Journal of King Saud University – Science 31 (2019) 902–912

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Review

Multi-product, multi-venders inventory models with different cases of the rational function under linear and non-linear constraints via geometric programming approach

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ARTICLE INFO

Article history:
Received 29 September 2017
Accepted 5 March 2018
Available online 12 March 2018

Keywords:
Multi-product
Multi-venders inventory
Geometric programming approach-maximum inventory level
Procurement cost
Zero lead-time
Linear and non-linear constraints

ABSTRACT

This research deal the probabilistic multi-product multi-vendor inventory model include varying order cost and zero lead-time under linear and non-linear constraints for the number of periods N_{rs}, the first linear constraint on the expected holding cost, the second nonlinear constraint on the buffer stock and the third linear constraint on the storage space. The goal is to limit the expected holding cost by an upper limit k_1, the limit for the buffer stock by an upper limit k_2 and the limit for the storage space by an upper limit k_3. The searchers' aim is to determine the minimum expected total cost, the optimal number of period N^*_{rs} and the optimal maximum inventory level Q^*_{mrs} by using a geometric programming approach. Then, applying the results of the models by a numerical example and graphs. Also, two special cases are deduced.

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2018.03.002

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1. Introduction

Many studies and research have emerged since more than ninety years to study the inventory. Harris (1915) was one of the first who managed to formulate effective inventory system by deriving the simple lot size formula and named (Wilson Formula) Proportion to Wilson who has published in 1930 in a way independent from Harris.

A lot of studies appeared to studying the unconstrained probabilistic inventory models, the first study in 1960's by Hadley and Whitin (1963) has Followed by many researches and studies. Duffin et al. (1967) debated the basic theories on GP with application in their book.

In 1965's Fabrycky and Banks (1965, 1967) treated some probabilistic inventory models and used the classical optimization for studied. It was Kotchenberger (1971) the first person who used Optimization by geometric programming on inventory problems. Zener (1971) used a geometric programming technique to solved a specific sort of non-linear problems. In 1989 Cheng (1989a, 1989b) used a geometric programming to studied an EOQ model and develop some inventory models. Ben-Daya and Raoyf (1994) presented unconstrained inventory model through GP method.

Also, appeared the more studies and researches for the probabilistic inventory models under linear and non linear constraints. Hariri and Abou-El-Ata (1995), Abou-El-Ata et al. (2003) and Fergany (2005) used a geometric programming approach to treated some of the constrained probabilistic inventory models with varying order cost. Similarly, Fergany and El-Wakeel (2004) applied geometric programming approach to studied the probabilistic inventory system with varying order cost. In (1997) Abou-el-ata and Kotb (1997) progress the restriction crisp inventory model by GP method. Teng and Yang (2007) treated deterministic Inventory Lot Size Models with time-Varying demand and Balkhi and Tadj (2008) were made a more dynamic models through the derivation of the EOQ model. Also El-Sodany (2011) studied the probabilistic safety stock model with varying holding cost by geometric programming approach. Islam (2015), applied a geometric programming approach to solved the multi-item, multi- criteria and multi-constraint level economic production planning inventory problem under the constraints of space capacity and the total allowable shortage cost.

In this paper we will discuss three probabilistic multi-product multi-vendor inventory models include varying order cost and zero lead-time under linear and nonlinear constraints for the number of periods N_{rs}, the first linear constraint on the holding cost, the second nonlinear constraint on the buffer stock and the third linear constraint on the storage space. The aim of the search to determine the minimum expected total cost, the optimal number of period N^*_{rs} and the optimal maximum inventory level Q^*_{mrs} by using a geometric programming approach (GPP). We discussed the model I in the case $g(N_{rs}) = \gamma$ for the probabilistic MIMS inventory model, and we got the same formulas for policy variables contained in Fabrycky and Banks (1967) in the same case $g(N_{rs}) = \gamma$ for the probabilistic SISS inventory models, this mean that the model I for the MIMS inventory models is a generalization of the probabilistic SISS inventory model for Fabrycky and Banks (1967). Also, we discussed the model II in the case $g(N_{rs}) = \frac{\nu+N_{rs}}{N_{rs}}$ for the probabilistic MIMS inventory model, and we got the same formulas for policy variables contained in Fabrycky and Banks (1967) in the same case $g(N_{rs}) = \frac{\nu+N_{rs}}{N_{rs}}$ for the probabilistic SISS, this mean that the model II for the MIMS inventory models is a generalization of the probabilistic SISS inventory model for Fabrycky and Banks (1967). The model III, we discussed it in the case $g(N_{rs}) = \gamma + \frac{\nu}{N}$ for the probabilistic MIMS inventory model and determined the optimal policy variables, and we deduced the optimal policy variables for the model I and model II as special cases from model III. Next, applying a numerical example for the three models, and finally, comparisons are done and conclusion is deduced.

2. Model's parameters and evolution

We adopted assumptions and notations for the model as follows

$\mathbf{C_{prs}}$	The production (purchase) cost for the $\mathbf{r^{th}}$ product and $\mathbf{s^{th}}$ vendor.
$\mathbf{C_{ors}(N_{rs})}$	The varying procurement cost for the $\mathbf{r^{th}}$ product per cycle and $\mathbf{s^{th}}$ vendor.
$\mathbf{C_{hr}}$	The holding cost for the $\mathbf{r^{th}}$ product per period.
$\mathbf{D_r}$	The annual demand rate for the $\mathbf{r^{th}}$ product per period. (Units)
$\mathbf{f(D_r)}$	The probability density function of the Demand with known average $\overline{D}_r$.
$\mathbf{\bar{I}}$	The expected level inventory for unit period. (Units/period)
$\mathbf{x_{ur}}$	The maximum demand for the $\mathbf{r^{th}}$ product during cycle. (Units/cycle)
$\mathbf{Q_{mrs}}$	The maximum inventory level of the $\mathbf{r^{th}}$ product and $\mathbf{s^{th}}$ vendor (Units)
$\mathbf{N_{rs}}$	The number of periods per cycle of the $\mathbf{r^{th}}$ product and $\mathbf{s^{th}}$ vendor, the review of the stock level of the $\mathbf{r^{th}}$ product is made every $\mathbf{N_{rs}}$ period.
$\mathbf{E(TC)}$	The expected total cost function.
$\mathbf{E(HC)}$	The expected annual holding cost.
$\mathbf{E(OC)}$	The expected annual procurement cost.
$\mathbf{E(PC)}$	The expected annual purchase cost.
$\mathbf{k_1}$	The limitation on the expected holding cost. (Units)
$\mathbf{k_2}$	The limitation on the expected buffer cost. (Units)
$\mathbf{k_3}$	The limitation on the area. (meter square m^2)
MIMS	Multi product (item), Multi-vendor. (source)

3. Assumptions for the model

1. A survey of stock level each N_{rs} periods.
2. An amount is ordered, so return the stock level to its initial posture specified Q_{rs}.
3. Suppose that Q_{rs} is a random variable representing the order amount of the r^{th} item and s^{th} source or vendor through cycle.
4. Shortages are not allowed.
5. The maximum inventory level of the r^{th} item and s^{th} source is Q_{mrs}, as follows:

$$Q_{mrs} = g(N_{rs})E(Q_{rs}) \quad Where: \quad E(Q_{rs}) = N_{rs}E(D_r) = N_{rs}\overline{D}_r \quad and \quad \overline{D}_r = E(D_r) = \int_0^\infty D_r f(D_r)$$

6. The procurement cost per unit is a varying function of N_{rs}, has the from:

$$C_{ors}(N_{rs}) = C_{ors}N_{rs}^{\beta} \quad where \quad C_{ors} > 0, \quad \beta > 0$$

4. Probabilistic (MIMS) inventory model with zero lead time under three constraints and varying order cost

We define the expected total cost for the period, that is the sum of the expected purchase cost for the period, the expected procurement cost for the period and the expected holding cost for the period as follows:

$$E(TC) = E(PC) + E(OC) + E(HO)$$

Where : $E(PC) = \sum_{r=1}^{n} C_{prs}E(D) = \sum_{r=1}^{n} C_{prs}\overline{D}_r$

$$E(OC) = \sum_{r=1}^{n} \frac{C_{ors}}{N_{rs}} \quad ; \quad E(HC) = \sum_{r=1}^{n} \frac{C_{hr}\bar{I}}{N_{rs}}$$

The expected level inventory $\bar{I}$ is given by:

$$\bar{I} = \frac{N_{rs}^2\overline{D}_r}{2}[2g(N_{rs}) - 1]$$

Now the holding cost component is given by:

$$E(HC) = \sum_{r}^{n} \frac{C_{hr}N_{rs}\overline{D}_r}{2}(2g(N_{rs}) - 1)$$

where $g(N_{rs})$ is the relational function just mentioned .The main variables for this model are Q_{mrs} and N_{rs} , then, we rewritten the total expected cost for the period as follows:

$$E(TC) = \sum_{r=1}^{n}\left[C_{prs}\overline{D}_r + \frac{C_{ors}}{N_{rs}} + \frac{C_{hr}N_{rs}\overline{D}_r}{2}(2g(N_{rs}) - 1)\right]$$
$$\text{where} \quad g(N_{rs}) > \frac{1}{2} \tag{1}$$

4.1. Model I: Consider the case $g(N_{rs}) = \gamma$ where $\gamma > \frac{1}{2}$

Substituting $g(N_{rs}) = \gamma$ where $\gamma$is constant, in the expected level inventory $\bar{I}$ and the expected holding cost $E(HC)$ are given by:

$$\bar{I} = \frac{N_{rs}^2\overline{D}_r}{2}(2\gamma - 1) \quad and \quad E(HC) = \sum_{r=1}^{n} \frac{C_{hr}N_{rs}\overline{D}_r}{2}(2\gamma - 1)$$

Also, the expected total cost in Eq. (1) is obtained as:

$$E(TC) = \sum_{r=1}^{n}\left[C_{prs}\overline{D}_r + C_{ors}N_{rs}^{\beta-1} + \frac{C_{hr}N_{rs}\overline{D}_r}{2}(2\gamma - 1)\right], \quad 0 < \beta < 1 \tag{2}$$

Subject to:

$$\sum_{r=1}^{n} \frac{C_{hr}\overline{D}_r N_{rs}}{2} \leqslant k_1$$
$$\sum_{r=1}^{n} \frac{C_{hr}\overline{D}_r v}{N_{rs}} \leqslant k_2$$
$$\sum_{r=1}^{n} S\overline{D}_r N_{rs} \leqslant k_3$$

The term $\sum_{r=1}^{n} C_{prs}\overline{D}_r$ is constant, then the expected total cost (2) can be written as following from:

$$\min\ E(TC) = \sum_{r=1}^{n}\left[C_{ors}N_{rs}^{\beta-1} + \frac{C_{hr}N_{rs}\overline{D}_r}{2}(2\gamma - 1)\right] \quad ; \quad 0 < \beta < 1 \tag{3}$$

Subject to:

$$\left.\begin{array}{l} \sum_{r=1}^{n} \frac{C_{hr}\overline{D}_r}{2k_1} N_{rs} \leqslant 1 \\ \sum_{r=1}^{n} \frac{C_{hr}\overline{D}_r v}{N_{rs}k_2} \leqslant 1 \\ \sum_{r=1}^{n} \frac{S\overline{D}_r N_{rs}}{k_3} \leqslant 1 \end{array}\right\} \tag{4}$$

Applying the geometric programming technique to the Eqs. (3) and (4), we obtain the primal geometric function:

$$G(\underline{w}) = \prod_{r=1}^{n}\left(\frac{C_{ors}}{w_{1rs}}\right)^{w_{1rs}} \cdot \left(\frac{C_{hr}\overline{D}_r(2\gamma - 1)}{2w_{2rs}}\right)^{w_{2rs}} \cdot \left(\frac{C_{hr}\overline{D}_r}{2k_1 w_{3rs}}\right)^{w_{3rs}} \cdot \left(\frac{C_{hr}\overline{D}_r v}{k_2 w_{4rs}}\right)^{w_{4rs}} \cdot \left(\frac{S\overline{D}_r}{k_3 w_{5rs}}\right)^{w_{5rs}} N_{rs}^{(\beta-1)w_{1rs}+w_{2rs}+w_{3rs}-w_{4rs}+w_{5rs}} \tag{5}$$

where $\underline{W} = w_{jrs},\ 0 < w_{jrs} < 1,\ r = 1, 2 \ldots n,\ s = 1, 2, \ldots \cdot m,\ j = 1, 2, 3, 4, 5$ are the weights that achieve orthogonal and natural condition as follows:

$$w_{1rs} + w_{2rs} = 1$$

$$(\beta - 1)w_{1rs} + w_{2rs} + w_{3rs} - w_{4rs} + w_{5rs} = 0$$

By solving the above equation, we get:

$$\left.\begin{array}{l} w_{1rs} = \dfrac{1 + w_{3rs} - w_{4rs} + w_{5rs}}{2 - \beta} \\ w_{2rs} = \dfrac{1 - \beta - w_{3rs} + w_{4rs} - w_{5rs}}{2 - \beta} \end{array}\right\} \tag{6}$$

The dual function is given by substitution from Eq. (6) into Eq. (5) as follows:

$$g(w_{3rs}, w_{4rs}, w_{5rs}) = \prod_{r=1}^{n}\left(\frac{(2 - \beta)C_{ors}}{1 + w_{3rs} - w_{4rs} + w_{5rs}}\right)^{\frac{1+w_{3rs}-w_{4rs}+w_{5rs}}{2-\beta}} \left(\frac{(2 - \beta)C_{hr}\overline{D}_r(2\gamma - 1)}{2(1 - \beta - w_{3rs} + w_{4rs} - w_{5rs})}\right)^{\frac{1-\beta-w_{3rs}+w_{4rs}-w_{5rs}}{2-\beta}} \cdot \left(\frac{C_{hr}\overline{D}_r}{2w_{3rs}}\right)^{w_{3rs}} \left(\frac{C_{hr}\overline{D}_r}{2k_1 w_{4rs}}\right)^{w_{4rs}} \left(\frac{S\overline{D}_r N_{rs}}{k_3 w_{5rs}}\right)^{w_{5rs}} \tag{7}$$

Now, take the logarithm of Eq. (7) and equate the first partial derivatives of $lng(w_{3rs}^*,\ w_{4rs}^*,\ w_{5rs}^*)$ to zero, respectively to calculate $w_{3rs}^*,\ w_{4rs}^*$ *and* w_{5rs}^* which maximize $g(w_{3rs}^*,\ w_{4rs}^*,\ w_{5rs}^*)$, we can obtain:

$$f(w_{5rs}) = w_{5rs}^{4-\beta} + A_1 w_{5rs}^{3-\beta} - A_1A_2 w_{5rs}^{2-\beta} + A_3 w_{5rs}^2 + (\beta - 1)A_1A_3 w_{5rs} - A_1A_2A_3 = 0 \tag{8}$$

$$f(w_{4rs}) = w_{4rs}^{4-\beta} + (1 - \beta)w_{4rs}^{3-\beta} - (A_2 + A_4)w_{4rs}^{2-\beta} + A_5 w_{4rs}^2 - A_5 w_{4rs} - A_5(A_2 + A_4) = 0 \tag{9}$$

$$f(w_{3rs}) = w_{3rs}^{4-\beta} + A_6 w_{3rs}^{3-\beta} - A_4A_6 w_{3rs}^{2-\beta} + A_7 w_{3rs}^2 + (\beta - 1)A_6A_7 w_{3rs} - A_4A_6A_7 = 0 \tag{10}$$

Where:

$$A_1 = \frac{2k_1S}{2k_1S + C_{hr}k_3} \quad ; \quad A_2 = \frac{C_{hr}\overline{D}_r^2 Sv}{k_2k_3e^2} \quad ;$$

$$A_3 = \frac{2C_{ors}\overline{D}_r^{1-\beta}S^{2-\beta}}{C_{hr}(2\gamma-1)(k_3e)^{2-\beta}}$$

$$A_4 = \frac{C_{hr}^2\overline{D}_r^2 v}{2k_1k_2e^2} \quad ; \quad A_5 = \frac{(2\gamma-1)(C_{hr}\overline{D}_r)^{3-\beta}v^{2-\beta}}{2C_{ors}(k_2e)^{2-\beta}}$$

$$A_6 = \frac{C_{hr}k_3}{C_{hr}k_3 + 2k_1S} \quad ; \quad A_7 = \frac{2C_{ors}(C_{hr}\overline{D}_r)^{1-\beta}}{(2\gamma-1)(2k_1e)^{2-\beta}}$$

It could easily prove that $f_j(0) < 0$ *and* $f_j(1) > 0$, $\forall j = 3,4,5$ this means that are three roots $w_{jrs} \in (0,1)\ \forall\ j = 3,4,5$. Any method such as the trial and error method could be used to calculate this root .We can verify that any root w_{3rs}^*, w_{4rs}^* *and* w_{5rs}^* calculated from Eqs. (8)–(10) maximize $(gw_{3rs}^*, w_{4rs}^*, w_{5rs}^*)$ respectively.

This is done by the second derivatives that verify Hessian matrix always negative as follows:

$$\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{irs}^2} = -\left[\frac{1}{(2-\beta)^2 w_{1rs}} + \frac{1}{(2-\beta)^2 w_{2rs}} + \frac{1}{w_{irs}}\right] < 0 \quad \forall\ i = 3,4,5$$

$$\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{irs}\partial w_{jrs}} = -\left[\frac{1}{(2-\beta)^2 w_{1rs}} + \frac{1}{(2-\beta)^2 w_{2rs}}\right] < 0 \quad i \neq j; \quad i,j = 3,5$$

$$\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{4rs}\partial w_{jrs}} = \left[\frac{1}{(2-\beta)^2 w_{1rs}} + \frac{1}{(2-\beta)^2 w_{2rs}}\right] > 0 \quad \forall\ j = 3,5$$

$$\text{and clearly} \left|\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{irs}\partial w_{jrs}}\right| < \left|\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial^2 w_{irs}}\right| \quad i \neq j;\ i,j = 3,4,5$$

$$\left(\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{irs}\partial w_{jrs}}\right)^2 < \left(\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial^2 w_{irs}}\right) \times \left(\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial^2 w_{jrs}}\right) \quad i \neq j;\ i,j = 3,4,5$$

Therefore from the Hessian matrix, we get:

$$\Delta = -\left[\frac{1}{(2-\beta)^2}\left(\frac{1}{w_{1rs}w_{3rs}w_{4rs}} + \frac{1}{w_{2rs}w_{3rs}w_{4rs}} + \frac{1}{w_{1rs}w_{3rs}w_{5rs}} + \frac{1}{w_{2rs}w_{3rs}w_{5rs}} + \frac{1}{w_{1rs}w_{4rs}w_{5rs}} + \frac{1}{w_{2rs}w_{4rs}w_{5rs}}\right) + \frac{1}{w_{3rs}w_{4rs}w_{5rs}}\right] < 0$$

Thus the roots w_{3rs}^*, w_{4rs}^* and w_{5rs}^* calculated from Eqs. (8)–(10) maximize the dual function $(gw_{3rs}^*, w_{4rs}^*, w_{5rs}^*)$ and the optimal solutions are $w_{jrs} \in (0,1)\forall j = 3,4,5$ where $w_{3rs}^*, w_{4rs}^* and w_{5rs}^*$ are obtained from Eqs. (8)–(10) respectively.

To find the optimal expected number of periods per cycle N_{rs}^* use the following relations due to Duffin and Peterson's theorem (Duffin et al., 1967) of geometric programming as follows:

$$C_{ors}N_{rs}^{\beta-1} = w_{1r}^* g(w_{3rs}^* w_{4rs}^*, w_{5rs}^*)$$

$$\frac{C_{hr}N_{rs}\overline{D}_r}{2}(2\gamma-1) = w_{2r}^* g(w_{3rs}^* w_{4rs}^*, w_{5rs}^*)$$

By solving the above equations, then substituting the values of w_{3rs}^*, w_{4rs}^* and w_{5rs}^* we get the optimal number of period per cycle N_{sr}^* as follows:

$$N_{rs}^* = \left(\frac{C_{hr}\overline{D}_r(2\gamma-1)(1 + w_{3rs}^* - w_{4rs}^* + w_{5rs}^*)}{2C_{ors}(1 - \beta - w_{3rs}^* + w_{4rs}^* - w_{5rs}^*)}\right)^{\frac{1}{\beta-2}} \tag{11}$$

Then, the maximum inventory level Q_{mrs}^* is given by:

$$Q_{mrs}^* = \gamma\,\overline{D}_r\left(\frac{C_{hr}\overline{D}_r(2\gamma-1)(1 + w_{3rs}^* - w_{4rs}^* + w_{5rs}^*)}{2C_{ors}(1 - \beta - w_{3rs}^* + w_{4rs}^* - w_{5rs}^*)}\right)^{\frac{1}{\beta-2}} \tag{12}$$

Substituting the value of N_{rs}^* from Eq. (11) into Eq. (3) after adding the constant term:

$$\min\ E(TC) = \sum_r^n \left[C_{prs}\overline{D}_r + C_{ors}\left(\frac{C_{hr}\overline{D}_r(2\gamma-1)(1 + w_{3rs}^* - w_{4rs}^* + w_{5rs}^*)}{2C_{ors}(1 - \beta - w_{3rs}^* + w_{4rs}^* - w_{5rs}^*)}\right)^{\frac{\beta-1}{\beta-2}} + \frac{C_{hr}\overline{D}_r(2\gamma-1)}{2}\left(\frac{C_{hr}\overline{D}_r(2\gamma-1)(1 + w_{3rs}^* - w_{4rs}^* + w_{5rs}^*)}{2C_{ors}(1 - \beta - w_{3rs}^* + w_{4rs}^* - w_{5rs}^*)}\right)^{\frac{1}{\beta-2}}\right] \tag{13}$$

4.2. Model II: The case $g(N_{rs}) = \frac{v+N_{rs}}{N_{rs}}$ where $v > 0$

The expected total cost in Eq. (1) will be:

$$E(TC) = \sum_{r=1}^n \left[C_{prs}\overline{D}_r + C_{ors}N_{rs}^{\beta-1} + C_{hr}\overline{D}_r v + \frac{C_{hr}\overline{D}_r N_{rs}}{2}\right]; \qquad 0 < \beta < 1 \tag{14}$$

Now, we defined the optimal minimum expected total cost min $E(TC)$ under the following constraints:

$$\sum_{r=1}^n \frac{C_{hr}\overline{D}_r N_{rs}}{2} \leqslant k_1$$
$$\sum_{r=1}^n \frac{C_{hr}\overline{D}_r v}{N_{rs}} \leqslant k_2$$
$$\sum_{r=1}^n S\overline{D}_r N_{rs} \leqslant k_3$$

Then Eq. (14) can be rewritten the annual expected total cost as following Whereas the term $\sum_{r=1}^n C_{prs}\overline{D}_r$ and $\sum_{r=1}^n C_{hr}\overline{D}_r v$ are constants:

$$\min\ E(TC) = \sum_{r=1}^n \left[C_{ors}N_{rs}^{\beta-1} + \frac{C_{hr}N_{rs}\overline{D}_r}{2}\right] \quad ; \qquad 0 < \beta < 1 \tag{15}$$

Subject to:

$$\left.\begin{array}{l} \sum_{r=1}^n \frac{C_{hr}\overline{D}_r N_{rs}}{2k_1} \leqslant 1 \\ \sum_{r=1}^n \frac{C_{hr}\overline{D}_r v}{N_{rs}k_2} \leqslant 1 \\ \sum_{r=1}^n \frac{S\overline{D}_r N_{rs}}{k_3} \leqslant 1 \end{array}\right\} \tag{16}$$

Applying the geometric programming technique to the Eq. (15) and (16), where $\underline{W} = w_{jrs}, 0 < w_{jrs} < 1, r = 1,2\ldots n, s = 1,2,\ldots m, j = 1,2,3,4,5$ are the weights that achieve orthogonal and natural condition whereas $w_{1rs} = \frac{1+w_{3rs}-w_{4rs}+w_{5rs}}{2-\beta}$ and $w_{2rs} = \frac{1-\beta-w_{3rs}+w_{4rs}-w_{5rs}}{2-\beta}$, we get:

$$g(w_{3rs}, w_{4rs}, w_{5rs}) = \prod_{r=1}^{n}\left(\frac{(2-\beta)C_{ors}}{1+w_{3rs}-w_{4rs}+w_{5rs}}\right)^{\frac{1+w_{3rs}-w_{4rs}+w_{5rs}}{2-\beta}} \left(\frac{(2-\beta)C_{hr}\overline{D}_r}{2(1-\beta-w_{3rs}+w_{4rs}-w_{5rs})}\right)^{\frac{1-\beta-w_{3rs}+w_{4rs}-w_{5rs}}{2-\beta}} \left(\frac{C_{hr}\overline{D}_r}{2k_1 w_{3rs}}\right)^{w_{3rs}} \left(\frac{C_{hr}\overline{D}_r}{k_2 w_{4rs}}\right)^{w_{4rs}} \left(\frac{S\overline{D}_r N_{rs}}{k_3 w_{5rs}}\right)^{w_{5rs}} \quad (17)$$

Now, take the logarithm of Eq. (17) and equate the first partial derivatives of $lng(w^*_{3rs}, w^*_{4rs}, w^*_{5rs})$ to zero, respectively to calculate w^*_{3rs}, $w^*_{4rs} and w^*_{5rs}$ which maximize $g(w^*_{3rs}, w^*_{4rs}, w^*_{5rs})$, we get:

$$f(w_{3rs}) = w_{3rs}^{4-\beta} + A_1 w_{3rs}^{3-\beta} - A_1A_2 w_{3rs}^{2-\beta} + A_3 w_{3rs}^2 + (\beta-1)A_1A_3 w_{3rs} - A_1A_2A_3 = 0 \quad (18)$$

$$f(w_{4rs}) = w_{4rs}^{4-\beta} + (1-\beta)w_{4rs}^{3-\beta} - (A_2+A_4)w_{4rs}^{2-\beta} + A_5 w_{4rs}^2 - A_5 w_{4rs} - A_5(A_2+A_4) = 0 \quad (19)$$

$$f(w_{5rs}) = w_{5rs}^{4-\beta} + A_6 w_{5rs}^{3-\beta} - A_4A_6 w_{5rs}^{2-\beta} + A_7 w_{5rs}^2 + (\beta-1)A_6A_7 w_{5rs} - A_4A_6A_7 = 0 \quad (20)$$

Where:

$$A_1 = \frac{C_{hr}k_3}{C_{hr}k_3 + 2k_1 S} \; ; \; A_2 = \frac{C_{hr}^2\overline{D}_r^2 v}{2k_1k_2e^2} \; ; \; A_3 = \frac{2C_{ors}(C_{hr}\overline{D}_r)^{1-\beta}}{(2k_1e)^{2-\beta}}$$

$$A_4 = \frac{C_{hr}\overline{D}_r^2 S v}{k_2k_3e^2} \; ; \; A_5 = \frac{(C_{hr}\overline{D}_r)^{3-\beta} v^{2-\beta}}{2C_{ors}(k_2e)^{2-\beta}}$$

$$A_6 = \frac{2k_1 S}{C_{hr}k_3 + 2k_1 S} \; ; \; A_7 = \frac{2C_{ors}\overline{D}_r^{1-\beta}S^{2-\beta}}{C_{hr}(k_3e)^{2-\beta}}$$

It could easily prove that $f_j(0) < 0 and f_j(1) > 0, \forall j = 3,4,5$ this means that are three roots $w_{jrs} \in (0,1) \forall j = 3,4,5$. Any method such as the trial and error method could be used to calculate this root . We can verify that any root w^*_{3rs}, w^*_{4rs} *and* w^*_{5rs} calculated from Eqs. (18)–(20) maximize $(gw^*_{3rs}, w^*_{4rs}, w^*_{5rs})$ respectively. This is done by the second derivative that verify Hessian matrix always negative as follows:

$$\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{irs}^2} = -\left[\frac{1}{(2-\beta)^2 w_{1rs}} + \frac{1}{(2-\beta)^2 w_{2rs}} + \frac{1}{w_{irs}}\right] < 0 \qquad \forall\, i = 3,4,5$$

$$\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{irs}\partial w_{jrs}} = -\left[\frac{1}{(2-\beta)^2 w_{1rs}} + \frac{1}{(2-\beta)^2 w_{2rs}}\right] < 0 \qquad i \neq j;\; i,j = 3,5$$

$$\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{4rs}\partial w_{jrs}} = \left[\frac{1}{(2-\beta)^2 w_{1rs}} + \frac{1}{(2-\beta)^2 w_{2rs}}\right] > 0 \qquad \forall\, j = 3,5$$

$$\text{and clearly } \left|\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{irs}\partial w_{jrs}}\right| < \left|\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial^2 w_{irs}}\right| \qquad i \neq j;\; i,j = 3,4,5$$

$$\left(\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{irs}\partial w_{jrs}}\right)^2 < \left(\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial^2 w_{irs}}\right) \times \left(\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial^2 w_{jrs}}\right) \qquad i \neq j;\; i,j = 3,4,5$$

Then:

$$\Delta = -\left[\frac{1}{(2-\beta)^2}\left(\frac{1}{w_{1rs}w_{3rs}w_{4rs}} + \frac{1}{w_{2rs}w_{3rs}w_{4rs}} + \frac{1}{w_{1rs}w_{3rs}w_{5rs}} + \frac{1}{w_{2rs}w_{3rs}w_{5rs}} + \frac{1}{w_{1rs}w_{4rs}w_{5rs}} + \frac{1}{w_{2rs}w_{4rs}w_{5rs}}\right) + \frac{1}{w_{3rs}w_{4rs}w_{5rs}}\right] < 0,$$

thus the roots $w^*_{3rs}, w^*_{4rs} and w^*_{5rs}$ calculated from Eqs. (18)–(20) maximize the dual function $(gw^*_{3rs}, w^*_{4rs}, w^*_{5rs})$ and the optimal solution is $w_{jrs} \in (0,1) \forall j = 3,4,5$ where w^*_{3rs}, $w^*_{4rs} and w^*_{5rs}$ are obtained from Eqs. (18)–(20).

By using the relations for Duffin and Peterson's theorem (Duffin et al., 1967) of geometric programming to find the optimal number of periods per cycle N^*_{rs} we get:

$$N^*_{rs} = \left(\frac{C_{hr}\overline{D}_r(1 + w^*_{3rs} - w^*_{4rs} + w^*_{5rs})}{2C_{ors}(1 - \beta - w^*_{3rs} + w^*_{4rs} - w^*_{5rs})}\right)^{\frac{1}{\beta-2}} \quad (21)$$

Then, the maximum inventory level Q^*_{mrs} as follows:

$$Q^*_{mrs} = \overline{D}_r\left[v + \left(\frac{C_{hr}\overline{D}_r(1 + w^*_{3rs} - w^*_{4rs} + w^*_{5rs})}{2C_{ors}(1 - \beta - w^*_{3rs} + w^*_{4rs} - w^*_{5rs})}\right)^{\frac{1}{\beta-2}}\right] \quad (22)$$

Substituting the value of N^*_{rs} from Eq. (21) into Eq. (15) after adding the constant term to get the minimum expected total cost as follows:

$$\min E(TC) = \sum_{r=1}^{n}\left[C_{prs}\overline{D}_r + C_{ors}\left(\frac{C_{hr}\overline{D}_r(1 + w^*_{3rs} - w^*_{4rs} + w^*_{5rs})}{2C_{ors}(1 - \beta - w^*_{3rs} + w^*_{4rs} - w^*_{5rs})}\right)^{\frac{\beta-1}{\beta-2}} + C_{hr}\overline{D}_r v + \frac{C_{hr}\overline{D}_r}{2}\left(\frac{C_{hr}\overline{D}_r(1 + w^*_{3rs} - w^*_{4rs} + w^*_{5rs})}{2C_{ors}(1 - \beta - w^*_{3rs} + w^*_{4rs} - w^*_{5rs})}\right)^{\frac{1}{\beta-2}}\right] \quad (23)$$

4.3. Model III: The case $g(N_{rs}) = \gamma + \frac{v}{N}$ where $v > 0, \gamma > \frac{1}{2}$

The expected total cost in Eq. (1) will be:

$$E(TC) = \sum_{r=1}^{n}\left[C_{prs}\overline{D}_r + C_{ors}N_{rs}^{\beta-1} + \frac{C_{hr}\overline{D}_r N_{rs}(2\gamma-1)}{2} + C_{hr}\overline{D}_r v\right]; \quad 0<\beta<1 \tag{24}$$

The optimal minimum E(TC) under the following constraints:

$$\sum_{r=1}^{n}\frac{C_{hr}\overline{D}_r N_{rs}}{2} \leqslant k_1$$
$$\sum_{r=1}^{n}\frac{C_{hr}\overline{D}_r v}{N_{rs}} \leqslant k_2$$
$$\sum_{r=1}^{n} S\overline{D}_r N_{rs} \leqslant k_3$$

Then can be rewritten the annual expected total cost as following whereas the term $\sum_{r=1}^{n} C_{prs}\overline{D}_r$ and $\sum_{r=1}^{n} C_{hr}\overline{D}_r v$ are constants:

$$\min\ E(TC) = \sum_{r=1}^{n}\left[C_{ors}N_{rs}^{\beta-1} + \frac{C_{hr}\overline{D}_r N_{rs}(2\gamma-1)}{2}\right]; 0<\beta<1 \tag{25}$$

Subject to:

$$\left.\begin{array}{l}\sum_{r=1}^{n}\frac{C_{hr}\overline{D}_r}{2k_1}N_{rs} \leqslant 1\\ \sum_{r=1}^{n}\frac{C_{hr}\overline{D}_r v}{N_{rs}k_2} \leqslant 1\\ \sum_{r=1}^{n}\frac{S\overline{D}_r N_{rs}}{k_3} \leqslant 1\end{array}\right\} \tag{26}$$

Applying the geometric programming technique to the Eqs. (25) and (26), where $\underline{W} = w_{jrs}, 0 < w_{jrs} < 1, r = 1,2\ldots n, s = 1,2,\ldots m$, $j = 1,2,3,4,5$ are the weights that achieve orthogonal and natural condition, we get:

$$g(w_{3rs}, w_{4rs}, w_{5rs}) = \prod_{r=1}^{n}\left(\frac{(2-\beta)C_{ors}}{1+w_{3rs}-w_{4rs}+w_{5rs}}\right)^{\frac{1+w_{3rs}-w_{4rs}+w_{5rs}}{2-\beta}} \left(\frac{(2-\beta)(2\gamma-1)C_{hr}\overline{D}_r}{2(1-\beta-w_{3rs}+w_{4rs}-w_{5rs})}\right)^{\frac{1-\beta-w_{3rs}+w_{4rs}-w_{5rs}}{2-\beta}} \cdot \left(\frac{C_{hr}\overline{D}_r}{2k_1 w_{3rs}}\right)^{w_{3rs}} \left(\frac{C_{hr}\overline{D}_r}{k_2 w_{4rs}}\right)^{w_{4rs}}\left(\frac{S\overline{D}_r}{k_3 w_{5rs}}\right)^{w_{5rs}} \tag{27}$$

Now, take the logarithm of Eq. (27) and equate the first partial derivatives of $lng(w_{3rs}^*, w_{4rs}^*, w_{5rs}^*)$ to zero, respectively to calculatew_{3rs}^* $w_{4rs}^* and w_{5rs}^*$ which maximize $g(w_{3rs}^*, w_{4rs}^*, w_{5rs}^*)$,we obtain:

$$f(w_{3rs}) = w_{3rs}^{4-\beta} + A_1 w_{3rs}^{3-\beta} - A_1A_2 w_{3rs}^{2-\beta} + A_3 w_{3rs}^2 + (\beta-1)A_1A_3 w_{3rs} - A_1A_2A_3 = 0 \tag{28}$$

$$f(w_{4rs}) = w_{4rs}^{4-\beta} + (1-\beta)w_{4rs}^{3-\beta} - (A_2+A_4)w_{4rs}^{2-\beta} + A_5 w_{4rs}^2 - A_5 w_{4rs} - A_5(A_2+A_4) = 0 \tag{29}$$

$$f(w_{5rs}) = w_{5rs}^{4-\beta} + A_6 w_{5rs}^{3-\beta} - A_4A_6 w_{5rs}^{2-\beta} + A_7 w_{5rs}^2 + (\beta-1)A_6A_7 w_{5rs} - A_4A_6A_7 = 0 \tag{30}$$

Where:

$$A_1 = \frac{C_{hr}k_3}{C_{hr}k_3 + 2k_1S} \quad ; \quad A_2 = \frac{C_{hr}^2\overline{D}_r^2 v}{2k_1k_2e^2} \quad ; \quad A_3 = \frac{2C_{ors}(C_{hr}\overline{D}_r)^{1-\beta}}{(2\gamma-1)(2k_1e)^{2-\beta}}$$

$$A_4 = \frac{C_{hr}\overline{D}_r^2 S v}{k_2k_3e^2} \quad ; \quad A_5 = \frac{(2\gamma-1)(C_{hr}\overline{D}_r)^{3-\beta}v^{2-\beta}}{2C_{ors}(k_2e)^{2-\beta}}$$

$$A_6 = \frac{2k_1S}{C_{hr}k_3 + 2k_1S} \quad ; \quad A_7 = \frac{2C_{ors}\overline{D}_r^{1-\beta}S^{2-\beta}}{(2\gamma-1)C_{hr}(k_3e)^{2-\beta}}$$

It could easily prove that $f_j(0) < 0$ *and* $f_j(1) > 0, \quad \forall j = 3,4,5$ this means that are three roots $w_{jrs} \in (0,1) \quad \forall j = 3,4,5$. Any method such as the trial and error method could be used to calculate this root .We can verify that any root $w_{3rs}^*, w_{4rs}^* and w_{5rs}^*$ calculated from Eqs. (28)–(30) maximize $(gw_{3rs}^*, w_{4rs}^*, w_{5rs}^*)$ respectively. This is done by the second derivative that verify Hessian matrix always negative as follows:

$$\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{irs}^2} = -\left[\frac{1}{(2-\beta)^2 w_{1rs}} + \frac{1}{(2-\beta)^2 w_{2rs}} + \frac{1}{w_{irs}}\right] < 0 \quad \forall\ i = 3,4,5$$

$$\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{irs}\partial w_{jrs}} = -\left[\frac{1}{(2-\beta)^2 w_{1rs}} + \frac{1}{(2-\beta)^2 w_{2rs}}\right] < 0 \quad i{\neq}j;\ i,j = 3,5$$

$$\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{4rs}\partial w_{jrs}} = \left[\frac{1}{(2-\beta)^2 w_{1rs}} + \frac{1}{(2-\beta)^2 w_{2rs}}\right] > 0 \quad \forall\ j = 3,5$$

Also, $$\left|\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{irs}\partial w_{jrs}}\right| < \left|\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial^2 w_{irs}}\right| \quad i{\neq}j;\ i,j = 3,4,5$$

$$\left(\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial w_{irs}\partial w_{jrs}}\right)^2 < \left(\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial^2 w_{irs}}\right)\left(\frac{\partial^2 \ln g(w_{3rs}, w_{4rs}, w_{5rs})}{\partial^2 w_{jrs}}\right) \quad i{\neq}j; i,j = 3,4,5$$

The Hessian matrix:

$$\Delta = -\left[\frac{1}{(2-\beta)^2}\left(\frac{1}{w_{1rs}w_{3rs}w_{4rs}} + \frac{1}{w_{2rs}w_{3rs}w_{4rs}} + \frac{1}{w_{1rs}w_{3rs}w_{5rs}} + \frac{1}{w_{2rs}w_{3rs}w_{5rs}} + \frac{1}{w_{1rs}w_{4rs}w_{5rs}} + \frac{1}{w_{2rs}w_{4rs}w_{5rs}}\right) + \frac{1}{w_{3rs}w_{4rs}w_{5rs}}\right] < 0$$

thus the roots $w_{3rs}^*, w_{4rs}^* and w_{5rs}^*$ calculated from Eqs. (28)–(30) maximize the dual function $(gw_{3rs}^*, w_{4rs}^*, w_{5rs}^*)$ and the optimal solution is $w_{jrs} \in (0,1) \forall j = 3,4,5$ where $w_{3rs}^*, w_{4rs}^* and w_{5rs}^*$ are obtained from Eqs. (28)–(30) respectively.

By using the relations for Duffin and Peterson's theorem (Duffin et al., 1967) of geometric programming to find the optimal number of periods per cycle N_{rs}^* we get:

Table 1
Input data.

Item		C_{ors}	C_{prs}	C_{hr}	$\overline{D}_r$
1	Source 1	50	20.12	0.27	3
	Source 2	65.35	30.10		
	Source 3	—	—		
2	Source 1	85	16.30	0.17	6
	Source 2	—	—		
	Source 3	40.12	18.50		
3	Source 1	15	14.20	0.24	4
	Source 2	15.35	11.08		
	Source 3	19.30	9.2		

The blank cells denote that the item is not available from the source indicated.

Table 2
Model I for different values of β with varying order cost when $\nu = 11;\ \gamma = 1.3$.

β	Source	Item 1			Item 2			Item 3		
		N^*_{rs}	Q^*_{mrs}	min E(TC)	N^*_{rs}	Q^*_{mrs}	min E(TC)	N^*_{rs}	Q^*_{mrs}	min E(TC)
0.1	1	2.349	77.529	103.521	2.669	176.122	161.518	1.152	50.684	81.619
	2	2.695	88.928	139.997	–	–	–	1.166	51.291	69.442
	3	–	–	–	1.830	120.787	153.888	1.312	57.723	65.14
0.2	1	2.306	76.105	105.599	2.635	173.888	165.177	1.088	47.88	81.788
	2	2.665	87.932	142.798	–	–	–	1.102	48.485	69.631
	3	–	–	–	1.772	116.966	155.364	1.248	54.915	65.545
0.3	1	2.238	73.863	107.843	2.573	169.842	169.221	1.012	44.529	81.876
	2	2.607	86.033	145.888	–	–	–	1.026	45.124	69.739
	3	–	–	–	1.694	111.821	156.884	1.170	51.473	65.885
0.4	1	2.137	70.514	110.237	2.474	163.299	173.657	0.921	40.532	81.843
	2	2.511	82.876	149.269	–	–	–	0.938	41.284	69.726
	3	–	–	–	1.591	104.979	158.404	1.074	47.265	66.117
0.5	1	1.990	65.677	112.729	2.323	153.327	178.448	0.813	45.788	81.631
	2	2.363	77.977	152.91	–	–	–	0.826	36.327	69.536
	3	–	–	–	1.454	95.972	159.844	0.958	42.143	66.175
0.6	1	1.783	58.852	115.199	2.101	138.68	183.462	0.686	30.203	81.156
	2	2.142	70.678	156.703	–	–	–	0.697	30.690	69.080
	3	–	–	–	1.276	84.237	161.059	0.817	35.959	65.961
0.7	1	1.497	49.417	117.392	1.784	117.754	188.356	0.539	23.735	80.289
	2	1.822	60.122	160.382	–	–	–	0.549	24.143	68.229
	3	–	–	–	1.048	69.154	161.784	0.650	28.597	65.315
0.8	1	1.112	36.710	118.767	1.344	88.709	192.314	0.375	16.485	78.831
	2	1.373	45.310	163.131	–	–	–	0.381	16.785	66.778
	3	–	–	–	0.761	50.229	161.523	0.457	20.089	63.979
0.9	1	0.6196	20.448	118.081	0.7632	50.368	193.302	0.204	8.992	76.441
	2	0.774	25.549	163.929	–	–	–	0.208	9.155	64.377
	3	–	–	–	0.422	27.833	159.255	0.249	10.958	61.489

$$N^*_{rs} = \left(\frac{(2\gamma-1)C_{hr}\overline{D}_r(1+w_{3rs}-w_{4rs}+w_{5rs})}{2C_{ors}(1-\beta-w_{3rs}+w_{4rs}-w_{5rs})}\right)^{\frac{1}{\beta-2}} \quad (31)$$

Then, the maximum inventory level Q^*_{mrs} is given by:

$$Q^*_{mrs} = \overline{D}_r\left[\nu+\gamma\left(\frac{(2\gamma-1)C_{hr}\overline{D}_r(1+w_{3rs}-w_{4rs}+w_{5rs})}{2C_{ors}(1-\beta-w_{3rs}+w_{4rs}-w_{5rs})}\right)^{\frac{1}{\beta-2}}\right] \quad (32)$$

Substituting the value of N^*_{rs} from Eq. (31) into Eq. (25) after adding the constant term to get the minimum expected total cost as follows:

$$\min E(TC) = \sum_r^n \left[C_{prs}\overline{D}_r + C_{ors}\left(\frac{(2\gamma-1)C_{hr}\overline{D}_r(1+w_{3rs}-w_{4rs}+w_{5rs})}{2C_{ors}(1-\beta-w_{3rs}+w_{4rs}-w_{5rs})}\right)^{\frac{\beta-1}{\beta-2}} + \frac{C_{hr}\overline{D}_r(2\gamma-1)}{2}\left(\frac{(2\gamma-1)C_{hr}\overline{D}_r(1+w_{3rs}-w_{4rs}+w_{5rs})}{2C_{ors}(1-\beta-w_{3rs}+w_{4rs}-w_{5rs})}\right)^{\frac{1}{\beta-2}} + C_{hr}\overline{D}_r\nu\right] \quad (33)$$

5. Special cases

5.1. Case 1

Substituting from $\nu = 0; \gamma > \frac{1}{2}$ in Eqs. (31)–(33) respectively we get the optimal policy variables for the model I

Table 3
Model II for different values of β with varying order cost when $\nu = 11;\ \gamma = 1.3$.

β	Source	Item 1			Item 2			Item 3		
		N^*_{rs}	Q^*_{mrs}	min E(TC)	N^*_{rs}	Q^*_{mrs}	min E(TC)	N^*_{rs}	Q^*_{mrs}	E min(TC)
0.1	1	10.724	65.172	79.865	12.559	141.354	124.645	5.204	64.816	73.454
	2	12.347	70.04	113.382	–	–	–	5.267	65.068	61.048
	3	–	–	–	8.460	116.756	132.745	5.942	67.768	54.318
0.2	1	11.366	67.097	81.457	13.428	146.568	126.19	5.298	65.192	73.855
	2	13.189	72.566	113.21	–	–	–	5.366	65.460	61.459
	3	–	–	–	8.848	119.088	133.745	6.094	68.376	55.122
0.3	1	11.984	68.951	83.550	14.298	151.788	129.473	5.341	65.364	74.903
	2	14.029	75.086	115.925	–	–	–	5.414	65.656	62.526
	3	–	–	–	9.193	121.157	135.399	6.195	68.780	56.108
0.4	1	12.502	70.505	86.257	15.081	156.480	133.323	5.297	65.188	75.891
	2	14.779	77.336	119.291	–	–	–	5.374	65.497	63.534
	3	–	–	–	9.433	122.598	138.36	6.201	68.804	57.346
0.5	1	12.771	71.312	89.937	15.599	159.594	140.809	5.110	64.440	77.161
	2	15.266	78.797	123.915	–	–	–	5.189	64.756	64.833
	3	–	–	–	9.456	122.739	140.089	6.045	68.180	58.955
0.6	1	12.508	70.524	95.324	15.499	158.994	145.323	4.688	62.752	78.934
	2	15.145	78.435	130.755	–	–	–	4.766	63.064	66.646
	3	–	–	–	9.065	120.389	143.455	5.613	66.452	61.217
0.7	1	11.189	66.566	98.030	14.094	150.564	154.641	3.888	59.552	79.207
	2	137.748	74.243	144.09	–	–	–	3.958	59.832	66.939
	3	–	–	–	7.911	113.466	147.827	4.721	62.884	61.742
0.8	1	7.830	56.489	105.571	10.055	126.330	167.721	2.492	53.968	81.053
	2	9.787	62.361	152.532	–	–	–	2.540	54.160	68.838
	3	–	–	–	5.378	98.268	153.620	3.074	56.296	64.253
0.9	1	0.575	34.724	122.348	0.756	70.530	196.062	0.165	44.660	85.727
	2	0.734	35.202	167.119		–	–	0.168	44.672	73.638
	3	–	–	–	0.382	68.292	167.015	0.207	44.828	70.410

Table 4
Model III for different values of β with varying order cost when $\nu = 11; \gamma = 1.3$.

β	Source	Item 1			Item 2			Item 3		
		N^*_{rs}	Q^*_{mrs}	min E(TC)	N^*_{rs}	Q^*_{mrs}	min E(TC)	N^*_{rs}	Q^*_{mrs}	min E(TC)
0.1	1	6.215	69.358	86.227	7.278	151.157	134.027	3.016	67.523	77.112
	2	7.155	74.859	118.733	–	–	–	3.053	67.811	64.751
	3	–	–	–	4.903	123.36	139.064	3.444	70.860	58.496
0.2	1	6.398	70.429	88.112	7.559	154.439	137.052	2.982	67.263	77.770
	2	7.424	76.432	121.074	–	–	–	3.021	67.563	65.424
	3	–	–	–	4.981	124.278	140.692	3.431	70.760	59.334
0.3	1	6.533	71.221	90.383	7.795	157.201	140.739	2.912	66.712	78.512
	2	7.648	77.740	123.924	–	–	–	2.952	67.023	66.184
	3	–	–	–	5.012	124.641	142.615	3.378	70.342	60.294
0.4	1	6.579	71.488	93.143	7.937	158.857	145.285	2.788	65.745	79.349
	2	7.778	78.499	127.432	–	–	–	2.828	66.060	67.043
	3	–	–	–	4.964	124.08	144.903	3.263	69.455	61.395
0.5	1	6.465	70.823	96.528	7.897	158.40	150.014	2.587	64.180	80.287
	2	7.729	78.214	131.794	–	–	–	2.627	64.492	68.007
	3	–	–	–	4.788	122.014	147.637	3.061	67.872	62.652
0.6	1	6.076	68.547	100.701	7.529	154.091	158.063	2.278	61.765	81.322
	2	7.357	76.039	137.266	–	–	–	2.315	62.060	69.074
	3	–	–	–	4.404	117.527	150.906	2.727	65.269	64.077
0.7	1	5.221	63.542	105.856	6.577	142.945	167.055	1.815	58.153	82.431
	2	6.415	70.527	144.163	–	–	–	1.847	58.407	70.220
	3	–	–	–	3.691	109.189	154.794	2.203	61.182	65.655
0.8	1	3.591	54.009	112.207	4.612	119.956	178.451	1.143	52.915	83.556
	2	4.489	59.260	152.879	–	–	–	1.165	53.088	71.390
	3	–	–	–	2.467	94.862	159.36	1.410	54.999	67.341
0.9	1	0.726	37.249	121.748	0.954	77.164	195.831	0.208	45.625	85.198
	2	0.926	38.420	166.149	–	–	–	0.213	45.659	73.096
	3	–	–	–	0.482	71.641	166.089	0.262	46.043	69.791

$$N_{rs}^{*} = \left(\frac{(2\gamma-1)C_{hr}\overline{D}_r(1+w_{3rs}-w_{4rs}+w_{5rs})}{2C_{ors}(1-\beta-w_{3rs}+w_{4rs}-w_{5rs})}\right)^{\frac{1}{\beta-2}}$$

$$Q_{mrs}^{*} = \gamma\,\overline{D}_r\left(\frac{(2\gamma-1)C_{hr}\overline{D}_r(1+w_{3rs}-w_{4rs}+w_{5rs})}{2C_{ors}(1-\beta-w_{3rs}+w_{4rs}-w_{5rs})}\right)^{\frac{1}{\beta-2}}$$

$$\min\,E(TC) = \sum_{r}^{n}\left[C_{prs}\overline{D}_r + C_{ors}\left(\frac{C_{hr}\overline{D}_r(2\gamma-1)(1+w_{3rs}^{*}-w_{4rs}^{*}+w_{5rs}^{*})}{2C_{ors}(1-\beta-w_{3rs}^{*}+w_{4rs}^{*}-w_{5rs}^{*})}\right)^{\frac{\beta-1}{\beta-2}} + \frac{C_{hr}\overline{D}_r(2\gamma-1)}{2}\left(\frac{C_{hr}\overline{D}_r(2\gamma-1)(1+w_{3rs}^{*}-w_{4rs}^{*}+w_{5rs}^{*})}{2C_{ors}(1-\beta-w_{3rs}^{*}+w_{4rs}^{*}-w_{5rs}^{*})}\right)^{\frac{1}{\beta-2}}\right]$$

5.2. Case 2

Substituting from $\gamma = 1$ in Eqs. (31)–(33) respectively we get:

$$N_{rs}^{*} = \left(\frac{C_{hr}\overline{D}_r(1+w_{3rs}-w_{4rs}+w_{5rs})}{2C_{ors}(1-\beta-w_{3rs}+w_{4rs}-w_{5rs})}\right)^{\frac{1}{\beta-2}} ;\ Q_{mrs}^{*} = \overline{D}_r\left[v + \left(\frac{C_{hr}\overline{D}_r(1+w_{3rs}-w_{4rs}+w_{5rs})}{2C_{ors}(1-\beta-w_{3rs}+w_{4rs}-w_{5rs})}\right)^{\frac{1}{\beta-2}}\right]$$

$$E(TC) = \sum_{r=1}^{n}\left[C_{prs}\overline{D}_r + C_{ors}\left(\frac{C_{hr}\overline{D}_r(1+w_{3rs}-w_{4rs}+w_{5rs})}{2C_{ors}(1-\beta-w_{3rs}+w_{4rs}-w_{5rs})}\right)^{\frac{\beta-1}{\beta-2}} + \frac{C_{hr}\overline{D}_r(\beta-1)}{2}\left(\frac{C_{hr}\overline{D}_r(1+w_{3rs}-w_{4rs}+w_{5rs})}{2C_{ors}(1-\beta-w_{3rs}+w_{4rs}-w_{5rs})}\right)^{\frac{1}{\beta-2}} + C_{hr}\overline{D}_r v\right]$$

These are the optimal policy variables for the model II.

6. Numerical application and analysis

A manager of probabilistic restricted MIMS inventory system considers the consequence of the minimum procurement cost and inventory policy of system involving three items and three vendors (sources). Source 1 and source 2 are manufacturing or remanufacturing alternatives while sources 3 are either vendors or intra firm transfer possibilities. The item dependent parameter of demand of the item is holding cost

parameters that depend upon the item as well as the source, C_{prs}, and also, the demand has uniform distribution with expected value for each item are given in Table 1. Addition parameters values needed are:

$K_1 = 3200$; $K_2 = 1100$; $K_3 = 5000$;
$S = 60\ m^2$; $\gamma = 1.3$; $v = 11$

By applying Eqs. (11)–(13) for the model I, Eqs. (21)–(23) for the model II and Eqs. (31)–(33) for the model III to each item and source, the minimum expected total cost as given in the Tables 2–4.

We can determine the optimal policy variables of the minimum total cost as Table 5:

For the model I, the number of periods and the maximum inventory level for each item each source are decreasing whenever $\boldsymbol{\beta}$ increased as Table 2, but the expected order cost is increasing

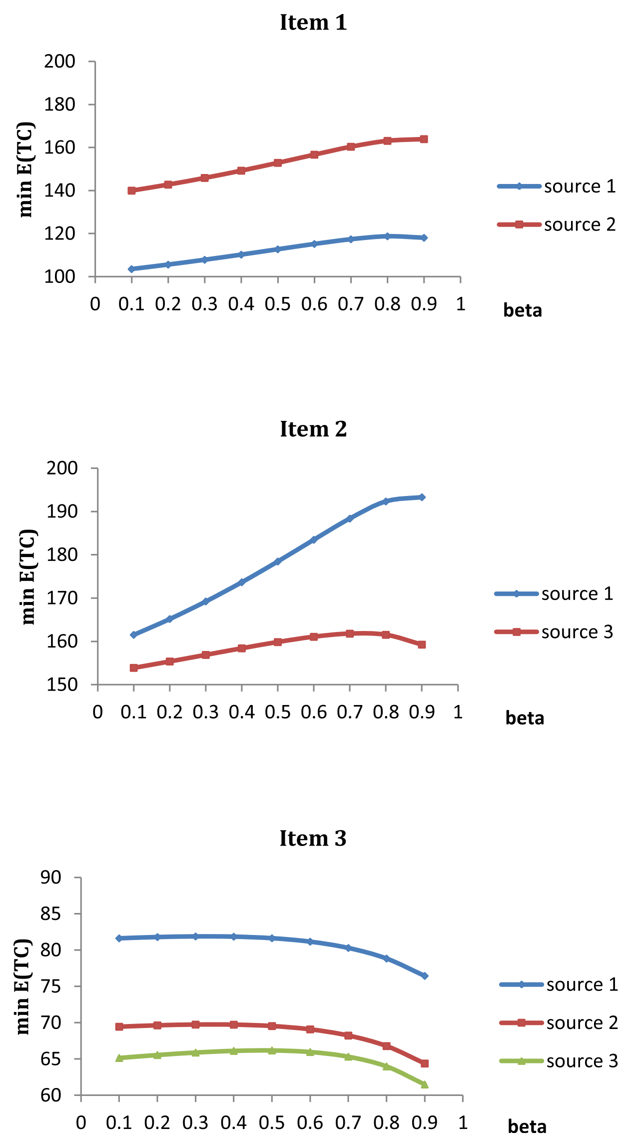

Fig. 1. Expected total cost for Model I at different values of β.

Table 5
The optimal policy variables of the three models.

		N_{rs}^{*}	Q_{mrs}^{*}	$\min\,E(TC)$	Item	source
Model I	$\beta = 0.1$	2.349	77.529	103.521	1	1
		1.830	120.787	153.888	2	3
		1.312	57.723	65.14	3	3
Model II	$\beta = 0.1$	10.724	65.172	79.865	1	1
		12.559	141.354	124.645	2	1
		5.942	67.768	54.318	3	3
Model III	$\beta = 0.1$	6.215	69.358	86.227	1	1
		7.278	151.157	134.027	2	1
		3.444	70.860	58.496	3	3

whenever β increased. The minimum expected total cost for item 1 is increasing whenever the value of the $\boldsymbol{\beta}$ increased for each source. The minimum expected total cost for item 2 source 1 is increasing whenever the value of the increased and the minimum expected total cost for item 2 source 2 is increasing in $\boldsymbol{\beta} \in [0.1, 0.7]$ but it is decreasing in $\boldsymbol{\beta} \in [0.8, 0.9]$ .The expected total cost for item 3 is decreasing whenever the value of the $\boldsymbol{\beta}$ increased for each source as show Fig. 1 and Table 2. In general, for the model I, the number of periods and the maximum inventory level are decreasing whenever $\boldsymbol{\beta}$ increased but the expected order cost and the optimal minimum expected total cost for the three items are increasing whenever the value of the $\boldsymbol{\beta}$ increased, also, we deduced the optimal policy variables of the model I when $\boldsymbol{\beta} = 0.1$.

For the model II, the minimum expected total cost for each items each source is increasing whenever the value of the $\boldsymbol{\beta}$ increased as show Fig. 2 and Table 3.

The expected number of periods and the maximum inventory level for item1 and item 2 are increasing in $\beta \in [0.1, 0.5]$ but they are decreasing in $\beta \in [0.6, 0.9]$ for each source. For item 3 (source1 –source2) the expected number of periods and the maximum inventory level are increasing in $\beta \in [0.1, 0.3]$ but they are decreasing in $\in [0.4, 0.9]$, also for item 3 (source3) the expected number of periods and the maximum inventory level are increasing in $\beta \in [0.1, 0.4]$ but they are decreasing in $\beta \in [0.5, 0.9]$. In general, for the model II, β the expected number of periods and the maximum inventory level are varying whenever β increased. The expected order cost and the minimum expected total cost are increasing whenever β increased for each item each source, we obtained the optimal policy variables of the model II when $\boldsymbol{\beta} = 0.1$.

For the model III, the expected number of periods and the maximum inventory level for item 1 are increasing β in$\beta \in [0.1, 0.4]$ but they are decreasing in $\beta \in [0.5, 0.9]$ for each source as Table 5. For item2 (source 1), the expected number of periods and the maximum inventory level are increasing in $\beta \in [0.1, 0.4]$ but they are decreasing in $\beta \in [0.5, 0.9]$, also, they are increasing in $\beta \in [0.1, 0.3]$ but they are decreasing in $\beta \in [0.4, 0.9]$ for source3 as Table 5. For item 3 each source, the expected number of periods and the maximum inventory level are decreasing whenever the value of the increased as Table 5. The minimum expected total cost for each items each source is increasing whenever the value of the β increased as show in Fig. 3 and Table 4, also, the optimal expected total cost for each items obtained when $\beta = 0.1$. In general, for the model III, the expected number of periods and the maximum inventory level are varying whenever β increased

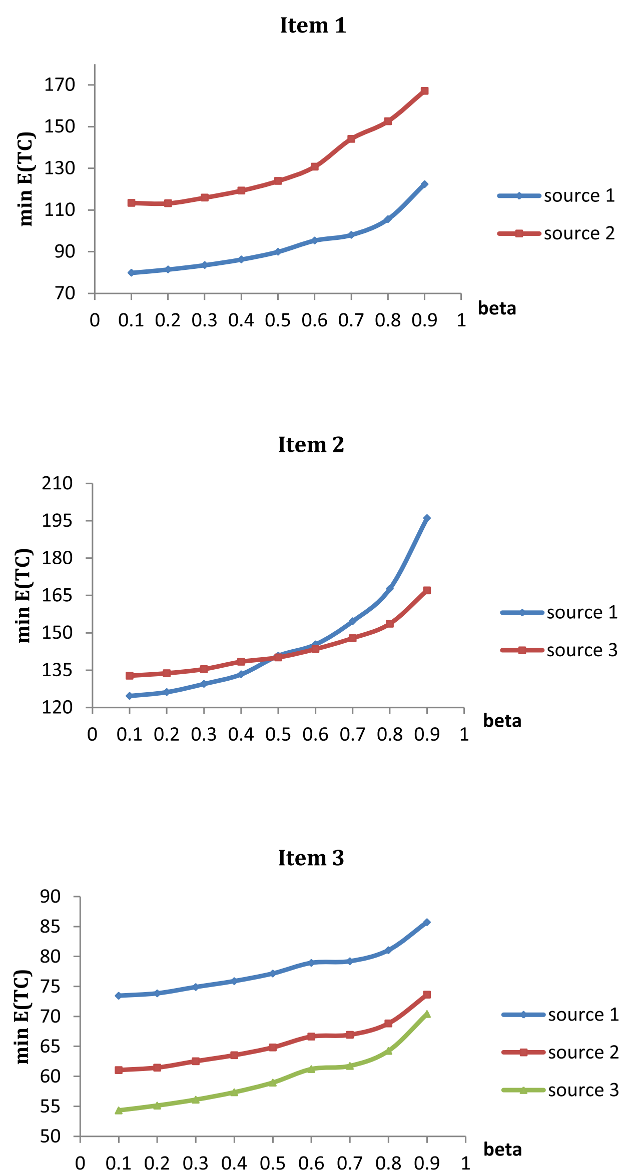

Fig. 2. Expected total cost for Model II at different values of β.

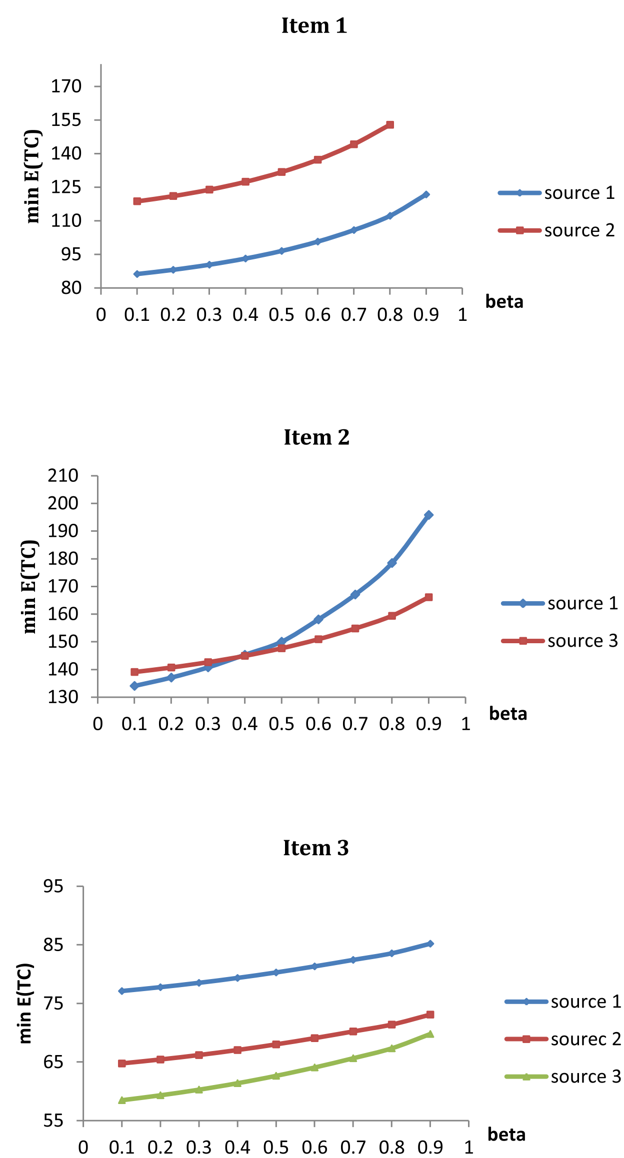

Fig. 3. Expected total cost for Model III at different values of β.

but the expected order cost and the minimum expected total cost are γ increasing whenever the value of the increased for each item each source. Also the expected number of periods is decreasing whenever γ increased but the maximum inventory level, the expected order cost and the minimum expected total cost are increasing whenever increased. We deduced the optimal policy variables of the III model III when $\boldsymbol{\beta} = 0.1$.

Finally, the minimum expected total cost of the model is equal to the summation of the minimum expected total cost for each item. Thus, $\mathbf{Min}\,\boldsymbol{E(TC)_I} = \mathbf{322.549}$, $\mathbf{Min}\,\boldsymbol{E(TC)_{II}} = \mathbf{258.828}$ and $\mathbf{Min}\,\boldsymbol{E(TC)_{III}} = \mathbf{278.75}$. From these results we can say that the model (II) is the best model of the three models because it has the optimal minimum expected total cost for the three items.

Furthermore, we can compare the optimal results of the three probabilistic MIMS inventory models without varying order cost(denoted crisp models) and the three probabilistic MIMS inventory models with varying order cost. For the crisp models: $\text{Min}\,E(TC)_{Crisp\ model\ I} = 319.784$, $\text{Min}\,E(TC)_{Crisp\ model\ II} = 261.432$ and $\text{Min}\,E(TC)_{Crisp\ model\ III} = 261.534$. One can deduce that the optimal expected total cost of the crisp model I and crisp model are better than models I and III with varying order cost, but the optimal expected total cost of the model II with varying order cost is better than the crisp model II. Now, we can conclude that the variation on the order cost increases the minimum expected total cost for the model I and model III but it reduces the minimum expected total cost for the model II. Also, the increasing values of the parameters ν and γ lead to increase the expected total cost of the three probabilistic MIMS inventory models.

7. Conclusion

In this paper we assumed three probabilistic multi-item multi-vender inventory models with varying order cost and zero lead-time under linear and nonlinear constraints for the number of periods N_{rs}, the first linear constraint on the holding cost, the second nonlinear constraint on the buffer stock and the third linear constraint on the storage space. Our objective is determining the minimum expected total cost. Using geometric programming approach (GPP), the exact solution of the optimal number of period N^*_{rs} and the optimal maximum inventory level Q^*_{mrs} are obtained for the three models. Next, we deduced some special simple SISS inventory models had been discussed by Fabrycky and Banks (1967).

For model III, we solved it and we figured out that it is a generalization of the models I and model II, and they can be special cases from model III. Finally, applying a numerical application to the three models, comparisons, analysis are done and as a result the system manager can use model II to obtain the minimization of the expected total cost for the given data of the items and vendors. We tend to use new methods like fuzzy numbers, SWARM and etc., to discover and decide which the best model of them is.

Acknowledgments

This research project was supported by a grant from the "Research Center of the Female Scientific and Medical Colleges", Deanship of Scientific Research, King Saud University.

Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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