Journal of King Saud University – Science 29 (2017) 401–412

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Review

A four-step implicit block method with three generalized off-step points for solving fourth order initial value problems directly

Ra'ft Abdelrahim [a,*], Zurni Omar [b]

[a] Department of Mathematics, College of Art and Sciences at Tabarjal, Al Jouf University, Saudi Arabia
[b] Department of mathematics, College of Art and Sciences, Univeristi Utara Malaysia, Malaysia

ARTICLE INFO

Article history:
Received 24 January 2017
Accepted 14 June 2017
Available online 16 June 2017

Keywords:
Block method
Hybrid block method
Fourth order ordinary differential equations
Collocation and interpolation
Three off-step points

ABSTRACT

The existing hybrid methods for solving ordinary differential equations were only derived using specific off-step points. Thus, this paper proposes a new four-step block method with three generalized off-step points for solving initial value problems of fourth order ordinary differential equations directly. The strategy employed to develop this method is interpolating the basis function at $y_{n+j}, j = 0(1)3$ and collocating the fourth derivative of the basis function at all points within the selected interval. The implementation of this method in a block-by-block fashion can overcome the setbacks of applying starting values and predictors which are created in predictor-corrector approach. The convergence analysis of the developed method is performed and the accuracy of the method is tested on several problems. The numerical results indicate that the new method outperforms the existing ones in terms of errors. In addition, the new method does not require much computation when compared with previous methods.

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1. Introduction

In this paper, we consider the numerical solution of the fourth order IVPs of the form

$$y'''' = f(x, y, y', y'', y'''), \quad x \in [a, b] \tag{1}$$

with initial conditions

$$y(a) = \omega_0, y'(a) = \omega_1, y''(a) = \omega_2, y'''(a) = \omega_3$$

Eq. (1) arises in wide fields of science and engineering. This equation can be solved by converting it into system of four

* Corresponding author.
E-mail addresses: rafatshaab@yahoo.com (R. Abdelrahim), zurni@uum.edu.my (Z. Omar).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2017.06.003
1018-3647/

equations of first-order IVPs and then suitable numerical methods for first order is employed (see Henrici, 1962; Lambert, 1973). However, this approach increases the number of equations and therefore some researchers have opted hybrid block methods for solving Eq. (1) directly, for instance, among of these researchers are (Yap and Ismail, 2015; Abdelrahim and Omar, 2015; Hussain et al., 2015; Omar and Abdelrahim, 2016; Jator, 2008; Awoyemi, 2005). In hybrid block methods, not only the numerical approximation can be computed at more than one point in the same time, but computational burden and zero stability barrier can also be avoided, see Adesanya et al. (2012) and Kayode (2008). However, the existing hybrid block methods only focus on the specific off step points. Hence, the aim of this work is to generalize three off step points of four step block method. As a results, the new method will be more robust and flexible.

2. Derivation of the method

This section shows the derivation of a four-step hybrid block method with three generalized off-step points x_{n+s_1}, x_{n+s_2} and x_{n+s_3} for solving (1).

Consider the polynomial of the form:

$$y(x) = \sum_{i=0}^{q+d-1} a_i \left(\frac{x - x_n}{h}\right)^i. \qquad (2)$$

as an approximate solution of (1), where $x \in [x_n, x_{n+4}]$ for $n = 0, 4, 8, \ldots, N-4$, q is the number of interpolation points which is equal to the order of differential equation, $d$is the number of collocation points and $h = x_n - x_{n-1}$ is constant step size of partition of interval $[a, b]$ which is given by $a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b$.

The next step in the derivation of the hybrid block method involves differentiating (2) four times to give

$$y''''(x) = f(x, y, y', y'', y''')$$
$$= \sum_{i=4}^{q+d-1} \frac{i(i-1)(i-2)(i-3)}{h^4} a_i \left(\frac{x - x_n}{h}\right)^{i-4}. \qquad (3)$$

Interpolating the approximate solution (2) at $x_{n+j}, j = 0(1)3$and collocating Eq. (3) at all points in the selected interval produces twelve equations which can be written as a system in matrix form $AX = B$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 & 512 & 1024 & 2048 \\ 1 & 3 & 9 & 27 & 81 & 243 & 729 & 2187 & 6561 & 19683 & 59049 & 177147 \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{(120s_1)}{h^4} & \frac{(360s_1^2)}{h^4} & \frac{(840s_1^3)}{h^4} & \frac{(1680s_1^4)}{h^4} & \frac{(3024s_1^5)}{h^4} & \frac{(5040s_1^6)}{h^4} & \frac{(7920s_1^7)}{h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120}{h^4} & \frac{360}{h^4} & \frac{840}{h^4} & \frac{1680}{h^4} & \frac{3024}{h^4} & \frac{5040}{h^4} & \frac{7920}{h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{(120s_2)}{h^4} & \frac{(360s_2^2)}{h^4} & \frac{(840s_2^3)}{h^4} & \frac{(1680s_2^4)}{h^4} & \frac{(3024s_2^5)}{h^4} & \frac{(5040s_2^6)}{h^4} & \frac{(7920s_2^7)}{h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{240}{h^4} & \frac{1440}{h^4} & \frac{6720}{h^4} & \frac{26880}{h^4} & \frac{96768}{h^4} & \frac{322560}{h^4} & \frac{1013760}{h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{(120s_3)}{h^4} & \frac{(360s_3^2)}{h^4} & \frac{(840s_3^3)}{h^4} & \frac{(1680s_3^4)}{h^4} & \frac{(3024s_3^5)}{h^4} & \frac{(5040s_3^6)}{h^4} & \frac{(7920s_3^7)}{h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{360}{h^4} & \frac{3240}{h^4} & \frac{22680}{h^4} & \frac{136080}{h^4} & \frac{734832}{h^4} & \frac{3674160}{h^4} & \frac{17321040}{h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{480}{h^4} & \frac{5760}{h^4} & \frac{53760}{h^4} & \frac{430080}{h^4} & \frac{3096576}{h^4} & \frac{20643840}{h^4} & \frac{129761280}{h^4} \end{pmatrix},$$

$$X = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}]^T \quad \text{and}$$

$$B = \left[y_n, y_{n+1}, y_{n+2}, y_{n+3}, f_n, f_{n+s_1}, f_{n+1}, f_{n+s_2}, f_{n+2}, f_{n+s_3}, f_{n+3}, f_{n+4}\right]^T.$$

The unknown values of $a_i's, i = 0(1)11$ can be obtained by matrix inverse approach, where $X = A^{-1}B$. The values obtained are then substituted back into Eq. (2) to produce a continuous implicit scheme of the form

$$y(x) = \sum_{i=0}^{3} \alpha_i(x) y_{n+i} + \sum_{i=0}^{4} \beta_i(x) f_{n+i} + \sum_{i=1}^{3} \beta_{s_i}(x) f_{n+s_i} \qquad (4)$$

Differentiating (4) once, twice and thrice gives

$$y^{(j)}(x) = \sum_{i=0}^{3} \alpha_i^{(j)}(x) y_{n+i} + \sum_{i=0}^{4} \beta_i^{(j)}(x) f_{n+i} + \sum_{i=1}^{3} \beta_{s_i}^{(j)}(x) f_{n+s_i};\ j = 1(1)3. \qquad (5)$$

Eq. (4) is evaluated at the non-interpolating point $x_{n+s_i},\ x_{n+4}, i = 1(1)3$ while Eqs. (5) is evaluated at x_n to give the discrete schemes and its derivatives at x_n. The discrete schemes and its derivatives at x_n are combined in a matrix form as below

$$AY_M^{[3]_4} = B^{[3]_4} R_1^{[3]_4} + h^4 [D^{[3]_4} R_2^{[3]_4} + E^{[3]_4} R_3^{[3]_4}] \qquad (6)$$

where,

$$A = \begin{pmatrix} 1 & \frac{-(s1(s1-2)(s1-3))}{2} & 0 & \frac{(s1(s1-1)(s1-3))}{2} & 0 & \frac{-(s1(s1-1)(s1-2))}{6} & 0 \\ 0 & \frac{-(s2(s2-2)(s2-3))}{2} & 1 & \frac{(s2(s2-1)(s2-3))}{2} & 0 & \frac{-(s2(s2-1)(s2-2))}{6} & 0 \\ 0 & \frac{-(s3(s3-2)(s3-3))}{2} & 0 & \frac{(s3(s3-1)(s3-3))}{2} & 1 & \frac{-(s3(s3-1)(s3-2))}{6} & 0 \\ 0 & -4 & 0 & 6 & 0 & -4 & 1 \\ 0 & \frac{-3}{h} & 0 & \frac{3}{(2h)} & 0 & \frac{-1}{(3h)} & 0 \\ 0 & \frac{5}{h^2} & 0 & \frac{-4}{h^2} & 0 & \frac{1}{h^2} & 0 \\ 0 & \frac{-3}{h^3} & 0 & \frac{3}{h^3} & 0 & \frac{-1}{h^3} & 0 \end{pmatrix},$$

$$Y_M^{[3]_4} = \begin{pmatrix} y_{n+s_1} \\ y_{n+1} \\ y_{n+s_2} \\ y_{n+2} \\ y_{n+s_3} \\ y_{n+3} \\ y_{n+4} \end{pmatrix},$$

$$B^{[3]_4} = \begin{pmatrix} \frac{-((s1-1)(s1-2)(s1-3))}{6} & 0 & 0 & 0 \\ \frac{-((s2-1)(s2-2)(s2-3))}{6} & 0 & 0 & 0 \\ \frac{-((s3-1)(s3-2)(s3-3))}{6} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \frac{-11}{(6h)} & -1 & 0 & 0 \\ \frac{2}{h^2} & 0 & -1 & 0 \\ \frac{-1}{h^3} & 0 & 0 & -1 \end{pmatrix}, \quad R_1^{[3]_4} = \begin{pmatrix} y_n \\ y'_n \\ y''_n \\ y'''_n \end{pmatrix}, \quad R_2^{[3]_4} = [f_n],$$

$$R_3^{[3]_4} = \begin{pmatrix} f_{n+s_1} \\ f_{n+1} \\ f_{n+s_2} \\ f_{n+2} \\ f_{n+s_3} \\ f_{n+3} \\ f_{n+4} \end{pmatrix}, \quad D^{[3]_4} = \begin{pmatrix} D_{11} \\ D_{21} \\ D_{31} \\ D_{41} \\ D_{51} \\ D_{61} \\ D_{71} \end{pmatrix}, \quad \text{and}$$

$$E^{[3]_4} = \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} & E_{17} \\ E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} & E_{27} \\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36} & E_{37} \\ E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & E_{46} & E_{47} \\ E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56} & E_{57} \\ E_{61} & E_{62} & E_{63} & E_{64} & E_{65} & E_{66} & E_{67} \\ E_{71} & E_{72} & E_{73} & E_{74} & E_{75} & E_{76} & E_{77} \end{pmatrix}.$$

The elements of $D^{[3]_4}$ and $E^{[3]_4}$ are given in A.

Multiplying Eq. (6) by the inverse of A gives a four-step hybrid block method (main block) with three generalized off-step points of the form

$$Y_M^{[3]_4} = A^{-1} B^{[3]_4} R_1^{[3]_4} + h^4 [A^{-1} D^{[3]_4} R_2^{[3]_4} + A^{-1} E^{[3]_4} R_3^{[3]_4}] \qquad (7)$$

which can be seen in Appendix B.

In order to get the derivatives (first, second and third) of the block, the value of the block at y_{n+1}, y_{n+2} and y_{n+3} are substituted into the first, second and third derivatives of the discrete schemes and this produces first, second and third derivatives of the block.

3. Analysis of the Method

3.1. Order of the Method

In finding order of the new block method (7), we define the linear difference operator L associated with (7)

$$L[y(x);h] = Y_m^{[3]_4} - A^{-1}B^{[3]_4}R_1^{[3]_4} - h^4\left[A^{-1}D^{[3]_4}R_2^{[3]_4} + A^{-1}E^{[3]_4}R_3^{[3]_4}\right] \quad (8)$$

Expanding $Y_m^{[3]_4}$ and $R_3^{[3]_4}$ components in Taylors series respectively and collecting their terms in powers of h yields

$$L[y(x),h] = \bar{C}_0 y(x) + \bar{C}_1 hy'(x) + \bar{C}_2 h^2 y''(x) + \cdots \quad (9)$$

Definition 3.1. The hybrid block method (7) and its linear operator (8) are said to have order p, if $\bar{C}_0 = \bar{C}_1 = \bar{C}_2 = \cdots = \bar{C}_{p+3} = 0$and $\bar{C}_{p+4} \neq 0$ with error constants vector $\bar{C}_{p+4}$.

Expanding (8) about $x = x_n$using Taylor series gives the orders of all integrators in the main block method which are $[8,8,8,8,8,8,8]^T$. Therefore, the new hybrid block method is consistent since each integrator has order greater than one.

3.2. Zero stability

Following Fatunla(1991), the zero stability of the method will obtain by finding the roots of the first characteristic function $\Pi(x)$ as given below, where I is 7×7 identity matrix and $\overline{B}^{[3]_4}$ is coefficients matrix of y_n,

$$\Pi(x) = |xI - \overline{B}^{[3]_4}|$$
$$= \left| x\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\end{pmatrix}\right|$$
$$= x^6(x-1)$$

Solving $\Pi(x) = 0$ yields $x = 0,0,0,0,0,0,1$. Hence, our method is zero stable since $\Pi(x)$having roots such that $|x_z| \leqslant 1$, and when $|x_z| = 1$, the multiplicity of x_z doesn't exceed four.

3.3. Convergence

Theorem 3.2 Henrici, 1962. *Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent.*

Following the theorem in the above the developed method (7) is convergent since it is zero stable and consistent.

3.4. Implementation of the method

The developed method is implemented by combining all the hybrid integrators for IVPs simultaneously without additional method to predict the starting values. We continue by explicitly obtaining initial conditions at x, $n = 0, 4, \ldots, N-4$ using the computed values $y(x_{n+4}) = y_{n+4}$ over subintervals $[x_0, x_4], \ldots, [x_{N-4}, x_N]$. For example, using (7), $n = 0,\ _M = 0,\ \left(y_{s_1}, y_1, y_{s_2}, y_2, y_{s_3}, y_3, y_4\right)$ where its first, second and third derivatives are simultaneously obtained over the subinterval $[x_0, x_4]$ as $y_0^{(j)}(x_0), j = 0(1)3$ are known from the IVP (1), for $n = 4,\ _M = 1,\ \left(y_{s_1+4}, y_5, y_{s_2+4}, y_6, y_7, y_{s_3+4}, y_8\right)$ with its derivatives are simultaneously obtained over the subinterval $[x_4, x_8]$ as $y_4^{(j)}(x_4),\ j = 0(1)3$ are known from the previous block and so on. Hence, the method is applied over non overlapping subintervals and the solutions derived are more accurate than those obtained in the existing method. We remind that, the system of these equations has a unique solution since the determinant of its matrix $\neq 0$ when $s_i \in (x_{n+(i-1)}, x_{n+i}),\ i = 1(1)3$. We also note that, for linear problems, we solve (7) directly from the start with Gaussian elimination using partial pivoting and for nonlinear problems, we use a modified Newton–Raphson method.

4. Numerical results

In this section, the following IVPs available in previous literatures which are solved to special case of the new method when $s_1 = \frac{1}{8}, s_2 = \frac{5}{3}$ and $s_3 = \frac{7}{2}$ (chosen arbitrarily) in order to compare the performance of new method with the existing ones in [3,4–9]. It is worth noting that the block method proposed is in a generalized form and hence can take varying off-step point values which gives room for flexibility. The absolute errors at different value of end point x with several h were carried out using flexible **Matlab code** as depicted in Tables 1,2,3,4.

The following notations are used in the tables.

h:step size.
Step: total number of steps taken to obtain solution.
$E(x_N)$: magnitude of the maximum error of the computed solution.
AE: absolute error.

Problem 1 : $y^{iv} = y''' + y'' + y' + 2y, \quad y(0) = 1,$
$y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 30,.$
Exact solution : $y(x) = 2e^{2x} - 5e^{-x} + 3\cos x - 9\sin x$

Problem 2 : $y^{iv} = -y'', \quad y(0) = 0, \quad y'(0) = \dfrac{-1.1}{72 - 50\pi},$
$y''(0) = \dfrac{1}{144 - 100\pi}, \quad y'''(0) = \dfrac{1.2}{144 - 100\pi}$
Exact solution : $y(x) = 1 - x - \cos x - \dfrac{1.2\sin x}{144 - 100\pi}$

In general, the results from Tables 1, show that the performance of four-step hybrid block methods for solving fourth order ODEs directly using three specific hybrids points is better than existing methods in terms of error. These are demonstrated in (Figs. 1 and 2).

Application:

The new developed method is applied for solving physical problem occurs in ship dynamics. In particular, this problem has been studied and solved numerically by Twizell (1988) and Cortell (1993) which describes how the sinusoidal wave of frequency Ω passes along a ship or offshore structure to lead to fourth order differential equation relates the fluids action with time x as below

Table 1
Comparison of the new method with some existing methods for solving problem 1.

h	Method	$E(x_N)$ at x = 2.	Steps
0.1	New method	$8.07e^{-10}$	5
	Yap and Ismail (2015)	$1.74e^{-8}$	5
	Adams	$2.11e^{-3}$	
	Jator	$1.26e^{-4}$	20
0.05	New method	$3.22e^{-12}$	10
	Yap and Ismail (2015)	$8.45e^{-11}$	10
	Adams	$5.37e^{-4}$	
	Jator	$1.91e^{-6}$	40
0.025	New method	$7.30e^{-16}$	20
	Yap and Ismail (2015)	$3.69e^{-13}$	20
	Adams	$5.09e^{-5}$	
	Jator	$2.96e^{-8}$	80

Table 2
Comparison of the new method with some existing methods for solving problem 2.

h	Method	$E(x_N)$ at x = 1.01325	Steps.
0.003125	New method	$1.65e^{-18}$	80
	Kayode (2008)	$1.58e^{-7}$	320
0.103125	New method	$2.95e^{-17}$	3
	Awoyemi et al. (2015)	$5.69e^{-6}$	2
h		$E(x_N)$ at x = 1	
0.1	New method	$1.45e^{-18}$	3
	Adesanya et al (2012)	$8.04e^{-16}$	2

Table 3
Comparison of the new method with Yap and Ismail (2015) for solving problem 1.

x	AE in new method	AE in Yap (2015).
0.2	3.512997×10^{-13}	2.318519×10^{-13}
0.4	4.183300×10^{-12}	2.260324×10^{-12}
0.6	1.430233×10^{-11}	1.965140×10^{-11}
0.8	3.592435×10^{-11}	9.914494×10^{-11}
1.0	7.276201×10^{-11}	3.311345×10^{-10}
1.2	1.336016×10^{-10}	9.000018×10^{-10}
1.4	2.234540×10^{-10}	2.117600×10^{-9}
1.6	3.579606×10^{-10}	4.550582×10^{-9}
1.8	5.433495×10^{-10}	9.117964×10^{-9}
2.0	8.079574×10^{-10}	1.740907×10^{-8}

Table 4
Comparison of the new method with Awoyemi et al. (2015) and Kayode (2008) for solving problem 2.

x	AE in Kayode (2008)	AE in Awoyemi et al. (2015)	AE in new method.
0.103125	$0.48355417 \times 10^{-16}$	2.11164×10^{-13}	2.269123×10^{-19}
0.206250	$0.13933299 \times 10^{-15}$	5.69866×10^{-12}	4.554620×10^{-19}
0.306250	$0.66893539 \times 10^{-15}$	6.80311×10^{-10}	3.891211×10^{-19}
0.406250	$0.20129384 \times 10^{-14}$	2.20723×10^{-9}	6.038588×10^{-19}
0.506250	$0.46736053 \times 10^{-14}$	1.27407×10^{-8}	9.727741×10^{-19}
0.603125	$0.91874598 \times 10^{-14}$	3.45612×10^{-6}	8.900363×10^{-19}
0.703125	$0.16069038 \times 10^{-13}$	6.55238×10^{-6}	8.843511×10^{-19}
0.803125	$0.25407974 \times 10^{-13}$	9.58653×10^{-6}	6.690965×10^{-19}
0.903125	$0.38108926 \times 10^{-13}$	1.04933×10^{-6}	2.258860×10^{-18}
1.003125	$0.54051538 \times 10^{-13}$	5.69624×10^{-6}	1.653371×10^{-18}

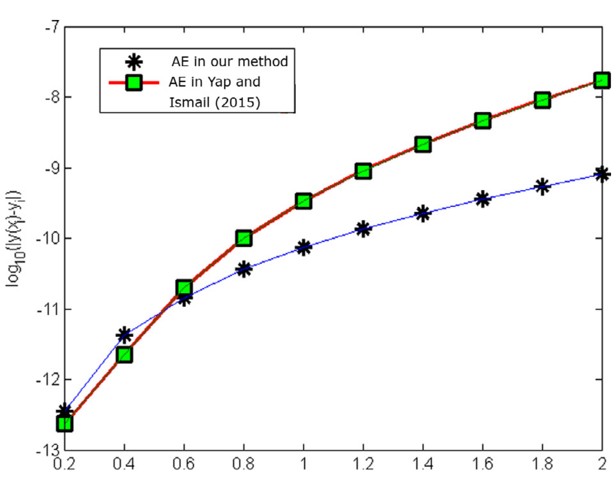

Fig. 1. Showing error in Problem 1.

$$y'''' = -3y'' - y(2 + \epsilon\cos(\Omega x)) \tag{10}$$

which is imposed to the following conditions: $y(0) = 1,\ y'(0) = 0, y''(0) = 0,\ y'''(0) = 0.$ where $\epsilon = 0$for the exis-

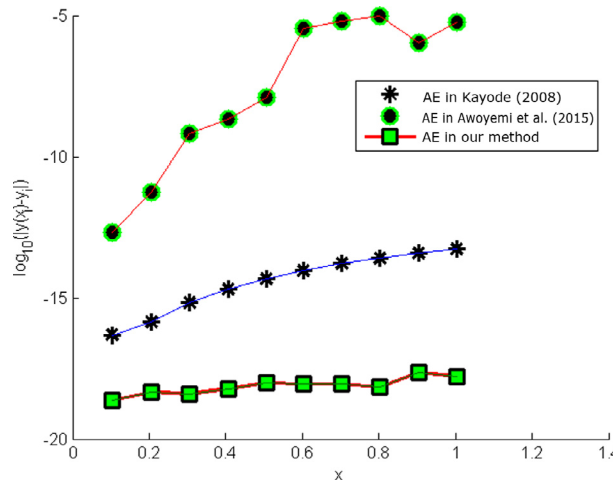

Fig. 2. Showing error in Problem 1.

Table 5
Comparison of the new method with Twizell (1988) and Cortell (1993) for solving Eq. (10).

h	Method	Error at x = 15.
0.25	New method	$8.15e^{-8}$
	Yap and Ismail (2015)	$5.2e^{-7}$
	Twizell	$1.9e^{-4}$
0.1	New method	$3.6e^{-11}$
	Yap and Ismail (2015)	$2.8e^{-10}$
	Cortell	$3.7e^{-5}$

tence of the theoretical solution, $y(x) = 2\cos x - \cos(x\sqrt{2})$. The theoretical solution is undefined when $\epsilon \neq 0$.

The comparison of the new method with Twizell (1988) and Cortell (1993) has been made at the end point $x = 15$ in terms of accuracy (refer to Table 5), where $h = 0.25$ and $h = 0.1$.

The block method (7) is self-starting and applied without the use of predictors which can reduce the accuracy of the method as well as increase programming time.

5. Conclusion

We have developed a four-step block method with three generalized off step points for the solution of fourth order initial value problems which was applied without the use of predictors that reduce the accuracy of the method. The properties of the developed block method which include: zero stability, order, consistency and convergence are established. The existing four-step hybrid block methods only focus on the specific off step points. Since the new method works for any three off-step points, it is therefore more flexible and robust. The application of the developed method was then made to several fourth order initial value problems. In general, the benefit of using the four-step hybrid block method for solving IVPs of fourth order ODEs is obvious. The new method displays its superiority by producing less error if compared to the present methods as shown in Tables 1 and 2. In addition, the developed method does not require much computation when compared with predictor corrector methods.

Author Contributions

Both authors equally contributed to this work.

Conflicts of Interest

The authors declare no conflict of interest.

Appendix A

$$D_{11}=\frac{-h^4(s_1-1)(s_1-2)(s_1-3)}{3991680s_2s_3}(2310s_2-220s_1+2310s_3+3718s_1s_2+3718s_1s_3$$
$$+924s_2s_3-3377s_1^2s_2-3377s_1^2s_3-1430s_1^3s_2-1430s_1^3s_3+1375s_1^4s_2+1375s_1^4s_3-308s_1^5s_2$$
$$-308s_1^5s_3+22s_1^6s_2+22s_1^6s_3+1762s_1^2+1052s_1^3+130s_1^4-520s_1^5+148s_1^6-12s_1^7$$
$$+11550s_1^2s_2s_3-4400s_1^3s_2s_3+726s_1^4s_2s_3-44s_1^5s_2s_3-11000s_1s_2s_3-5040)$$

$$D_{12}=\frac{-h^4(s_2-1)(s_2-2)(s_2-3)}{3991680s_1s_3}(2310s_1-220s_2+2310s_3+3718s_1s_2+924s_1s_3$$
$$+3718s_2s_3-3377s_1s_2^2-1430s_1s_2^3+1375s_1s_2^4-3377s_2^2s_3-308s_1s_2^5-1430s_2^3s_3$$
$$+22s_1s_2^6+1375s_2^4s_3-308s_2^5s_3+22s_2^6s_3+1762s_2^2+1052s_2^3+130s_2^4-520s_2^5$$
$$+148s_2^6-12s_2^7+11550s_1s_2^2s_3-4400s_1s_2^3s_3+726s_1s_2^4s_3-44s_1s_2^5s_3-11000s_1s_2s_3-5040)$$

$$D_{13}=\frac{-h^4(s_3-1)(s_3-2)(s_3-3)}{3991680s_1s_2}(2310s_1+2310s_2-220s_3+924s_1s_2+3718s_1s_3$$
$$+3718s_2s_3-3377s_1s_3^2-1430s_1s_3^3-3377s_2s_3^2+1375s_1s_3^4-1430s_2s_3^3-308s_1s_3^5+1375s_2s_3^4$$
$$+22s_1s_3^6-308s_2s_3^5+22s_2s_3^6+1762s_3^2+1052s_3^3+130s_3^4-520s_3^5+148s_3^6-12s_3^7$$
$$+11550s_1s_2s_3^2-4400s_1s_2s_3^3+726s_1s_2s_3^4-44s_1s_2s_3^5-11000s_1s_2s_3-5040)$$

$$D_{14}=\frac{h^4(50s_1+50s_2+50s_3-7s_1s_2s_3-200)}{5040s_1s_2s_3}$$

$$D_{15}=\frac{h^3(770s_1+770s_2+770s_3+308s_1s_2+308s_1s_3+308s_2s_3-2123s_1s_2s_3-1680)}{221760s_1s_2s_3}$$

$$D_{16}=\frac{-h^2(262s_1+262s_2+262s_3+1150s_1s_2+1150s_1s_3+1150s_2s_3-4463s_1s_2s_3-1220)}{60480s_1s_2s_3}$$

$$D_{17}=\frac{-h(850s_1+850s_2+850s_3-4415s_1s_2-4415s_1s_3-4415s_2s_3+20034s_1s_2s_3+782)}{60480s_1s_2s_3}$$

$$E_{11}=\frac{-h^4}{((166320s_1-665280)(s_1-s_2)(s_1-s_3))}(2530s_1-2310s_2-2310s_3-2794s_1s_2$$
$$-2794s_1s_3-924s_2s_3-1254s_1^2s_2-1254s_1^2s_3+110s_1^3s_2+110s_1^3s_3+990s_1^4s_2+990s_1^4s_3$$
$$-352s_1^5s_2-352s_1^5s_3+33s_1^6s_2+33s_1^6s_3+1032s_1^2+202s_1^3-240s_1^4-470s_1^5+204s_1^6-21s_1^7$$
$$+1320s_1^2s_2s_3-2365s_1^3s_2s_3+660s_1^4s_2s_3-55s_1^5s_2s_3+4631s_1s_2s_3+5040)$$

$$E_{12}=\frac{-(h^4s_1(s_1-2)(s_1-3))}{((997920s_2-997920)(s_3-1))}(24684s_1s_2-31251s_2-31251s_3-24464s_1$$
$$+24684s_1s_3+28941s_2s_3+2684s_1^2s_2+2684s_1^2s_3-1188s_1^3s_2-1188s_1^3s_3-748s_1^4s_2-748s_1^4s_3$$
$$+264s_1^5s_2+264s_1^5s_3-22s_1^6s_2-22s_1^6s_3-4446s_1^2+136s_1^3+618s_1^4+256s_1^5-126s_1^6+12s_1^7$$
$$+693s_1^2s_2s_3+2618s_1^3s_2s_3-627s_1^4s_2s_3+44s_1^5s_2s_3-28402s_1s_2s_3+36291)$$

$$E_{13}=\frac{-(h^4s_1(s_1-1)(s_1-2)(s_1-3))}{(166320s_2(s_1-s_2)(s_2-s_3)(s_2-1)(s_2-2)(s_2-3)(s_2-4))}(2310s_3-220s_1$$
$$+3718s_1s_3-3377s_1^2s_3-1430s_1^3s_3+1375s_1^4s_3-308s_1^5s_3+22s_1^6s_3+1762s_1^2$$
$$+1052s_1^3+130s_1^4-520s_1^5+148s_1^6-12s_1^7-5040))$$

$$E_{14}=\frac{(h^4s_1(s_1-1)(s_1-3))}{((665280s_2-1330560)(s_3-2))}(11000s_1+27126s_2+27126s_3-5390s_1s_2$$
$$-5390s_1s_3-14718s_2s_3-2959s_1^2s_2-2959s_1^2s_3-1298s_1^3s_2-1298s_1^3s_3-319s_1^4s_2-319s_1^4s_3$$
$$+220s_1^5s_2+220s_1^5s_3-22s_1^6s_2-22s_1^6s_3+4156s_1^2+1544s_1^3+508s_1^4+80s_1^5-104s_1^6$$
$$+12s_1^7+3168s_1^2s_2s_3+1364s_1^3s_2s_3-528s_1^4s_2s_3+44s_1^5s_2s_3$$
$$+836s_1s_2s_3-49212)$$

$$E_{15}=\frac{(h^4s_1(s_1-1)(s_1-2)(s_1-3))}{(166320s_3(s_1-s_3)(s_2-s_3)(s_3-1)(s_3-2)(s_3-3)(s_3-4))}(2310s_2-220s_1$$
$$+3718s_1s_2-3377s_1^2s_2-1430s_1^3s_2+1375s_1^4s_2-308s_1^5s_2+22s_1^6s_2+1762s_1^2+1052s_1^3+130s_1^4$$
$$-520s_1^5+148s_1^6-12s_1^7-5040)$$

$$E_{16} = \frac{(h^4 s_1(s_1-1)(s_1-2))}{((997920s_2-2993760)(s_3-3))}(484s_1s_2 - 2541s_2 - 2541s_3 - 1672s_1 + 484s_1s_3 + 1617s_2s_3 + 880s_1^2s_2 + 880s_1^2s_3 + 484s_1^3s_2 + 484s_1^3s_3 + 88s_1^4s_2 + 88s_1^4s_3 - 176s_1^5s_2 - 176s_1^5s_3 + 22s_1^6s_2 + 22s_1^6s_3 - 878s_1^2 - 400s_1^3 - 134s_1^4 + 8s_1^5 + 82s_1^6 - 12s_1^7 - 1419s_1^2s_2s_3 - 638s_1^3s_2s_3 + 429s_1^4s_2s_3 - 44s_1^5s_2s_3 + 1078s_1s_2s_3 + 2583)$$

$$E_{17} = \frac{-(h^4 s_1(s_1-1)(s_1-2)(s_1-3))}{((3991680s_2-15966720)(s_1-4)(s_3-4))}(198s_1s_2 - 1914s_2 - 1914s_3 - 1012s_1 + 198s_1s_3 + 1056s_2s_3 + 583s_1^2s_2 + 583s_1^2s_3 + 330s_1^3s_2 + 330s_1^3s_3 + 55s_1^4s_2 + 55s_1^4s_3 - 132s_1^5s_2 - 132s_1^5s_3 + 22s_1^6s_2 + 22s_1^6s_3 - 570s_1^2 - 268s_1^3 - 90s_1^4 + 8s_1^5 + 60s_1^6 - 12s_1^7 - 990s_1^2s_2s_3 - 440s_1^3s_2s_3 + 330s_1^4s_2s_3 - 44s_1^5s_2s_3 + 880s_1s_2s_3 + 2616)$$

$$E_{21} = \frac{(h^4 s_2(s_2-1)(s_2-2)(s_2-3))}{(166320s_1(s_1-s_2)(s_1-s_3)(s_1-1)(s_1-2)(s_1-3)(s_1-4))}(2310s_3 - 220s_2 + 3718s_2s_3 - 3377s_2^2s_3 - 1430s_2^3s_3 + 1375s_2^4s_3 - 308s_2^5s_3 + 22s_2^6s_3 + 1762s_2^2 + 1052s_2^3 + 130s_2^4 - 520s_2^5 + 148s_2^6 - 12s_2^7 - 5040)$$

$$E_{22} = \frac{-(h^4 s_2(s_2-2)(s_2-3))}{((997920s_1-997920)(s_3-1))}(24684s_1s_2 - 24464s_2 - 31251s_3 - 31251s_1 + 28941s_1s_3 + 24684s_2s_3 + 2684s_1s_2^2 - 1188s_1s_2^3 - 748s_1s_2^4 + 2684s_2^2s_3 + 264s_1s_2^5 - 1188s_2^3s_3 - 22s_1s_2^6 - 748s_2^4s_3 + 264s_2^5s_3 - 22s_2^6s_3 - 4446s_2^2 + 136s_2^3 + 618s_2^4 + 256s_2^5 - 126s_2^6 + 12s_2^7 + 693s_1s_2^2s_3 + 2618s_1s_2^3s_3 - 627s_1s_2^4s_3 + 44s_1s_2^5s_3 - 28402s_1s_2s_3 + 36291)$$

$$E_{23} = \frac{-(h^4)}{((166320s_2-665280)(s_1-s_2)(s_2-s_3))}(2310s_1 - 2530s_2 + 2310s_3 + 2794s_1s_2 + 924s_1s_3 + 2794s_2s_3 + 1254s_1s_2^2 - 110s_1s_2^3 - 990s_1s_2^4 + 1254s_2^2s_3 + 352s_1s_2^5 - 110s_2^3s_3 - 33s_1s_2^6 - 990s_2^4s_3 + 352s_2^5s_3 - 33s_2^6s_3 - 1032s_2^2 - 202s_2^3 + 240s_2^4 + 470s_2^5 - 204s_2^6 + 21s_2^7 - 1320s_1s_2^2s_3 + 2365s_1s_2^3s_3 - 660s_1s_2^4s_3 + 55s_1s_2^5s_3 - 4631s_1s_2s_3 - 5040)$$

$$E_{24} = \frac{(h^4 s_2(s_2-1)(s_2-3))}{((665280s_1-1330560)(s_3-2))}(27126s_1 + 11000s_2 + 27126s_3 - 5390s_1s_2 - 14718s_1s_3 - 5390s_2s_3 - 2959s_1s_2^2 - 1298s_1s_2^3 - 319s_1s_2^4 - 2959s_2^2s_3 + 220s_1s_2^5 - 1298s_2^3s_3 - 22s_1s_2^6 - 319s_2^4s_3 + 220s_2^5s_3 - 22s_2^6s_3 + 4156s_2^2 + 1544s_2^3 + 508s_2^4 + 80s_2^5 - 104s_2^6 + 12s_2^7 + 3168s_1s_2^2s_3 + 1364s_1s_2^3s_3 - 528s_1s_2^4s_3 + 44s_1s_2^5s_3 + 836s_1s_2s_3 - 49212)$$

$$E_{25} = \frac{(h^4 s_2(s_2-1)(s_2-2)(s_2-3))}{(166320s_3(s_1-s_3)(s_2-s_3)(s_3-1)(s_3-2)(s_3-3)(s_3-4))}(2310s_1 - 220s_2 + 3718s_1s_2 - 3377s_1s_2^2 - 1430s_1s_2^3 + 1375s_1s_2^4 - 308s_1s_2^5 + 22s_1s_2^6 + 1762s_2^2 + 1052s_2^3 + 130s_2^4 - 520s_2^5 + 148s_2^6 - 12s_2^7 - 5040)$$

$$E_{26} = \frac{(h^4 s_2(s_2-1)(s_2-2))}{((997920s_1-2993760)(s_3-3))}(484s_1s_2 - 1672s_2 - 2541s_3 - 2541s_1 + 1617s_1s_3 + 484s_2s_3 + 880s_1s_2^2 + 484s_1s_2^3 + 88s_1s_2^4 + 880s_2^2s_3 - 176s_1s_2^5 + 484s_2^3s_3 + 22s_1s_2^6 + 88s_2^4s_3 - 176s_2^5s_3 + 22s_2^6s_3 - 878s_2^2 - 400s_2^3 - 134s_2^4 + 8s_2^5 + 82s_2^6 - 12s_2^7 - 1419s_1s_2^2s_3 - 638s_1s_2^3s_3 + 429s_1s_2^4s_3 - 44s_1s_2^5s_3 + 1078s_1s_2s_3 + 2583)$$

$$E_{27} = \frac{-(h^4 s_2(s_2-1)(s_2-2)(s_2-3))}{((3991680s_2-15966720)(s_1-4)(s_3-4))}(198s_1s_2 - 1012s_2 - 1914s_3 - 1914s_1 + 1056s_1s_3 + 198s_2s_3 + 583s_1s_2^2 + 330s_1s_2^3 + 55s_1s_2^4 + 583s_2^2s_3 - 132s_1s_2^5 + 330s_2^3s_3 + 22s_1s_2^6 + 55s_2^4s_3 - 132s_2^5s_3 + 22s_2^6s_3 - 570s_2^2 - 268s_2^3 - 90s_2^4 + 8s_2^5 + 60s_2^6 - 12s_2^7 - 990s_1s_2^2s_3 - 440s_1s_2^3s_3 + 330s_1s_2^4s_3 - 44s_1s_2^5s_3 + 880s_1s_2s_3 + 2616)$$

$$E_{31} = \frac{(h^4 s_3(s_3-1)(s_3-2)(s_3-3))}{(166320s_1(s_1-s_2)(s_1-s_3)(s_1-1)(s_1-2)(s_1-3)(s_1-4))}(2310s_2 - 220s_3 + 3718s_2s_3 - 3377s_2s_3^2 - 1430s_2s_3^3 + 1375s_2s_3^4 - 308s_2s_3^5 + 22s_2s_3^6 + 1762s_3^2 + 1052s_3^3 + 130s_3^4 - 520s_3^5 + 148s_3^6 - 12s_3^7 - 5040)$$

$$E_{32} = \frac{-(h^4 s_3(s_3-2)(s_3-3))}{((997920s_1-997920)(s_2-1))}(28941s_1s_2 - 31251s_2 - 24464s_3 - 31251s_1 + 24684s_1s_3 + 24684s_2s_3 + 2684s_1s_3^2 - 1188s_1s_3^3 + 2684s_2s_3^2 - 748s_1s_3^4 - 1188s_2s_3^3 + 264s_1s_3^5 - 748s_2s_3^4 - 22s_1s_3^6 + 264s_2s_3^5 - 22s_2s_3^6 - 4446s_3^2 + 136s_3^3 + 618s_3^4 + 256s_3^5 - 126s_3^6 + 12s_3^7 + 693s_1s_2s_3^2 + 2618s_1s_2s_3^3 - 627s_1s_2s_3^4 + 44s_1s_2s_3^5 - 28402s_1s_2s_3 + 36291)$$

$$E_{33} = \frac{-(h^4 s_3(s_3-1)(s_3-2)(s_3-3))}{(166320s_2(s_1-s_2)(s_2-s_3)(s_2-1)(s_2-2)(s_2-3)(s_2-4))}(2310s_1 - 220s_3 + 3718s_1s_3 - 3377s_1s_3^2 - 1430s_1s_3^3 + 1375s_1s_3^4 - 308s_1s_3^5 + 22s_1s_3^6 + 1762s_3^2 + 1052s_3^3 + 130s_3^4 - 520s_3^5 + 148s_3^6 - 12s_3^7 - 5040)$$

$$E_{34} = \frac{(h^4 s_3(s_3-1)(s_3-3))}{((665280s_1-1330560)(s_2-2))}(27126s_1 + 27126s_2 + 11000s_3 - 14718s_1s_2 - 5390s_1s_3 - 5390s_2s_3 - 2959s_1s_3^2 - 1298s_1s_3^3 - 2959s_2s_3^2 - 319s_1s_3^4 - 1298s_2s_3^3 + 220s_1s_3^5 - 319s_2s_3^4 - 22s_1s_3^6 + 220s_2s_3^5 - 22s_2s_3^6 + 4156s_3^2 + 1544s_3^3 + 508s_3^4 + 80s_3^5 - 104s_3^6 + 12s_3^7 + 3168s_1s_2s_3^2 + 1364s_1s_2s_3^3 - 528s_1s_2s_3^4 + 44s_1s_2s_3^5 + 836s_1s_2s_3 - 49212)$$

$$E_{35} = \frac{(h^4)}{((166320s_3-665280)(s_1-s_3)(s_2-s_3))}(2310s_1 + 2310s_2 - 2530s_3 + 924s_1s_2 + 2794s_1s_3 + 2794s_2s_3 + 1254s_1s_3^2 - 110s_1s_3^3 + 1254s_2s_3^2 - 990s_1s_3^4 - 110s_2s_3^3 + 352s_1s_3^5 - 990s_2s_3^4 - 33s_1s_3^6 + 352s_2s_3^5 - 33s_2s_3^6 - 1032s_3^2 - 202s_3^3 + 240s_3^4 + 470s_3^5 - 204s_3^6 + 21s_3^7 - 1320s_1s_2s_3^2 + 2365s_1s_2s_3^3 - 660s_1s_2s_3^4 + 55s_1s_2s_3^5 - 4631s_1s_2s_3 - 5040)$$

$$E_{36} = \frac{(h^4 s_3(s_3-1)(s_3-2))}{((997920s_1-2993760)(s_2-3))}(1617s_1s_2 - 2541s_2 - 1672s_3 - 2541s_1 + 484s_1s_3 + 484s_2s_3 + 880s_1s_3^2 + 484s_1s_3^3 + 880s_2s_3^2 + 88s_1s_3^4 + 484s_2s_3^3 - 176s_1s_3^5 + 88s_2s_3^4 + 22s_1s_3^6 - 176s_2s_3^5 + 22s_2s_3^6 - 878s_3^2 - 400s_3^3 - 134s_3^4 + 8s_3^5 + 82s_3^6 - 12s_3^7 - 1419s_1s_2s_3^2 - 638s_1s_2s_3^3 + 429s_1s_2s_3^4 - 44s_1s_2s_3^5 + 1078s_1s_2s_3 + 2583)$$

$$E_{37} = \frac{-(h^4 s_3(s_3-1)(s_3-2)(s_3-3))}{((3991680s_3-15966720)(s_1-4)(s_2-4))}(1056s_1s_2 - 1914s_2 - 1012s_3 - 1914s_1 + 198s_1s_3 + 198s_2s_3 + 583s_1s_3^2 + 330s_1s_3^3 + 583s_2s_3^2 + 55s_1s_3^4 + 330s_2s_3^3 - 132s_1s_3^5 + 55s_2s_3^4 + 22s_1s_3^6 - 132s_2s_3^5 + 22s_2s_3^6 - 570s_3^2 - 268s_3^3 - 90s_3^4 + 8s_3^5 + 60s_3^6 - 12s_3^7 - 990s_1s_2s_3^2 - 440s_1s_2s_3^3 + 330s_1s_2s_3^4 - 44s_1s_2s_3^5 + 880s_1s_2s_3 + 2616)$$

$$E_{41} = \frac{-(5h^4(s_2+s_3-4))}{(21s_1(s_1-s_2)(s_1-s_3)(s_1-1)(s_1-2)(s_1-3)(s_1-4))}$$

$$E_{42} = \frac{(h^4(167s_1 + 167s_2 + 167s_3 - 217s_1s_2 - 217s_1s_3 - 217s_2s_3 + 217s_1s_2s_3 + 33))}{((1260s_3-1260)(s_1-1)(s_2-1))}$$

$$E_{43} = \frac{(5h^4(s_1+s_3-4))}{(21s_2(s_1-s_2)(s_2-s_3)(s_2-1)(s_2-2)(s_2-3)(s_2-4))}$$

$$E_{44} = \frac{(h^4(2262s_1 + 2262s_2 + 2262s_3 - 1106s_1s_2 - 1106s_1s_3 - 1106s_2s_3 + 553s_1s_2s_3 - 4724))}{((840s_3-1680)(s_1-2)(s_2-2))}$$

$$E_{45} = \frac{-(5h^4(s_1+s_2-4))}{(21s_3(s_1-s_3)(s_2-s_3)(s_3-1)(s_3-2)(s_3-3)(s_3-4))}$$

$$E_{46} = \frac{(h^4(1903s_1 + 1903s_2 + 1903s_3 - 651s_1s_2 - 651s_1s_3 - 651s_2s_3 + 217s_1s_2s_3 - 5509))}{((1260s_3-3780)(s_1-3)(s_2-3))}$$

$$E_{47} = \frac{-(h^4(62s_1 + 62s_2 + 62s_3 - 28s_1s_2 - 28s_1s_3 - 28s_2s_3 + 7s_1s_2s_3 - 48))}{((5040s_3-20160)(s_1-4)(s_2-4))}$$

$$E_{51} = \frac{-(h^3(55s_2 + 55s_3 + 22s_2s_3 - 120))}{(660s_1(s_1-s_2)(s_1-s_3)(s_1-1)(s_1-2)(s_1-3)(s_1-4))}$$

$$E_{52} = -(h^3(10417s_1 + 10417s_2 + 10417s_3 - 9647s_1s_2 - 9647s_1s_3 - 9647s_2s_3 + 9955s_1s_2s_3 - 12097))/((55440s_1-55440)(s_2-1)(s_3-1))$$

$$E_{53} = \frac{(h^3(55s_1 + 55s_3 + 22s_1s_3 - 120))}{(660s_2(s_1 - s_2)(s_2 - s_3)(s_2 - 1)(s_2 - 2)(s_2 - 3)(s_2 - 4))}$$

$$E_{54} = -(h^3(9042s_1 + 9042s_2 + 9042s_3 - 4906s_1s_2 - 4906s_1s_3 - 4906s_2s_3$$
$$+ 2299s_1s_2s_3 - 16404))((36960s_1 - 73920)(s_2 - 2)(s_3 - 2))$$

$$E_{55} = \frac{-(h^3(55s_1 + 55s_2 + 22s_1s_2 - 120))}{(660s_3(s_1 - s_3)(s_2 - s_3)(s_3 - 1)(s_3 - 2)(s_3 - 3)(s_3 - 4))}$$

$$E_{56} = \frac{(h^3(121s_1 + 121s_2 + 121s_3 - 77s_1s_2 - 77s_1s_3 - 77s_2s_3 + 11s_1s_2s_3 - 123))}{((7920s_1 - 23760)(s_2 - 3)(s_3 - 3))}$$

$$E_{57} = \frac{-(h^3(638s_1 + 638s_2 + 638s_3 - 352s_1s_2 - 352s_1s_3 - 352s_2s_3 + 11s_1s_2s_3 - 872))}{((221760s_1 - 887040)(s_2 - 4)(s_3 - 4))}$$

$$E_{61} = \frac{(h^2(131s_2 + 131s_3 + 575s_2s_3 - 610))}{(1260s_1(s_1 - s_2)(s_1 - s_3)(s_1 - 1)(s_1 - 2)(s_1 - 3)(s_1 - 4))}$$

$$E_{62} = \frac{(h^2(9643s_1 + 9643s_2 + 9643s_3 - 9381s_1s_2 - 9381s_1s_3 - 9381s_2s_3 + 10531s_1s_2s_3 - 10863))}{((15120s_1 - 15120)(s_2 - 1)(s_3 - 1))}$$

$$E_{63} = \frac{-(h^2(131s_1 + 131s_3 + 575s_1s_3 - 610))}{(1260s_2(s_1 - s_2)(s_2 - s_3)(s_2 - 1)(s_2 - 2)(s_2 - 3)(s_2 - 4))}$$

$$E_{64} = \frac{(h^2(7346s_1 + 7346s_2 + 7346s_3 - 3804s_1s_2 - 3804s_1s_3 - 3804s_2s_3 + 1327s_1s_2s_3 - 13472))}{((10080s_1 - 20160)(s_2 - 2)(s_3 - 2))}$$

$$E_{65} = \frac{(h^2(131s_1 + 131s_2 + 575s_1s_2 - 610))}{(1260s_3(s_1 - s_3)(s_2 - s_3)(s_3 - 1)(s_3 - 2)(s_3 - 3)(s_3 - 4))}$$

$$E_{66} = \frac{(h^2(301s_1s_2 - 641s_2 - 641s_3 - 641s_1 + 301s_1s_3 + 301s_2s_3 + 283s_1s_2s_3 + 703))}{((15120s_1 - 45360)(s_2 - 3)(s_3 - 3))}$$

$$E_{67} = \frac{-(h^2(186s_1s_2 - 482s_2 - 482s_3 - 482s_1 + 186s_1s_3 + 186s_2s_3 + 241s_1s_2s_3 + 708))}{((60480s_1 - 241920)(s_2 - 4)(s_3 - 4))}$$

$$E_{71} = \frac{(h(850s_2 + 850s_3 - 4415s_2s_3 + 782))}{(2520s_1(s_1 - s_2)(s_1 - s_3)(s_1 - 1)(s_1 - 2)(s_1 - 3)(s_1 - 4))}$$

$$E_{72} = -(h(12123s_1 + 12123s_2 + 12123s_3 - 12973s_1s_2 - 12973s_1s_3 - 12973s_2s_3$$
$$+ 17388s_1s_2s_3 - 12905))/((15120s_1 - 15120)(s_2 - 1)(s_3 - 1))$$

$$E_{73} = \frac{-(h(850s_1 + 850s_3 - 4415s_1s_3 + 782))}{(2520s_2(s_1 - s_2)(s_2 - s_3)(s_2 - 1)(s_2 - 2)(s_2 - 3)(s_2 - 4))}$$

$$E_{74} = \frac{(h(3155s_1s_2 - 7160s_2 - 7160s_3 - 7160s_1 + 3155s_1s_3 + 3155s_2s_3 + 630s_1s_2s_3 + 13538))}{((10080s_1 - 20160)(s_2 - 2)(s_3 - 2))}$$

$$E_{75} = \frac{(h(850s_1 + 850s_2 - 4415s_1s_2 + 782))}{(2520s_3(s_1 - s_3)(s_2 - s_3)(s_3 - 1)(s_3 - 2)(s_3 - 3)(s_3 - 4))}$$

$$E_{76} = \frac{(h(487s_1 + 487s_2 + 487s_3 + 121s_1s_2 + 121s_1s_3 + 121s_2s_3 - 1512s_1s_2s_3 - 679))}{((15120s_1 - 45360)(s_2 - 3)(s_3 - 3))}$$

$$E_{77} = \frac{-(h(366s_1 + 366s_2 + 366s_3 + 121s_1s_2 + 121s_1s_3 + 121s_2s_3 - 1134s_1s_2s_3 - 682))}{((60480s_1 - 241920)(s_2 - 4)(s_3 - 4))}$$

Appendix B

$$y_{n+s_1} = y_n + \frac{h^2 s_1^2 y_n''}{2} + \frac{h^3 s_1^3 y_n'''}{6} + hs_1 y_n'$$
$$- \frac{h^4 s_1^4}{(3991680 s_2 s_3)}(22176s_1s_2 + 22176s_1s_3 - 133056s_2s_3 - 13200s_1^2s_2 - 13200s_1^2s_3 + 3465s_1^3s_2$$
$$+ 3465s_1^3s_3 - 440s_1^4s_2 - 440s_1^4s_3 + 22s_1^5s_2 + 22s_1^5s_3 - 6336s_1^2 + 4950s_1^3 - 1540s_1^4$$
$$- 9240s_1^2s_2s_3 + 46200s_1s_2s_3 - 12s_1^6 + 990s_1^3s_2s_3 - 44s_1^4s_2s_3 + 220s_1^5)f_n$$
$$+ \frac{h^4 s_1^6}{(3991680(s_1 - 4)(s_2 - 4)(s_3 - 4))}(1584s_1s_2 + 1584s_1s_3 - 5544s_2s_3 - 1089s_1^2s_2 - 1089s_1^2s_3$$
$$+ 264s_1^3s_2 + 264s_1^3s_3 - 22s_1^4s_2 - 22s_1^4s_3 - 594s_1^2 + 484s_1^3 - 132s_1^4 + 12s_1^5 - 594s_1^2s_2s_3$$
$$+ 44s_1^3s_2s_3 + 2904s_1s_2s_3)f_{n+4}$$

$$- \frac{h^4 s_1^6}{(997920(s_1-3)(s_2-3)(s_3-3))}(2112s_1s_2 + 2112s_1s_3 - 7392s_2s_3 - 1386s_1^2s_2 - 1386s_1^2s_3$$
$$+ 308s_1^3s_2 + 308s_1^3s_3 - 22s_1^4s_2 - 22s_1^4s_3 - 792s_1^2 + 616s_1^3 - 154s_1^4 + 12s_1^5$$
$$- 693s_1^2s_2s_3 + 44s_1^3s_2s_3 + 3696s_1s_2s_3)f_{n+3}$$
$$+ \frac{h^4 s_1^6}{(665280(s_1-2)(s_2-2)(s_3-2))}(3168s_1s_2 + 3168s_1s_3 - 11088s_2s_3 - 1881s_1^2s_2 - 1881s_1^2s_3$$
$$+ 352s_1^3s_2 + 352s_1^3s_3 - 22s_1^4s_2 - 22s_1^4s_3 - 1188s_1^2 + 836s_1^3 - 176s_1^4 + 12s_1^5$$
$$- 792s_1^2s_2s_3 + 44s_1^3s_2s_3 + 5016s_1s_2s_3)f_{n+2}$$
$$- \frac{h^4 s_1^6}{(997920(s_1-1)(s_2-1)(s_3-1))}(6336s_1s_2 + 6336s_1s_3 - 22176s_2s_3 - 2574s_1^2s_2 - 2574s_1^2s_3$$
$$+ 396s_1^3s_2 + 396s_1^3s_3 - 22s_1^4s_2 - 22s_1^4s_3 - 2376s_1^2 + 1144s_1^3 - 198s_1^4 + 12s_1^5 - 891s_1^2s_2s_3$$
$$+ 44s_1^3s_2s_3 + 6864s_1s_2s_3)f_{n+1}$$
$$- \frac{h^4 s_1^4}{(166320(s_1-s_2)(s_1-s_3)(s_1-1)(s_1-2)(s_1-3)(s_1-4))}(11088s_1s_2 + 11088s_1s_3$$
$$- 33264s_2s_3 - 21s_1^6 - 9900s_1^2s_2 - 9900s_1^2s_3 + 3465s_1^3s_2 + 3465s_1^3s_3 - 550s_1^4s_2$$
$$- 550s_1^4s_3 + 33s_1^5s_2 + 33s_1^5s_3 - 4752s_1^2 + 4950s_1^3 - 1925s_1^4 + 330s_1^5 - 6930s_1^2s_2s_3$$
$$+ 990s_1^3s_2s_3 - 55s_1^4s_2s_3 + 23100s_1s_2s_3)f_{n+s_1}$$
$$+ h^4s_1^6(6336s_1 - 22176s_3 + 13200s_1s_3 - 3465s_1^2s_3 + 440s_1^3s_3 - 22s_1^4s_3 - 4950s_1^2 + 1540s_1^3$$
$$- 220s_1^4 + 12s_1^5)/(166320s_2(s_1-s_2)(s_2-s_3)(s_2-1)(s_2-2)(s_2-3)(s_2-4))f_{n+s_2}$$
$$- h^4s_1^6(6336s_1 - 22176s_2 + 13200s_1s_2 - 3465s_1^2s_2 + 440s_1^3s_2 - 22s_1^4s_2 - 4950s_1^2$$
$$+ 1540s_1^3 - 220s_1^4 + 12s_1^5)/(166320s_3(s_1-s_3)(s_2-s_3)(s_3-1)(s_3-2)(s_3-3)(s_3-4))f_{n+s_3}$$

$$y_{n+1} = y_n + +hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n$$
$$- \frac{h^4(4136s_1 + 4136s_2 - 16159s_1s_2 - 1418)}{(166320s_3(s_1-s_3)(s_2-s_3)(s_3-1)(s_3-2)(s_3-3)(s_3-4))}f_{n+s_3}$$
$$+ \frac{h^4(4136s_1 + 4136s_2 + 4136s_3 - 16159s_1s_2 - 16159s_1s_3 - 16159s_2s_3 + 111309s_1s_2s_3 - 1418)}{(3991680s_1s_2s_3}f_n$$
$$- \frac{h^4(396s_1 + 396s_2 + 396s_3 - 1133s_1s_2 - 1133s_1s_3 - 1133s_2s_3 + 4323s_1s_2s_3 - 166)}{(3991680(s_1-4)(s_2-4)(s_3-4))}f_{n+4}$$
$$+ \frac{h^4(550s_1 + 550s_2 + 550s_3 - 1562s_1s_2 - 1562s_1s_3 - 1562s_2s_3 + 5907s_1s_2s_3 - 232)}{(997920(s_1-3)(s_2-3)(s_3-3))}f_{n+3}$$
$$- \frac{h^4(902s_1 + 902s_2 + 902s_3 - 2519s_1s_2 - 2519s_1s_3 - 2519s_2s_3 + 9339s_1s_2s_3 - 386)}{(665280(s_1-2)(s_2-2)(s_3-2))}f_{n+2}$$
$$+ \frac{h^4(2640s_1 + 2640s_2 + 2640s_3 - 6776s_1s_2 - 6776s_1s_3 - 6776s_2s_3 + 22935s_1s_2s_3 - 1222)}{(997920(s_1-1)(s_2-1)(s_3-1))}f_{n+1}$$
$$- \frac{h^4(4136s_2 + 4136s_3 - 16159s_2s_3 - 1418)}{(166320s_1(s_1-s_2)(s_1-s_3)(s_1-1)(s_1-2)(s_1-3)(s_1-4))}f_{n+s_1}$$
$$+ \frac{h^4(4136s_1 + 4136s_3 - 16159s_1s_3 - 1418)}{(166320s_2(s_1-s_2)(s_2-s_3)(s_2-1)(s_2-2)(s_2-3)(s_2-4))}f_{n+s_2}$$
$$y_{n+s_2} = y_n + hs_2y'_n + \frac{h^2s_2^2}{2y''_n} + \frac{h^3s_2^3w0}{6}y'''_n$$
$$- \frac{h^4s_2^4}{(3991680s_1s_3)}(22176s_1s_2 - 133056s_1s_3 + 22176s_2s_3 - 13200s_1s_2^2 + 3465s_1s_2^3 - 440s_1s_2^4$$
$$- 13200s_2^2s_3 + 22s_1s_2^5 + 3465s_2^3s_3 - 440s_2^4s_3 + 22s_2^5s_3 - 6336s_2^2 + 4950s_2^3$$
$$- 1540s_2^4 + 220s_2^5 - 12s_2^6 - 9240s_1s_2^2s_3 + 990s_1s_2^3s_3 - 44s_1s_2^4s_3 + 46200s_1s_2s_3)f_n$$
$$+ \frac{h^4s_2^6}{(3991680(s_1-4)(s_2-4)(s_3-4))}(1584s_1s_2 - 5544s_1s_3 + 1584s_2s_3 - 1089s_1s_2^2$$
$$+ 264s_1s_2^3 - 22s_1s_2^4 - 1089s_2^2s_3 + 264s_2^3s_3 - 22s_2^4s_3 - 594s_2^2 + 484s_2^3$$
$$- 132s_2^4 + 12s_2^5 - 594s_1s_2^2s_3 + 44s_1s_2^3s_3 + 2904s_1s_2s_3))f_{n+4}$$
$$- \frac{h^4s_2^6}{(997920(s_1-3)(s_2-3)(s_3-3))}(2112s_1s_2 - 7392s_1s_3 + 2112s_2s_3 - 1386s_1s_2^2$$

$$+308s_1s_2^3-22s_1s_2^4-1386s_2^2s_3+308s_2^3s_3-22s_2^4s_3-792s_2^2+616s_2^3-154s_2^4$$
$$+12s_2^5-693s_1s_2^2s_3+44s_1s_2^3s_3+3696s_1s_2s_3)f_{n+3}$$
$$+\frac{h^4s_2^6}{(665280(s_1-2)(s_2-2)(s_3-2))}(3168s_1s_2-11088s_1s_3+3168s_2s_3-1881s_1s_2^2$$
$$+352s_1s_2^3-22s_1s_2^4-1881s_2^2s_3+352s_2^3s_3-22s_2^4s_3-1188s_2^2+836s_2^3-176s_2^4+12s_2^5$$
$$-792s_1s_2^2s_3+44s_1s_2^3s_3+5016s_1s_2s_3)f_{n+2}$$
$$-\frac{h^4s_2^6}{(997920(s_1-1)(s_2-1)(s_3-1))}(6336s_1s_2-22176s_1s_3+6336s_2s_3-2574s_1s_2^2$$

$$+396s_1s_2^3-22s_1s_2^4-2574s_2^2s_3+396s_2^3s_3-22s_2^4s_3-2376s_2^2+1144s_2^3-198s_2^4+12s_2^5$$
$$-891s_1s_2^2s_3+44s_1s_2^3s_3+6864s_1s_2s_3)f_{n+1}$$
$$+\frac{h^4s_2^4}{(166320(s_1-s_2)(s_2-s_3)(s_2-1)(s_2-2)(s_2-3)(s_2-4))}(11088s_1s_2-33264s_1s_3$$
$$+11088s_2s_3-9900s_1s_2^2+3465s_1s_2^3-550s_1s_2^4-9900s_2^2s_3+33s_1s_2^5+3465s_2^3s_3$$
$$-550s_2^4s_3+33s_2^5s_3-4752s_2^2+4950s_2^3-1925s_2^4+330s_2^5-21s_2^6-6930s_1s_2^2s_3+990s_1s_2^3s_3$$
$$-55s_1s_2^4s_3+23100s_1s_2s_3)f_{n+s_2}$$
$$-(h^4s_2^6(6336s_2-22176s_3+13200s_2s_3-3465s_2^2s_3+440s_2^3s_3-22s_2^4s_3-4950s_2^2+1540s_2^3$$
$$-220s_2^4+12s_2^5))/(166320s_1(s_1-s_2)(s_1-s_3)(s_1-1)(s_1-2)(s_1-3)(s_1-4))f_{n+s_1}$$
$$+(h^4s_2^6(22176s_1-6336s_2-13200s_1s_2+3465s_1s_2^2-440s_1s_2^3+22s_1s_2^4+4950s_2^2-1540s_2^3$$
$$+220s_2^4-12s_2^5))/(166320s_3(s_1-s_3)(s_2-s_3)(s_3-1)(s_3-2)(s_3-3)(s_3-4))f_{n+s_3}$$

$$y_{n+2}=y_n++2hy'_n+2h^2y''_n+\frac{4h^3}{3y'''_n}$$
$$+\frac{2h^4(319s_1+319s_2+319s_3-968s_1s_2-968s_1s_3-968s_2s_3+4884s_1s_2s_3-124)}{(31185s_1s_2s_3}f_n$$
$$-\frac{2h^4(33s_1+33s_2+33s_3-88s_1s_2-88s_1s_3-88s_2s_3+264s_1s_2s_3-8)}{(31185(s_1-4)(s_2-4)(s_3-4))}f_{n+4}$$
$$+\frac{8h^4(44s_1+44s_2+44s_3-121s_1s_2-121s_1s_3-121s_2s_3+363s_1s_2s_3-8)}{(31185(s_1-3)(s_2-3)(s_3-3))}f_{n+3}$$
$$-\frac{2h^4(110s_1+110s_2+110s_3-374s_1s_2-374s_1s_3-374s_2s_3+1155s_1s_2s_3+28)}{(10395(s_1-2)(s_2-2)(s_3-2))}f_{n+2}$$
$$+\frac{8h^4(660s_1+660s_2+660s_3-979s_1s_2-979s_1s_3-979s_2s_3+1947s_1s_2s_3-536)}{(31185(s_1-1)(s_2-1)(s_3-1))}f_{n+1}$$
$$-\frac{16fns_1h^4(319s_2+319s_3-968s_2s_3-124)}{(10395s_1(s_1-s_2)(s_1-s_3)(s_1-1)(s_1-2)(s_1-3)(s_1-4))}f_{n+s_1}$$
$$+\frac{16h^4(319s_1+319s_3-968s_1s_3-124)}{(10395s_2(s_1-s_2)(s_2-s_3)(s_2-1)(s_2-2)(s_2-3)(s_2-4))}f_{n+s_2}$$
$$-\frac{16h^4(319s_1+319s_2-968s_1s_2-124)}{(10395s_3(s_1-s_3)(s_2-s_3)(s_3-1)(s_3-2)(s_3-3)(s_3-4))}f_{n+s_3}$$

$$y_{n+s_3}=y_n+hs_3y'_n+\frac{h^2s_3^2}{2}y''_n+\frac{h^3s_3^3}{6}y'''_n$$
$$+\frac{h^4s_3^4}{(3991680s_1s_2}(133056s_1s_2-22176s_1s_3-22176s_2s_3+13200s_1s_3^2-3465s_1s_3^3$$
$$+13200s_2s_3^2+440s_1s_3^4-3465s_2s_3^3-22s_1s_3^5+440s_2s_3^4-22s_2s_3^5+6336s_3^2-4950s_3^3+1540s_3^4$$
$$-220s_3^5+12s_3^6+9240s_1s_2s_3^2-990s_1s_2s_3^3+44s_1s_2s_3^4-46200s_1s_2s_3)f_n$$
$$-\frac{h^4s_3^6}{(3991680(s_1-4)(s_2-4)(s_3-4))}(5544s_1s_2-1584s_1s_3-1584s_2s_3+1089s_1s_3^2$$
$$-264s_1s_3^3+1089s_2s_3^2+22s_1s_3^4-264s_2s_3^3+22s_2s_3^4+594s_3^2-484s_3^3+132s_3^4-12s_3^5$$
$$+594s_1s_2s_3^2-44s_1s_2s_3^3-2904s_1s_2s_3)f_{n+4}$$

$$+\frac{h^4 s_3^6}{(997920(s_1-3)(s_2-3)(s_3-3))}(7392s_1s_2-2112s_1s_3-2112s_2s_3+1386s_1s_3^2$$
$$-308s_1s_3^3+1386s_2s_3^2+22s_1s_3^4-308s_2s_3^3+22s_2s_3^4+792s_3^2-616s_3^3+154s_3^4-12s_3^5$$
$$+693s_1s_2s_3^2-44s_1s_2s_3^3-3696s_1s_2s_3)f_{n+3}$$
$$-\frac{h^4 s_3^6}{(665280(s_1-2)(s_2-2)(s_3-2))}(11088s_1s_2-3168s_1s_3-3168s_2s_3+1881s_1s_3^2$$
$$-352s_1s_3^3+1881s_2s_3^2+22s_1s_3^4-352s_2s_3^3+22s_2s_3^4+1188s_3^2-836s_3^3+176s_3^4-12s_3^5$$
$$+792s_1s_2s_3^2-44s_1s_2s_3^3-5016s_1s_2s_3)f_{n+2}$$
$$+\frac{h^4 s_3^6}{(997920(s_1-1)(s_2-1)(s_3-1))}(22176s_1s_2-6336s_1s_3-6336s_2s_3+2574s_1s_3^2$$
$$-396s_1s_3^3+2574s_2s_3^2+22s_1s_3^4-396s_2s_3^3+22s_2s_3^4+2376s_3^2-1144s_3^3+198s_3^4-12s_3^5$$
$$+891s_1s_2s_3^2-44s_1s_2s_3^3-6864s_1s_2s_3)f_{n+1}$$
$$+\frac{h^4 s_3^4}{(166320(s_1-s_3)(s_2-s_3)(s_3-1)(s_3-2)(s_3-3)(s_3-4))}(33264s_1s_2-11088s_1s_3$$
$$-11088s_2s_3+9900s_1s_3^2-3465s_1s_3^3+9900s_2s_3^2+550s_1s_3^4-3465s_2s_3^3-33s_1s_3^5+550s_2s_3^4$$
$$-33s_2s_3^5+4752s_3^2-4950s_3^3+1925s_3^4-330s_3^5+21s_3^6+6930s_1s_2s_3^2-990s_1s_2s_3^3$$
$$+55s_1s_2s_3^4-23100s_1s_2s_3)f_{n+s_3}$$
$$+(h^4s_3^6(22176s_2-6336s_3-13200s_2s_3+3465s_2s_3^2-440s_2s_3^3+22s_2s_3^4+4950s_3^2-1540s_3^3$$
$$+220s_3^4-12s_3^5)/(166320s_1(s_1-s_2)(s_1-s_3)(s_1-1)(s_1-2)(s_1-3)(s_1-4))f_{n+s_1}$$
$$-(h^4s_3^6(22176s_1-6336s_3-13200s_1s_3+3465s_1s_3^2-440s_1s_3^3+22s_1s_3^4+4950s_3^2-1540s_3^3$$
$$+220s_3^4-12s_3^5)/(166320s_2(s_1-s_2)(s_2-s_3)(s_2-1)(s_2-2)(s_2-3)(s_2-4))f_{n+s_2}$$

$$y_{n+3}=y_n+3hy'_n+\frac{9h^2}{2}y''_n+\frac{9h^3}{2}y'''_n$$
$$+\frac{27h^4(33s_1s_2+33s_1s_3+33s_2s_3-121s_1s_2s_3-18)}{(49280(s_1-4)(s_2-4)(s_3-4))}f_{n+4}$$
$$-\frac{27h^4(44s_1s_2+44s_1s_3+44s_2s_3-165s_1s_2s_3-18)}{(12320(s_1-3)(s_2-3)(s_3-3))}f_{n+3}$$
$$+\frac{27h^4(132s_1+132s_2+132s_3-451s_1s_2-451s_1s_3-451s_2s_3+2167s_1s_2s_3-18)}{(49280s_1s_2s_3}f_n$$
$$-\frac{81h^4(33s_1s_2-198s_2-198s_3-198s_1+33s_1s_3+33s_2s_3+209s_1s_2s_3+414)}{(24640(s_1-2)(s_2-2)(s_3-2))}f_{n+2}$$
$$+\frac{27h^4(594s_1+594s_2+594s_3-726s_1s_2-726s_1s_3-726s_2s_3+1177s_1s_2s_3-576)}{(12320(s_1-1)(s_2-1)(s_3-1))}f_{n+1}$$
$$-\frac{81fns_1h^4(132s_2+132s_3-451s_2s_3-18)}{(6160s_1(s_1-s_2)(s_1-s_3)(s_1-1)(s_1-2)(s_1-3)(s_1-4))}f_{n+S_1}$$
$$+\frac{81h^4(132s_1+132s_3-451s_1s_3-18)}{(6160s_2(s_1-s_2)(s_2-s_3)(s_2-1)(s_2-2)(s_2-3)(s_2-4))}f_{n+s_2}$$
$$-\frac{81h^4(132s_1+132s_2-451s_1s_2-18)}{(6160s_3(s_1-s_3)(s_2-s_3)(s_3-1)(s_3-2)(s_3-3)(s_3-4))}f_{n+s_3}$$

$$y_{n+4}=y_n+4hy'_n+8h^2y''_n+\frac{32h^3}{3}y'''_n$$
$$+\frac{32h^4(44s_1s_2+44s_1s_3+44s_2s_3-165s_1s_2s_3-32)}{(31185(s_1-4)(s_2-4)(s_3-4))}f_{n+4}$$
$$+\frac{256h^4(22s_1+22s_2+22s_3-77s_1s_2-77s_1s_3-77s_2s_3+363s_1s_2s_3-4)}{(31185s_1s_2s_3}f_n$$
$$+\frac{1024h^4(44s_1+44s_2+44s_3-22s_1s_2-22s_1s_3-22s_2s_3+33s_1s_2s_3-128)}{(31185(s_1-3)(s_2-3)(s_3-3))}f_{n+3}$$
$$+\frac{1024h^4(132s_1+132s_2+132s_3-154s_1s_2-154s_1s_3-154s_2s_3+231s_1s_2s_3-128)}{(31185(s_1-1)(s_2-1)(s_3-1))}f_{n+1}$$
$$-\frac{256h^4(88s_1s_2-220s_2-220s_3-220s_1+88s_1s_3+88s_2s_3+33s_1s_2s_3+448)}{(10395(s_1-2)(s_2-2)(s_3-2))}f_{n+2}$$

$$
-\frac{2048h^4(22s_2+22s_3-77s_2s_3-4)}{(10395s_1(s_1-s_2)(s_1-s_3)(s_1-1)(s_1-2)(s_1-3)(s_1-4))}f_{n+s_1}
+\frac{2048h^4(22s_1+22s_3-77s_1s_3-4)}{(10395s_2(s_1-s_2)(s_2-s_3)(s_2-1)(s_2-2)(s_2-3)(s_2-4))}f_{n+s_3}
-\frac{2048h^4(22s_1+22s_2-77s_1s_2-4)}{(10395s_3(s_1-s_3)(s_2-s_3)(s_3-1)(s_3-2)(s_3-3)(s_3-4))}f_{n+s_3}
$$

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