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Original article

Robust estimators for circular regression models

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ABSTRACT

The problem of robust estimation in circular regression models has not been studied well. This paper considers the *JS* circular regression model due to its interesting properties and its sensitivity for existence and detection of outliers. We extend the robust estimators such as *M*-estimation, least-trimmed squares (*LTS*), and least-median squares (*LMS*) estimators, which have been successfully used in the linear regression models, to the *JS* circular regression model.

The robustness of the proposed estimators are studied through its influence function, and via simulation study. The results show that the proposed robust circular *M*-estimation is effective in estimating circular models' parameters in the presence of vertical outliers. However, circular *LTS* and *LMS* are highly robust estimators in case of circular leverage points. An application of the proposed robust circular estimators is illustrated using a real eye data set.

1. Introduction

The applications on circular variables have increased in last two decades. It varied in many fields including biology, meteorology and medicine. Although the first circular regression model backs to Gould, (1969) and various versions of these models have been proposed; the study of outliers and robustness of circular regression models still not well considered. Most of outliers' detection procedures were derived based on the simple circular regression model (Hussin et al., 2004) by extending the common methods from linear regression (Abuzaid et al., 2008, 2013). Model (Hussin et al., 2004) assumed a linear relationship between the two circular variables, which is a conservative condition, moreover, it is not applicable to be extended for multiple regression settings. Alternatively, Ibrahim (2013) investigated the robustness of the *JS* model which is proposed by Sarma and Jammalamadaka, (1993) with one independent circular variable based on the least squares estimation (*LS*), and some outliers' detection procedures were proposed by Ibrahim et al. (2013). Moreover, Alkasadi et al. (2019) derived an outlier detection procedure for the multiple *JS* model with two independent circular variables. Recently, Jha and Biswas (2017) have studied the robustness of Kato et al. (2008) circular regression model based on wrapped Cauchy distribution settings by proposing the maximum trimmed cosine estimator.

Robust estimation methods received a great deal of interest to improve estimator performance in linear regression models. These estimators limit the influence of outliers. In this regard, Huber and Lovric (2011), Hampel et al. (2011) and Birkes and Dodge (2011) showed that the robust M-estimation is highly robust to vertical, but leverage point can break them down completely. Several robust alternatives have been investigated in the literature, among those, least median squares (*LMS*) estimator Rousseeuw (1984) and least trimmed of squares (*LTS*) introduced in Rousseeuw and Leroy (2005), which are not much affected with leverage points.

This study is proposing *M* estimator and high breakdown point estimators, *LTS* and *LMS* for *JS* circular regression, to reduce the effect of vertical outliers and leverage point.

The rest of the article is organized as follows: Section 2 reviews the formulation of the *JS* circular regression model, and its *LS* parameters estimates. Section 3 formulates the effect of outliers in the *JS* model. Section 4 proposes the robust *M*-estimators, studies the influence function for the proposed estimators and introduces bounded influence of the *JS* circular regression estimators. An extensive simulation study is conducted to study the perfor-

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mance of the proposed robust estimators in Section 5. Section 6 applies the robust estimators to the eye data set.

2. The *JS* circular regression model

2.1. Model formulation

For any two circular random variables $\mathbf{U}$ and $\mathbf{V}$, Sarma and Jammalamadaka (1993) proposed a regression model to predict v for a given u, by considering the conditional expectation of the vector e^{iv} given u such that

$$E(e^{iv}|u) = \rho(u)e^{i\mu(u)} = g_1(u) + ig_2(u), \tag{1}$$

where, $e^{iv} = \cos v + i\sin v, \mu(u)$ represents the conditional mean direction of v given u and $\rho(u)$ represents the conditional concentration parameter. Equivalently, we may write

$$E(\cos v|u) = g_1(u) \quad \text{and} \quad E(\sin v|u) = g_2(u). \tag{2}$$

Then, v can be predicted such that

$$\mu(u) = \hat{v} = \arctan\frac{g_2(u)}{g_1(u)} = \begin{cases} \arctan\frac{g_2(u)}{g_1(u)} & \text{if } g_1(u) \geqslant 0, \\ \pi + \arctan\frac{g_2(u)}{g_1(u)} & \text{if } g_1(u) \leqslant 0, \\ \textit{undefined} & \text{if } g_1(u) = g_2(u) = 0. \end{cases} \tag{3}$$

Due to the fact that $g_1(u)$ and $g_2(u)$ are periodic functions, thus they are approximated for a suitable degree m (Kufner and Kadlec, 1971), which have the following two observational regression-like models

$$V_{1j} = \cos v_j = g_1(u) \simeq \sum_{k=0}^{m}(A_k\cos ku_j + B_k\sin ku_j) + \varepsilon_{1j}$$

$$\text{and} \quad V_{2j} = \sin v_j = g_2(u) \simeq \sum_{k=0}^{m}(C_k\cos ku_j + D_k\sin ku_j) + \varepsilon_{2j}. \tag{4}$$

for $j = 1,\ldots,n$, where, $\boldsymbol{\varepsilon} = (\varepsilon_1,\varepsilon_2)$ is the vector of random errors following the bivariate normal distribution with mean vector $\mathbf{0}$ and unknown dispersion matrix $\boldsymbol{\Sigma}$. The parameters A_k, B_k, C_k, and $D_k, k = 0,1,\ldots,m$, the standard errors as well as the matrix $\boldsymbol{\Sigma}$ can then be estimated, by assuming that $B_0 = D_0 = 0$ to ensure model's identifiability.

2.2. Least squares estimation

Let $(u_1,v_1),\ldots,(u_n,v_n)$ be a random circular sample of size n. Therefore, the observational Eqs. (4) can be summarized as

$$\mathbf{V}^{(1)} = (V_{11},\ldots,V_{1n})', \mathbf{V}^{(2)} = (V_{21},\ldots,V_{2n})',$$

$$\boldsymbol{\varepsilon}^{(1)} = (\varepsilon_{11},\ldots,\varepsilon_{1n}), \boldsymbol{\varepsilon}^{(2)} = (\varepsilon_{21},\ldots,\varepsilon_{2n}). \tag{5}$$

$$\mathbf{U}_{n\times(2m+1)} = \begin{bmatrix} 1 & \cos u_1 & \cdots & \cos mu_1 & \sin u_1 & \cdots & \sin mu_1 \\ 1 & \cos u_2 & \cdots & \cos mu_1 & \sin u_2 & \cdots & \sin mu_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cos u_n & \cdots & \cos mu_n & \sin u_n & \cdots & \sin mu_n \end{bmatrix}, \tag{6}$$

$$\lambda^{(1)} = (A_0,A_1,\ldots,A_m,B_1,\ldots,B_m)' \text{ and } \quad \lambda^{(2)} = (C_0,C_1,\ldots,C_m,D_1,\ldots,D_m)'. \tag{7}$$

The observational Eqs. (4) can be written in matrix form

$$\mathbf{V}^{(1)} = \mathbf{U}\boldsymbol{\lambda}^{(1)} + \boldsymbol{\varepsilon}^{(1)} \text{and} \quad \mathbf{V}^{(2)} = \mathbf{U}\boldsymbol{\lambda}^{(2)} + \boldsymbol{\varepsilon}^{(2)}. \tag{8}$$

The least squares estimates turn out to be

$$\hat{\lambda}^{(1)} = \min\sum_{i=1}^{n}\left(\mathbf{V}_i^{(1)} - \mathbf{U}\lambda^{(1)}\right)^2 \text{and}$$

$$\hat{\lambda}^{(2)} = \min\sum_{i=1}^{n}\left(\mathbf{V}_i^{(2)} - \mathbf{U}\lambda^{(2)}\right)^2. \tag{9}$$

These equations can be combined into the following single matrices

$$\hat{\lambda}^{(1)} = (\mathbf{U}'\mathbf{U})^{-1}\mathbf{U}'\mathbf{V}^{(1)} \quad \text{and} \quad \hat{\lambda}^{(2)} = (\mathbf{U}'\mathbf{U})^{-1}\mathbf{U}'\mathbf{V}^{(2)} \tag{10}$$

The following section explains the effect of outliers on the *JS* circular regression model.

3. Outliers in the *JS* circular regression model

Outliers are a common problem in the statistical analysis. It is defined as observations that are very different to the other observations in a set of data. Ibrahim (2013) investigated the robustness of the *JS* model by simulation study, and concluded that *JS* model is sensitive for outliers existence, and the presence of outliers has potentially serious effects on *LS* estimation. Then Ibrahim et al. (2013) proposed a *COVRATIO* statistic to define outliers in the y-vertical. In this paper, we define two types of outliers, namely outliers in $\mathbf{V}$, it is so called (circular vertical outliers), and outliers with respect to $\mathbf{U}$ and it is so called (circular leverage points). The effect of outliers on *LS* estimation is introduced by two ways:

1.*Circular Vertical Outliers*: if $\mathbf{V}_{1j}$ and $\mathbf{V}_{2j}$ are replaced by $\mathbf{V}_{1j}^{\star}$ and $\mathbf{V}_{2j}^{\star}$, respectively, where, $\mathbf{V}_{1j}^{\star} = Z_1\mathbf{V}_{1j}$ and $\mathbf{V}_{2j}^{\star} = Z_2\mathbf{V}_{2j}$, which implies $\mathbf{V}_{1j} = Z_1^{-1}\mathbf{V}_{1j}^{\star}$ and $\mathbf{V}_{2j} = Z_2^{-1}\mathbf{V}_{2j}^{\star}$, then the circular regression in (4) can be rewritten as follows:

$$Z_1^{-1}\mathbf{V}_{1j}^{\star} = \cos v_j = \sum_{k=0}^{m}(A_k\cos ku_j + B_k\sin ku_j) + \varepsilon_{1j}$$

$$\text{and} \quad Z_2^{-1}\mathbf{V}_{2j}^{\star} = \sin v_j = \sum_{k=0}^{m}(C_k\cos ku_j + D_k\sin ku_j) + \varepsilon_{2j}. \tag{11}$$

Thus, $\hat{\lambda}^{(1)}\left(\mathbf{U},\mathbf{V}_{1j}^{\star}\right) = Z_1^{-1}\hat{\lambda}^{(1)}(\mathbf{U},\mathbf{V}_{1j})$, and $\hat{\lambda}^{(2)}\left(\mathbf{U},\mathbf{V}_{2j}^{\star}\right) = Z_2^{-1}\hat{\lambda}^{(2)}(\mathbf{U},\mathbf{V}_{2j})$.

2.*Circular Leverage Points*: if $\mathbf{U}$ is replaced by $\mathbf{U}^{\star}$, where $\mathbf{U}^{\star} = Z\mathbf{U}$, then $\hat{\lambda}^{(1)}(\mathbf{U}^{\star},\mathbf{V}_{1j}) = Z^{-1}\hat{\lambda}^{(1)}(\mathbf{U},\mathbf{V}_{1j})$, and $\hat{\lambda}^{(2)}(\mathbf{U}^{\star},\mathbf{V}_{2j}) = Z^{-1}\hat{\lambda}^{(2)}(\mathbf{U},\mathbf{V}_{2j})$.

The following section derives robust estimators of the *JS* model parameters.

4. Robust estimation of the *JS* circular regression parameters

In this section the robust estimation is extended to the *JS* circular regression model instead of the classical *LS* estimator defined in (10) to improve estimation precision. We will use *CM*, *CLTS* and *CLMS* abbreviations for circular *M*-estimation, circular least-trimmed squares, and circular least-median squares,respectively, as derived in the following subsections:

4.1. Robust *CM*-estimation of *JS* circular regression parameters

In Eq. (9), if $\left(\mathbf{V}_i - \mathbf{U}\lambda^{(p)}\right)^2$, where $p = 1,2$, has been replaced by $F(\mathbf{V}_i - \mathbf{U}\lambda^{(p)})$, where, F is symmetric, non decreasing function on $[0,\infty)$, and almost continuously differentiable anywhere, where $F(0) = 0$. Furthermore, F is a function, which is less sensitive to outliers than squares, then it yields an estimating equation, which result is the same idea of *M*-estimation of a linear regression model as described by Huber and Lovric (2011).

We define *CM*-estimates on the *JS* circular regression as follows:

$$\hat{\boldsymbol{\lambda}}^{(1)} = \min\sum_{i=1}^{n} F\left(\mathbf{V}_i^{(1)} - \mathbf{U}\boldsymbol{\lambda}^{(1)}\right) \quad \text{and}$$

$$\hat{\boldsymbol{\lambda}}^{(2)} = \min\sum_{i=1}^{n} F\left(\mathbf{V}_i^{(2)} - \mathbf{U}\boldsymbol{\lambda}^{(2)}\right). \tag{12}$$

To solve these equations, let the influence curve be $\psi = F'$, if this exists then we will have:

$$\sum_{i=1}^{n}\psi\left(\mathbf{V}_i^{(1)} - \mathbf{U}\boldsymbol{\lambda}^{(1)}\right) = 0 \quad \text{and} \quad \sum_{i=1}^{n}\psi\left(\mathbf{V}_i^{(2)} - \mathbf{U}\boldsymbol{\lambda}^{(2)}\right) = 0. \tag{13}$$

Whereas, if $F(.) = (\mathbf{V} - \mathbf{U}\lambda)^2$, then the solution becomes the *LS* estimate. Generally, Biweight or Huber functions have been widely used as $F(.)$. However, for Biweight function we define the weight matrix $W = diag(w)$ with $w = \frac{\psi(\varepsilon)}{\varepsilon}$ then (13) can be reformulated as

$$W^{(1)}\left(\mathbf{V}^{(1)} - \mathbf{U}\lambda^{(1)}\right) = 0 \quad \text{and} \quad W^{(2)}\left(\mathbf{V}^{(2)} - \mathbf{U}\lambda^{(2)}\right) = 0, \tag{14}$$

where,

$$W^{(1)} = \frac{\psi\left(\mathbf{V}^{(1)} - \mathbf{U}\lambda^{(1)}\right)}{\left(\mathbf{V}^{(1)} - \mathbf{U}\lambda^{(1)}\right)} \quad \text{and} \quad W^{(2)} = \frac{\psi\left(\mathbf{V}^{(2)} - \mathbf{U}\lambda^{(2)}\right)}{\left(\mathbf{V}^{(2)} - \mathbf{U}\lambda^{(2)}\right)}. \tag{15}$$

Then (13) can be written as:

$$\sum_{i=1}^{n} W^{(1)} F\left(\mathbf{V}_i^{(1)} - \mathbf{U}\lambda^{(1)}\right)\mathbf{U} \quad \text{and} \quad \sum_{i=1}^{n} W^{(2)} F\left(\mathbf{V}_i^{(2)} - \mathbf{U}\lambda^{(2)}\right)\mathbf{U}. \tag{16}$$

These equations can be combined into the following single matrices:

$$\mathbf{U}^T W^{(1)}\mathbf{U}\lambda^{(1)} = \mathbf{U}^T W^{(1)}\mathbf{V}^{(1)} \quad \text{and} \quad \mathbf{U}^T W^{(2)}\mathbf{U}\lambda^{(2)} = \mathbf{U}^T W^{(2)}\mathbf{V}^{(2)}, \tag{17}$$

therefore, the robust estimators are given by

$$\hat{\lambda}^{(1)} = \left(\mathbf{U}^T W^{(1)}\mathbf{U}\right)^{-1}\mathbf{U}^T W^{(1)}\mathbf{V}^{(1)} \quad \text{and}$$

$$\hat{\lambda}^{(2)} = \left(\mathbf{U}^T W^{(2)} U\right)^{-1}\mathbf{U}^T W^{(2)}\mathbf{V}^{(2)}, \tag{18}$$

where, $W^{(1)}$ and $W^{(2)}$ are $n \times n$ matrix of weight matrix. The covariance matrix of $\hat{\lambda}^{(p)}$ is

$$\Sigma = \frac{E(\psi^2)}{[E(\psi')]^2}(\mathbf{U}'\mathbf{U})^{-1}.$$

A popular possibility of $F(.)$ is to use the Huber's function as introduced in Huber and Lovric (2011). For a positive real $M = 1.345$, Huber introduced the following objective function

$$F(\varepsilon) = \begin{cases} \varepsilon^2, & \text{for} |\varepsilon| \leqslant M, \\ 2M|\varepsilon| - M^2, & \text{for} |\varepsilon| > M. \end{cases} \tag{19}$$

So the ψ is given as

$$\psi(\varepsilon) = \begin{cases} 2\varepsilon, & \text{for} |\varepsilon| \leqslant M, \\ 2Msgn(\varepsilon), & \text{for} |\varepsilon| > M. \end{cases} \tag{20}$$

Then the weight function W is given by:

$$W(\varepsilon) = \begin{cases} 1, & \text{for} |\varepsilon| \leqslant M, \\ \frac{M}{|\varepsilon|}, & \text{for} |\varepsilon| > M. \end{cases} \tag{21}$$

4.2. Influence Function of Circular Regression Estimators

Let T be an estimator of ψ- type, then the influence function (*IF*) describes the effect of an infinitesimal contamination at $\mathbf{U}$ on the estimator T, and it is defined as:

$$IF(\mathbf{U}, \mathbf{V}; T, G) = \frac{-\psi(\mathbf{U}, T(G))}{\int \left[\frac{\partial\psi(\mathbf{V}, \lambda^{(p)})}{\partial\lambda^{(p)}}\right] f(\mathbf{U}, \mathbf{V}) dy}, \tag{22}$$

where, $f(\mathbf{U}, \mathbf{V})$ is the density function. That is, the *IF* of $T = LS$ circular estimate, and it is given by:

$$IF(\mathbf{U}, \mathbf{V}; LS, G) = \frac{-2\mathbf{U}\left(\mathbf{V} - \mathbf{U}'\lambda^{(p)}\right)}{\int \psi'\left(\mathbf{V} - \mathbf{U}'\lambda^{(p)}\right) f(\mathbf{U}, \mathbf{V}) d(\mathbf{U}, \mathbf{V})}$$

$$= \frac{-2\mathbf{U}\left(\mathbf{V} - \mathbf{U}'\lambda^{(p)}\right)}{\int \psi'\left(\mathbf{V} - \mathbf{U}'\lambda^{(p)}\right) f(\mathbf{U}, \mathbf{V}) d(\mathbf{U}, \mathbf{V})} \cdot \frac{\mathbf{U}\mathbf{U}'}{\mathbf{U}\mathbf{U}'}$$

$$= \frac{-2\mathbf{U}\mathbf{U}'\mathbf{U}\left(\mathbf{V} - \mathbf{U}'\lambda^{(p)}\right)}{E(\psi(\varepsilon)\varepsilon)}. \tag{23}$$

It worth to remark that, *IF* of circular *LS* estimator is an unbounded function in $\mathbf{U}$ and $\mathbf{V}$. On the other hand, the influence function of robust *CM* estimator is given by:

$$IF(\mathbf{U}, \mathbf{V}; M, G) = \frac{\mathbf{U}\mathbf{U}'\psi\left(\mathbf{V} - \mathbf{U}'\lambda^{(p)}\right)}{E(\psi(\varepsilon)\varepsilon)}, \tag{24}$$

where, ψ is a bounded function. Thus, the *IF* of *CM* estimator is bounded with respect to vertical $\mathbf{V}$, but its unbounded with respect to leverage $\mathbf{U}$.

4.3. Bounded-influence circular estimator

The *CM* estimators are sensitive to circular leverage observations so, we propose a bounded-influence circular estimator named robust *CLTS* estimator. By ordering the squared residuals ε_1^2 and ε_2^2 ascendingly:

$$\varepsilon^2_{1(1)}, \varepsilon^2_{1(2)}, \ldots, \varepsilon^2_{1(n)} \quad \text{and} \quad \varepsilon^2_{2(1)}, \varepsilon^2_{2(2)}, \ldots, \varepsilon^2_{2(n)}.$$

Then the *CLTS* circular estimator choose the circular coefficients $\hat{\lambda}^{(1)}$ and $\hat{\lambda}^{(2)}$ which minimize the sum of the smallest $\frac{n}{2}$ of the squared residuals,

$$CLTS\left(\hat{\lambda}^{(1)}\right) = \sum_{i=1}^{\frac{n}{2}}\left(e^{(1)}\right)_{(i)} \quad \text{and} \quad CLTS\left(\hat{\lambda}^{(2)}\right) = \sum_{i=1}^{\frac{n}{2}}\left(e^{(2)}\right)_{(i)},$$

which is equivalent to find the circular estimates corresponding to the half circular sample having the smallest sum of squares of residuals. As such, breakdown point is 50%. Replacing $\frac{n}{2}$ by$[n + (2m+1) + 1/2]$, we get a robust *CLMS* estimator.

The following section investigates the performance of the proposed robust estimators via simulation.

5. Simulation study

5.1. Settings

A simulation study was carried out to investigate the performance of the proposed robust estimators for the *JS* circular regression model, namely *CM*, *CLTS*, and *CLMS*. Furthermore, to compare

Table 1
Simulation results of different estimators with no contamination data.

		Estimators			
		LS	*CM*	*CLTS*	*CLMS*
$n = 20$	median *MSE*	**3.0700**	3.0996	3.4295	3.4159
	median SE	6.8847	6.9177	7.2380	7.2353
	median $A(\kappa)$	0.9979	0.9978	0.9879	0.9890
$n = 50$	median *MSE*	**3.06375**	3.1055	3.3815	3.3915
	median *SE*	2.7494	2.7686	2.8726	2.8765
	median $A(\kappa)$	0.9975	0.9976	0.9853	0.9848
$n = 100$	median *MSE*	**3.0341**	3.0780	3.3166	3.3301
	median SE	1.3673	1.3775	1.4220	1.4241
	median $A(\kappa)$	0.9975	0.9976	0.9850	0.9837

Table 2
Simulation results of different estimators, with different percentages of vertical outliers.

		Estimators			
		LS	*CM*	*CLTS*	*CLMS*
10% vertical					
$n = 20$	median *MSE*	4.1001	**3.3024**	3.3155	3.3664
	median SE	7.9386	7.1251	7.1456	7.1705
	median $A(\kappa)$	0.9061	0.9977	0.9899	0.9888
$n = 50$	median *MSE*	4.3319	**3.3278**	3.2751	3.3469
	median *SE*	3.2712	2.8444	2.8511	2.8591
	median $A(\kappa)$	0.9051	0.9975	0.9866	0.9871
$n = 100$	median *MSE*	4.4382	**3.2669**	3.3229	3.2979
	median *SE*	1.6572	1.4200	1.4233	1.4187
	median $A(\kappa)$	0.9027	0.9975	0.9849	0.9866
20% vertical					
$n = 20$	median *MSE*	4.4997	**3.3534**	3.3750	3.5734
	median *SE*	8.2735	7.1689	7.2018	7.4142
	median $A(\kappa)$	0.8198	0.9974	0.9914	0.9892
$n = 50$	median *MSE*	5.0477	**3.5431**	3.3549	3.3567
	median *SE*	3.5196	2.8652	2.8656	2.9596
	median $A(\kappa)$	0.8114	0.9975	0.9878	0.9872
$n = 100$	median *MSE*	5.3299	**3.3107**	3.3394	3.5284
	median *SE*	1.8108	1.4230	1.4278	1.4769
	median $A(\kappa)$	0.8109	0.9975	0.9875	0.9857
30% vertical					
$n = 20$	median *MSE*	4.8460	**3.3397**	3.3416	4.1199
	median *SE*	8.5369	7.1550	7.1677	7.9297
	median $A(\kappa)$	0.7392	0.9950	0.9936	0.9907
$n = 50$	median *MSE*	5.3271	**3.3031**	3.3244	4.2309
	median *SE*	3.5931	2.8447	3.2352	2.8531
	median $A(\kappa)$	0.7299	0.9971	0.9899	0.9900
$n = 100$	median *MSE*	5.4952	**3.2587**	3.3031	4.1668
	median *SE*	1.8271	1.4122	1.4211	1.6069
	median $A(\kappa)$	0.7214	0.9974	0.9898	0.9881

these estimators with classical estimator *LS*. For simplicity, we consider the case when $m = 1$. Hence, we have the following set of parameters to be estimated:

$$\boldsymbol{\lambda} = \left(\boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}\right) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$$

$$= (A_0, A_1, B_1, C_0, C_1, D_1). \tag{25}$$

We consider the set of uncorrelated random errors $(\varepsilon_1, \varepsilon_2)$ from the bivariate Normal distribution with mean vector **0** and variances $\left(\sigma_1^2, \sigma_2^2\right)$ to be (0.03,0.03). The independent circular variable **V** is generated from von Mises distribution with mean π and concentration parameter equals 2 i.e. $vM(\pi, 2)$.

For simplicity, we set the true values of A_0 and C_0 of the *JS* model to be 0, while A_1, B_1, C_0 and D_1 are obtained by using the standard additive trigonometric polynomial equations $\cos(a + u)$ and $\sin(a + u)$ when $a = 2$. For example, $\cos(2 + u) = -0.0416 \cos u - 0.9093 \sin u$ and $\sin(2 + u) = 0.9093 \cos u - 0.04161 \sin u$. Then by comparison with Eq. (4), the true values of A_1, B_1, C_1 and D_1 are -0.0 4161, -0.09093, 0.09093 and -0.04161 respectively. Similarly, we can also get different sets of true values by choosing different values of *a*.

We then introduce vertical and leverage outliers into the data such that the percentages of contamination used are c%= 5%, 10%, 20%, 30%, 40% and 50% from the different sample sizes, namely *n* = 20, 50 and 100.

To investigate the robustness of the estimators against vertical and leverage circular outliers, the following scenarios were considered:

Table 3
Simulation results of different estimators, with different percentages of leverage points.

		Estimators			
		LS	*CM*	*CLTS*	*CLMS*
10% leverage					
$n = 20$	median *MSE*	3.3900	3.3715	**2.7337**	**2.8885**
	median *SE*	7.2103	7.1795	6.3544	6.6290
	median $A(\kappa)$	0.8364	0.8401	0.9328	0.8995
$n = 50$	median *MSE*	3.3310	3.3289	**2.7125**	**2.8507**
	median *SE*	2.8518	2.8517	2.5201	2.6391
	median $A(\kappa)$	0.8401	0.8409	0.8940	0.8704
$n = 100$	median *MSE*	3.3235	3.3213	**2.6144**	**2.8353**
	median *SE*	1.4229	1.4243	1.2460	1.3141
	median $A(\kappa)$	0.8351	0.8370	0.9070	0.8687
20% leverage					
$n = 20$	median *MSE*	3.3816	3.3181	**2.7845**	**2.8622**
	median *SE*	7.2014	7.1410	6.4384	6.4432
	median $A(\kappa)$	0.7103	0.7108	0.8748	0.8652
$n = 50$	median *MSE*	3.3159	3.3115	**2.6904**	**2.8177**
	median *SE*	2.8488	2.8500	2.5284	2.5409
	median $A(\kappa)$	0.6989	0.6991	0.8485	0.8415
$n = 100$	median *MSE*	3.2980	3.2661	**3.1258**	**3.0405**
	median *SE*	1.4186	1.4138	1.3575	1.3629
	median $A(\kappa)$	0.7788	0.7812	0.8688	0.8540
30% leverage					
$n = 20$	median *MSE*	3.3077	3.2467	**2.7804**	**2.9814**
	median *SE*	7.1256	7.0743	6.3662	6.5228
	median $A(\kappa)$	0.5691	0.5767	0.8287	0.8298
$n = 50$	median *MSE*	3.2844	3.2708	**2.7492**	**2.9900**
	median *SE*	2.8401	2.8333	2.4930	2.6046
	median $A(\kappa)$	0.5655	0.5671	0.8044	0.8016
$n = 100$	median *MSE*	3.3032	3.2740	**2.9956**	**3.0928**
	median *SE*	1.4228	1.4173	1.3373	1.3492
	median $A(\kappa)$	0.7029	0.7047	0.8233	0.8346

1. No contamination
2. Vertical (outliers in the **V** only)
3. Leverage points (Outliers in some **U** only).

For vertical outliers scenario, the observation at position *d*, say v_d, is contaminated as follows; $v_d^{\star} = v_d + \gamma\pi \quad mod(2\pi)$, where $v_d^{\star}$ is the value after contamination and γ is the degree of contamination in the range $0 \leqslant \gamma \leqslant 1$. The generated data of **U** and **V** are then fitted by the *JS* circular regression model to give the estimates of $\widehat{A}_0, \widehat{A}_1, \widehat{B}_0, \widehat{B}_1, \widehat{C}_1$, and $\widehat{D}_1$.

For leverage point scenario, different percentages of observation at position *d*, say $u_d \sim vM(2\pi, 6)$ instead of the original generated data from $vM(\pi, 2)$. The performance of the proposed estimators were then determined by assessing summary of three statistics based on $s = 1000$ Monte Carlo trials.

The first statistic is the median of standard error (*SE*) of the six parameters and it is obtained by

$$SE\left(\hat{\lambda}_j\right) = \sqrt{\frac{\sum_{j=1}^{s}\left(\hat{\lambda}_{i,j} - \bar{\lambda}_i\right)^2}{s}}, i = 1, 2, \ldots, 6$$

where $\bar{\lambda}$ is the mean of the estimates which is obtained by, $\frac{\sum_{j=1}^{s}\hat{\lambda}_{i,j}}{s}$. The second statistic is the median of mean errors of the estimators given by, $\frac{\varepsilon_{i,j}}{s}$. Finally, the median of mean of the cosines of the circular residuals $A(\kappa)$.

The simulations were performed by the statistical software *R*. To run the simulation, the function *rlm*, *ltsreg* and *lmsreg* from library *MASS* were used for *M*-estimation, *LTS* and *LMS*, respectively.

5.2. Results and discussion

Table 1 shows that the median (*MSE*) of the *LS* is relatively smaller than other estimators when the data are uncontaminated, So the *LS* gave the best estimator.

Table 2 shows the results for contaminated data with vertical outliers, where the *MSE* for *CM* was the smallest, and estimated the associated $A(\kappa)$ were larger than others estimators. Thus, we concluded that the robust *CM* is better than *LS*. The *CLTS* and *CLMS* performance are almost the same.

According to Table 3, *CLTS* and *CLMS* perform better than all the other estimators. They estimated models parameters with smallest *MSE*, but suffer from small values of $A(\kappa)$ when the leverage percentages are increased in the data set. The *CM* does poorly as worse as *LS*, and has higher median (*MSE*) than other robust estimators.

6. Practical example (Eye Data)

As an application of the proposed robust estimators, we consider the eye data which are consisting of 23 observations. The selected measurements are the angle of the posterior corneal curvature (u) and the angle of the eye (between posterior corneal curvature to iris) (v).

The Mean Circular Error(*MCE*) statistic was applied on the data after fitting the *JS* model (Ibrahim, 2013), and they showed that there are two vertical outliers with observation numbers 2 and

Table 4
Results of fitting the *JS* circular regression model for eye data.

	Estimators			
Parameters	*LS*	*CM*	*CLTS*	*CLMS*
$\widehat{A}_0$	1.0821	1.0516	1.2020	1.2450
$\widehat{A}_1$	−0.1497	−0.1579	−1.1748	−0.1834
$\widehat{B}_1$	−0.3836	−0.3383	−0.4383	−0.4748
$\widehat{C}_0$	0.0986	0.0855	−1.4844	81.3559
$\widehat{C}_1$	0.2533	0.2711	−0.0555	−0.0752
$\widehat{D}_1$	0.5935	0.6125	2.1307	2.0032
SSE	5.7828	4.9763	3.7041	**3.6986**
$A(\kappa)$	0.9775	0.9863	0.9249	0.9288

15 were identified. The interest here is to compare the fitted model using different estimators and to check the $SSE = \sum(SE)$.

The results based on classical *LS* and robust estimators are reported in Table 4. The *SSE* for *LS* is the largest (SSE_{LS}=5.7828) compare to other estimators. Thus, the *CLMS* is the superior estimator.

7. Conclusion

This paper has revisited the *JS* circular regression model by deriving a set of robust estimators including *M*, *LTS* and *LMS* estimators to improve the robustness of the *LS* estimator. Simulation results and the application on real data clearly show that robust circular estimators perform better than the classical estimator mentioned earlier. Thus, it is recommend to obtain robust estimators for other circular regression models to increase the accuracy of its predictability.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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