then

$$\int_0^1 \dot{u}v = uv|_0^1 - \int_0^1 u\dot{v}. \tag{5}$$

Proof. Let $u, v \in W_{HK}$. Then by Remark 2, $\tilde{u}, \tilde{v} \in ACG_*([0,1])$, $\dot{u} = \tilde{u}'$ and $\dot{v} = \tilde{v}'$. From Sánchez-Perales and Mendoza-Torres (2020, Proposition 2.5), $\tilde{u}\tilde{v} \in ACG_*([0,1])$. Thus by Proposition 3.5, $uv \in W_{HK}$ and

$$(uv)^\cdot = (\tilde{u}\tilde{v})' = \tilde{v}'\tilde{v} + \tilde{u}\tilde{v}' = \dot{u}v + u\dot{v}. \tag{6}$$

Integrating (6) we obtain (5). □

4. Existence and uniqueness of a solution of a boundary value problem

Define the spaces

$$W_{HK}^2 = \{u \in W_{HK} : \dot{u} \in W_{HK}\}$$

and

$$W_{HK_0} = \{u \in W_{HK} : u(0) = u(0+) = u(1) = u(1-) = 0\}.$$

Let $f \in HK([0,1])$ and let q, ρ be real valued functions such that $q \in L^2([0,1])$, $\rho \in ACG_*([0,1]) \cap BV([0,1])$ and $|\rho(x)| \geqslant \alpha$ for all $x \in [0,1]$ and some $\alpha > 0$. Consider the following problems:

I. Find $u \in W_{HK}^2$ that satisfies

$$\begin{aligned} &-[\rho\dot{u}]^\cdot + qu = f \text{ a.e. on } (0,1); \\ &u(0) = u(0+) = 0,\ u(1) = u(1-) = 0. \end{aligned} \tag{7}$$

II. Find $u \in W_{HK_0}$ that satisfies

$$\int_0^1 \rho\dot{u}\varphi' + \int_0^1 qu\varphi = \int_0^1 f\varphi \tag{8}$$

for all $\varphi \in V$.

The boundary value problem (I) and the variational problem (II) are equivalent. Indeed, suppose that $u \in W_{HK}^2$ is a solution of the boundary value problem (I). Multiplying both sides of the differential equation in (7) by $\varphi \in V$ and integrating it from 0 to 1, we obtain that

$$-\int_0^1 [\rho\dot{u}]^\cdot\varphi + \int_0^1 qu\varphi$$

[1] v' denotes de usual derivative of v. From $u_1 \in ACG_*([0,1])$ it follows that u_1' exists a.e on $[0,1]$.

Therefore

$$\int_0^1 \rho \dot{u}\varphi' + \int_0^1 qu\varphi = \int_0^1 f\varphi$$

for all $\varphi \in V$. Conversely, suppose that $u \in W_{HK_0}$ satisfies (8) for all $\varphi \in V$. Then,

$$\int_0^1 \rho \dot{u}\varphi' = -\int_0^1 (qu - f)\varphi$$

for all $\varphi \in V$. Thus $\rho \dot{u} \in W_{HK}$. Since $\frac{1}{\rho} \in ACG_*([0,1])$, it follows by Proposition 3.5 and Corollary 3.7 that $\dot{u} \in W_{HK}$, consequently $u \in W^2_{HK}$. On the other hand, since $\rho\dot{u} \in W_{HK}$ we have that

$$\int_0^1 \rho \dot{u}\varphi' = -\int_0^1 (\rho\dot{u})^{\cdot}\varphi$$

for all $\varphi \in V$. Consequently, from (8),

$$\int_0^1 [-(\rho\dot{u})^{\cdot} + qu - f]\varphi = 0$$

for all $\varphi \in V$. Hence, by Corollary 3.2, $-(\rho\dot{u})^{\cdot} + qu = f$ a.e. on $[0,1]$.

To find a solution to the boundary value problem (I), we demonstrate the existence and uniqueness of the variational problem (II). Define B on $W_{HK_0} \times V$ and l_u, l_f on V by

$$\begin{aligned} B(u,\varphi) &= \textstyle\int_0^1 \rho\dot{u}\varphi', \\ l_u(\varphi) &= \textstyle\int_0^1 qu\varphi, \\ l_f(\varphi) &= \textstyle\int_0^1 f\varphi. \end{aligned}$$

It is clear that l_u and l_f are linear operators and B is a bilinear operator. The variational problem (8) is equivalent to find $u \in W_{HK_0}$ such that

$$B(u,\varphi) + l_u(\varphi) = l_f(\varphi) \tag{9}$$

for all $\varphi \in V$.

Affirmation 1. *There exists an operator $A : W_{HK_0} \to W_{HK_0}$ such that $l_u(\varphi) = B(A(u),\varphi)$ for all $u \in W_{HK_0}$ and $\varphi \in V$.*

Proof. Define the functions

$$h_u(t) = -\frac{1}{\rho(t)}\int_0^t qu$$

and

$$z_u(t) = \int_0^t \left(h_u - \alpha_u \frac{1}{\rho(t)}\right),$$

where $\alpha_u = \frac{\int_0^1 h_u}{\int_0^1 \frac{1}{\rho}}$. Observe $z_u \in W_{HK}$ and $z_u(0) = z_u(0+) = z_u(1) = z_u(1-) = 0$. Define the operator $A : W_{HK_0} \to W_{HK_0}$ by

$$A(u) = z_u. \tag{10}$$

Then A is a linear operator, and for each $\varphi \in V$,

$$\begin{aligned} B(A(u),\varphi) &= \int_0^1 \rho\dot{z}_u\varphi' = \int_0^1 \rho\left(h_u - \alpha_u\frac{1}{\rho}\right)\varphi' = \int_0^1 (\rho h_u - \alpha_u)\varphi' \\ &= \int_0^1 \left[\left(\int_0^t -qu\right) - \alpha_u\right]\varphi'(t)dt = \int_0^1 qu\varphi = l_u(\varphi). \end{aligned}$$

Affirmation 2. *There exists a function $v_f \in W_{HK_0}$ for which $l_f(\varphi) = B(v_f,\varphi)$ for all $\varphi \in V$.*

Proof. Let F be the primitive of f, observe that $\dot{F} = f$. Then v_f defined by

$$v_f(t) = \int_0^t \frac{1}{\rho}(F - \beta_u),$$

where $\beta_u = \frac{\int_0^1 \frac{F}{\rho}}{\int_0^1 \frac{1}{\rho}}$, satisfies that $v_f \in W_{HK_0}$ and

$$B(v_f,\varphi) = \int_0^1 \rho\dot{v}_f\varphi' = \int_0^1 (F - \beta_u)\varphi' = \int_0^1 f\varphi = l_f(\varphi),$$

for all $\varphi \in V$.

Therefore, the problem (9) is equivalent to find $u \in W_{HK_0}$ such that

$$B(u,\varphi) + B(A(u),\varphi) = B(v_f,\varphi),$$

for all $\varphi \in V$, or

$$B(u + A(u) - v_f,\varphi) = 0,$$

for all $\varphi \in V$.

Remember that an operator $T : X \to Y$ between two normed spaces is compact if and only if for any bounded sequence (x_n) in X, the sequence (Tx_n) contains a converging subsequence. The following is the Arzelá-Ascoli Theorem.

Theorem 4.1. *A subset $\mathcal{H}$ of $C([0,1])$ is relatively compact on $C([0,1])$ if and only if:*

(i) *$\mathcal{H}$ is pointwise bounded, i.e. for every $x \in [0,1]$,*

$$\sup_{v\in\mathcal{H}}|v(x)| < \infty.$$

(ii) *$\mathcal{H}$ is equicontinuous, i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|v(x) - v(y)| < \epsilon,$$

for all $x, y \in [0,1]$ with $|x - y| < \delta$, and for each $v \in \mathcal{H}$.

Affirmation 3. *The operator $A : (W_{HK_0}, \|\cdot\|) \to (W_{HK_0}, \|\cdot\|)$ is compact, where $\|u\| = \|u\|_2 + \|\dot{u}\|_A$.*

Proof. Let $\Omega \subset W_{HK_0}$ be bounded. Then there exists $M > 0$ such that $\|u\| \leqslant M$ for all $u \in \Omega$. Let

$$\mathcal{H} = \left\{h_u - \alpha_u\frac{1}{\rho} : u \in \Omega\right\}.$$

We shall use the Arzelá-Ascoli theorem to prove that $\mathcal{H}$ is relatively compact in $C([0,1])$ with the uniform norm.

(i) First observe that if $u \in \Omega$, then

$$|\alpha_u| = \frac{\left|\int_0^1 h_u\right|}{\left|\int_0^1 \frac{1}{\rho}\right|} \leqslant \frac{\int_0^1 \frac{1}{|\rho(t)|}\left|\int_0^t qu\right|dt}{\left|\int_0^1 \frac{1}{\rho}\right|} \leqslant \frac{\|q\|_2\|u\|_2}{\alpha\left|\int_0^1 \frac{1}{\rho}\right|} \leqslant \frac{\|q\|_2 M}{\alpha\left|\int_0^1 \frac{1}{\rho}\right|}.$$

Let $x \in [0,1]$. Then for every $u \in \Omega$,

$$\begin{aligned} \left|h_u(x) - \alpha_u\frac{1}{\rho(x)}\right| &\leqslant \frac{1}{|\rho(x)|}\left[\left(\int_0^x |qu|\right) + |\alpha_u|\right] \\ &\leqslant \frac{1}{\alpha}(\|q\|_2\|u\|_2 + |\alpha_u|) \\ &\leqslant \frac{\|q\|_2 M}{\alpha}\left(1 + \frac{1}{\alpha\left|\int_0^1 \frac{1}{\rho}\right|}\right). \end{aligned}$$

(ii) We set $Q(t) = \int_0^t |q|^2$. Let $\epsilon > 0$, since Q and $\frac{1}{\rho}$ are continuous on $[0,1]$, there exists $\delta > 0$ such that if $|y - x| < \delta$, then

$$\left|\frac{1}{\rho(y)}-\frac{1}{\rho(x)}\right|<\frac{\epsilon}{2\|q\|_2M\left(1+\frac{1}{\alpha\left|\int_0^1\frac{1}{\rho}\right|}\right)}$$

and

$$\sqrt{|Q(y)-Q(x)|}<\frac{\alpha\epsilon}{2M}.$$

Let $x,y\in[0,1]$ be such that $|y-x|<\delta$. Suppose that $x<y$. Then

$$\begin{aligned}&\left|h_u(y)-\alpha_u\frac{1}{\rho(y)}-\left(h_u(x)-\alpha_u\frac{1}{\rho(x)}\right)\right|\\&\leqslant|h_u(y)-h_u(x)|+|\alpha_u|\left|\frac{1}{\rho(y)}-\frac{1}{\rho(x)}\right|\\&\leqslant\left|\frac{1}{\rho(y)}-\frac{1}{\rho(x)}\right|(\|q\|_2M+|\alpha_u|)+\frac{1}{\alpha}\int_x^y|qu|\\&\leqslant\left|\frac{1}{\rho(y)}-\frac{1}{\rho(x)}\right|\|q\|_2M\left(1+\frac{1}{\alpha\left|\int_0^1\frac{1}{\rho}\right|}\right)+\frac{M}{\alpha}\sqrt{Q(y)-Q(x)}<\epsilon.\end{aligned}$$

Therefore, by the Arzelá-Ascoli theorem, $\mathcal{H}$ is relatively compact in $C([0,1])$ with the uniform norm. Now, let (u_n) be a bounded sequence in W_{HK_0}, then for the above, there exists a subsequence (u_{n_k}) of (u_n) and $g\in C([0,1])$ such that $h_{u_{n_k}}-\alpha_{u_{n_k}}\frac{1}{\rho}\to g$ uniformly. This implies that

$$z_{u_{n_k}}(t)=\int_0^t\left(h_{u_{n_k}}-\alpha_{u_{n_k}}\frac{1}{\rho}\right)\to\int_0^t g$$

uniformly for all $t\in[0,1]$. We set $z(t)=\int_0^t g$ Then $z\in ACG_*([0,1])$ and $z(0)=z(0+)=z(1)=z(1-)=0$. Therefore, $z\in W_{HK_0}$. It is clear that $\|z_{u_{n_k}}-z\|_2\to 0$ and $\|\dot{z}_{u_{n_k}}-\dot{z}\|_A=\left\|h_{u_{n_k}}-\alpha_{u_{n_k}}\frac{1}{\rho}-g\right\|_A\to 0$. Thus $\|A(u_{n_k})-z\|\to 0$.

Therefore A is a compact operator.

We shall use the Fredholm's Alternative Theorem:

Theorem 4.2. (Kress, 1989, Theorem 3.4) *If $A:X\to X$ is a compact operator on a normed space X. The equation $(A+I)u=f$ has a unique solution for all $f\in X$ if and only if the homogeneous equation $(A+I)u=0$ has only the trivial solution.*

Affirmation 4. *Let $A:W_{HK_0}\to W_{HK_0}$ be the operator defined in (10). If $\rho>0$ and $q>0$ then the equation $(A+I)u=0$ has only the trivial solution in W_{HK_0}.*

Proof. Let $u\in W_{HK_0}$ be such that $(A+I)u=0$ a.e. on $[0,1]$. Then $z_u=-u$ hence by Proposition 3.5 $\dot{u}=-h_u+\alpha_u\frac{1}{\rho}$. This implies that

$$\rho(t)\dot{u}(t)=\left(\int_0^t qu\right)+\alpha_u$$

for almost all $t\in[0,1]$. Again, by Proposition 3.5,

$$(\rho\dot{u})^{\cdot}=qu. \tag{11}$$

Denote by u^* the conjugate of u, it is clear that $u^*\in W_{HK_0}$. Then by Corollary 3.7,

$$\int_0^1 quu^*=\int_0^1(\rho\dot{u})^{\cdot}u^*=\rho\tilde{\dot{u}}u^*|_0^1-\int_0^1\rho\dot{u}(u^*)\dot{=}-\int_0^1\rho\dot{u}(\dot{u})^*.$$

Thus

$$\int_0^1 q|u|^2+\int_0^1\rho|\dot{u}|^2=0$$

which implies that $u=0$ a.e. on $[0,1]$.

Affirmation 5. *If $\rho>0$ and $q>0$, then the uniqueness of the variational problem (8) holds.*

Proof. Suppose that there exist $u,v\in W_{HK_0}$ such that satisfy the variational problem (8). Then

$$\int_0^1\rho(\dot{u}-\dot{v})\varphi'=-\int_0^1 q(u-v)\varphi$$

for all $v\in V$. Therefore

$$[\rho(\dot{u}-\dot{v})]^{\cdot}=q(u-v). \tag{12}$$

Thus in similar way to Affirmation 4, we have that

$$\int_0^1 q|u-v|^2+\int_0^1\rho|\dot{u}-\dot{v}|^2=0.$$

Therefore $u=v$ a.e. on $[0,1]$.

Remark 3. From (11) and (12) it follows that the conclusions of Affirmation 4 and Affirmation 5 are also fulfilled if we assume that the homogeneous problem

$$\begin{aligned}&-[\rho\dot{u}]^{\cdot}+qu=0\text{ a.e. on }(0,1);\\&u(0)=u(0+)=0,\ u(1)=u(1-)=0\end{aligned}$$

has only the trivial solution.

Finally, we present the existence and uniqueness theorem.

Theorem 4.3. *Let $f\in HK([0,1])$ and let q,ρ be real valued functions such that $q\in L^2([0,1]),\rho\in ACG_*([0,1])\cap BV([0,1])$ and $|\rho(x)|\geqslant\alpha$for all $x\in[0,1]$ and some $\alpha>0$. If one of the following conditions holds:*

(i) *$\rho>0$ and $q>0$,*
(ii) *the homogeneous problem*

$$\begin{aligned}&-[\rho\dot{u}]^{\cdot}+qu=0\text{ a.e. on }(0,1);\\&u(0)=u(0+)=0,\ u(1)=u(1-)=0\end{aligned}$$

has only the trivial solution,

then there exists a unique $u\in W^2_{HK}$ such that

$$\begin{aligned}&-[\rho\dot{u}]^{\cdot}+qu=f\text{ a.e. on }(0,1);\\&u(0)=u(0+)=0,\ u(1)=u(1-)=0.\end{aligned}$$

Example 1. The unique solution of the problem

$$\begin{aligned}&-[\rho\dot{u}]^{\cdot}+qu=f\text{ a.e. on }(0,1);\\&u(0)=u(0+)=0,\ u(1)=u(1-)=0;\end{aligned}$$

where $\rho(x)=\sqrt{x}+1, q(x)=1$ and $f(x)=3x^{\frac{1}{2}}+2-\frac{1}{2\sqrt{x}}+x-x^2$, is given by

$$u(x)=x(x-1).$$

In this case $f\in HK([0,1])$.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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