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Original article

Effects of a quantity-based discount frame in inventory planning under time-dependent demand: A case study of mango businesses in Bangladesh

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ABSTRACT

This work studies an inventory planning problem for a practitioner selling a non-immediate perishable item after purchasing it from a supplier within a quantity-based discount frame and assesses the impact of the discount frame on the practitioner's costs. Due to perishability, consumption behavior is directly interconnected with the storage time of the items. The foremost objective of this work is to provide, for the first time, proper inventory planning to minimize the practitioner's total cost under a quantity-based discount frame when demand changes with the product's storage time. To make this inventory planning problem more realistic, both zero-ending and stock-out situations are investigated. Under both scenarios, the optimal inventory plan for the practitioner is examined analytically, and then, integrating the analytical results, two solution algorithms are presented under the quantity-based discount frame. In order to verify the investigated inventory problem and scrutinize the efficiency of the algorithms as well, a real example of a mango business is inspected. Several insights are gained for the practitioner by running a sensitivity analysis on the optimal inventory plan against the system input parameters. The results suggest that the practitioner can take advantage of a reduced acquisition price by making a larger number of purchases under the order-based rebate program, greatly reducing its cost.

1. Introduction

The profitability of a company relies heavily on effective stock volume management through proper inventory planning to minimize costs while meeting client needs. However, perishable products pose challenges as client demand varies over the stocking period. In classical inventory planning, the unit acquisition cost is typically fixed regardless of quantity, but suppliers often offer volume-based discounts, especially for seasonal food items. Deterioration is a common issue for food items during stocking, resulting in lost investments and complicating optimal inventory planning. Delayed deteriorated items, including many seasonal foods, exhibit deterioration after a certain period. Additionally, suppliers may provide rebate schemes based on purchased quantities, offering an opportunity for the company to reduce total acquisition costs significantly by increasing quantities purchased. Effective inventory management, considering varying demand, deterioration, and rebate schemes, is crucial for companies to maximize profits and minimize losses.

Mango is the most popular fruit all over the world because of its taste, and Bangladesh is no exception. Bangladesh is the 7th largest mango-producing country in the world. In the fiscal year 2019–20, Bangladesh had 17,686 ha of mango orchards, yielded about 0.18 million metric tons of mangoes, and sold them for $7.05 million.

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Moreover, in the last fiscal year 2020–2021, 17,943 ha of land were under mango cultivation and yielded about 0.217 million metric tons of mangoes, which were sold for $8.525 million, according to the Department of Agriculture Extension, Bangladesh. Bangladeshi mangoes are in high demand in the local and international markets for their great taste and aroma. In the last three decades, a significant number of mango-based industries have sprung up across the country. Fig. 1 presents a graphic illustration of a mango business under a purchased-quantity-based discount facility for a company in Bangladesh. However, since products are perishable, a high purchase volume may not always be profitable and may even be a nightmare for the company. As a result, the following research questions are raised:

(i) How does the order-based rebate scheme affect the optimal inventory plan for the company?
(ii) What is the consequence of a storage time-dependent consumption rate at the company's optimal inventory?
(iii) How does the non-immediate deterioration feature under a rebate scheme on unit acquisition cost based on purchased quantity affect the company's inventory planning?

To address the aforementioned research questions, this work investigates, for the first time, the impacts of a rebate scheme on the unit acquisition price when items are non-immediately deteriorating and their consumption rate depends upon the stocking period. For that, the company's various costs are derived based on the inventory behavior over the business cycle, and then, the company's total cost per unit of time is calculated. Second, the theoretical outcomes are presented with theorems, and then, combining the analytical results, two solution algorithms are presented under the quantity-based discount frame. Third, a real example of mango businesses in Bangladesh is introduced to verify the inventory procedure. Finally, the key findings for the company are delivered.

1.1. Literature review

According to economists and health specialists, consuming seasonal fruits or vegetables is beneficial both from a health perspective for consumers and from an economic perspective for local producers and consumers. Furthermore, Li et al. (2021) revealed that perishable foods account for nearly half of the total stocked items in grocery sales in the United States and, therefore, grocery companies place increasing importance on proper inventory planning for perishable food items. An important and unavoidable characteristic of these perishable foods is deterioration. Because of their storage environment and physical structure, the deterioration of many perishable foods commences late from the storing moment and these items are recognized as non-instantaneous or non-immediately deteriorating items. Wu et al. (2006) described, for the first time, the best inventory planning for non-immediately deteriorating items. Rabbani et al. (2015) discussed a company's integrated replenishment and marketing strategies for non-immediately deteriorating items, in which an exponential discount is offered on the selling price after deterioration appears. Ai et al. (2017) determined the optimal inventory planning for a company replenishing multiple non-immediately decay items where the consumption rates of all items are constant. To reduce the deterioration cost by slowing down deterioration rates, Li et al. (2019) found the optimal preservation investment cost for non-immediately decay items. Later, Khan et al. (2020) investigated the impacts of disparate deterioration beginning time due to different storage environments in a two-warehouse system on a company's total. Duary et al. (2022) derived the best selling price decision for a company dealing with inventory problems for non-immediate deteriorating items under a prepayment mechanism imposed by the supplier. Recently, Khan et al. (2022a) analytically characterized when a company's optimal replenishment policy for non-immediately perishable items involves the effect of deterioration. In the studies described above, no one considers the effects of stocking duration on item consumption rates, but storage duration has a significant effect on consumer demand, especially for all food items.

When a company deals with perishable food items, its inventory planning greatly depends on the reasonable storage duration of the items. Prasad and Mukherjee (2016) studied the resultant consequences of storage duration and currently available stock volume on the consumption rate and then, analyzed the best inventory planning for a decay item. Adopting a decreasing (exponentially) client demand with items' storage durations along with

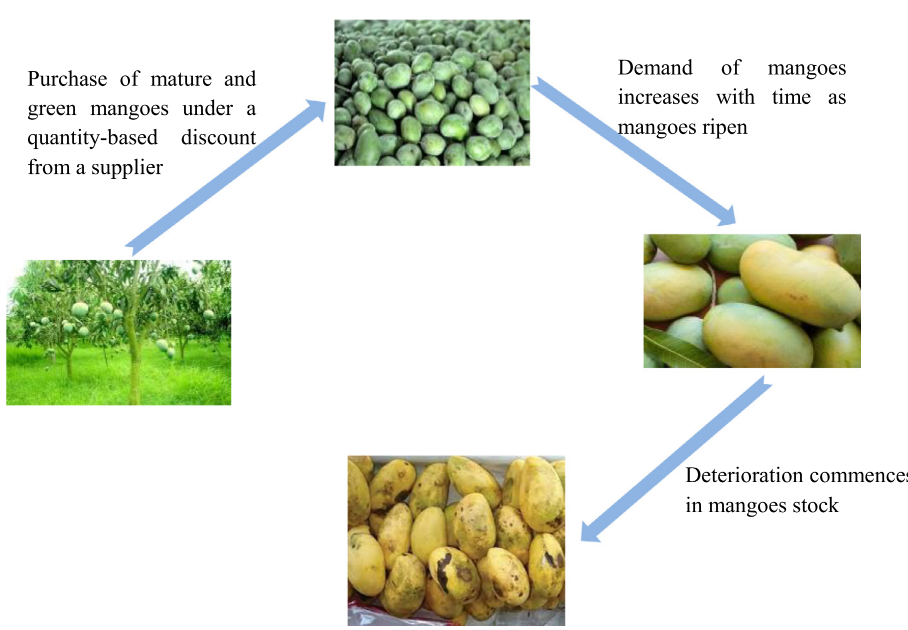

Fig. 1. Graphic illustration of a mango business under a quantity-based discount frame.

the impact of selling price reversely on demand, Kumar et al. (2016) further discussed the consequence of time on a company's overall finances and inventory planning for decay items. Dobson et al. (2017) assumed that consumers evaluate the quality of items by comparing the lifetime and storage time of the items, and, therefore, the demand falls linearly with the age of the items. Integrating a linearly increasing unit holding cost with storage time, Pervin et al. (2018) discussed an inventory planning problem for decay items where the demand also changes linearly with the storage time of the items. To determine the consumption rate of items more accurately, San-José et al. (2019) replaced the linear storage time-dependent demand with a power form and then obtained the best inventory plan analytically. In addition, Cárdenas-Barrón et al. (2021) further investigated the impacts of time on demand in power form after allowing a stock-out scenario. Khan et al. (2022b) found the best inventory policies for both positive stock-ending and stock-out scenarios for a non-immediate decay item when demand is interlinked with the item's storage duration. Recently, Kumar et al. (2023) further investigated the best replenishment strategy under a time-associated demand pattern by adopting a fractional differential equation approach.

Most seasonal foods have a short shelf life, and therefore, suppliers are very cautious about selling these items in the early stages. For that reason, suppliers of seasonal food items allow several concessions on the unit acquisition cost based on order quantity. Taleizadeh and Pentico (2014) discussed the consequences of an all-unit rebate scheme on a retailer's inventory decision by adopting a constant demand rate. Again, Taleizadeh et al. (2015) further investigated the results of an incremental rebate scheme on a retailer's inventory planning. However, both of these studies were conducted at a constant consumption rate. Replacing constant demand with a variable price-sensitive consumption rate, Alfares and Ghaithan (2016) found the best pricing and inventory planning strategies for a retailer under an all-unit rebate frame. Allowing a discount for only imperfectly produced items instead of all items, Cunha et al. (2018) determined the best inventory planning for a production system. Qiu and Lee (2019) studied rail transportation pricing policy under a sole breakpoint-dependent rebate scheme. Shaikh et al. (2019) found the best inventory planning for a company under an all-unit rebate frame for decay items when the demand varies with warehouse stock levels. Taking into consideration the stored items' maximum lifetime on deterioration, Khan et al. (2019) investigated the consequences of an all-unit rebate frame on both pricing and inventory decisions. Lesmono et al. (2020) described an inventory procedure for a retailer dealing with multiple perishable items under multiple all-unit rebate schemes. Afterward, Khan et al. (2021) proposed a quantity-based integrated advance payment, delay payment, and unit acquisition price scheme and found the best inventory planning for a company. Recently, Khan et al. (2022c) found the best pricing and inventory strategies for a deteriorating item under a purchased quantity-sensitive rebate scheme on the unit acquisition price. Rahman et al. (2022) further scrutinized the significance of an all-unit concession arrangement on the optimal inventory policy in the interval environment. All the above works on the rebate scheme for the unit acquisition price focused on constant, price-dependent, or stock volume-dependent demand. However, the optimal inventory planning for seasonal foods with stocking period-sensitive demand and order-based rebates remains unexplored.

Due to supply disruptions or demand fluctuations, inventory planners sometimes face stock-out situations when customers may go elsewhere for their needs or wait for new items to arrive. If all the customers wait for their needs, then the situation is referred to as 'complete backordering' in inventory literature. Prasad and Mukherjee (2016) formulated a deterministic inventory planning problem for time-varying demand when shortages are completely backlogged. San-José et al. (2018) further investigated the complete backordering situation under non-linear stocking duration sensitive demand. Some mentionable recent studies in complete backordering consideration in the literature are: San-José et al. (2020), Lin (2021), and De-la-Cruz-Márquez et al. (2021). In addition, if a fraction of all customers move elsewhere during a stock-out situation and the remaining portion of the customers wait for new items to arrive, then the situation is referred to in the inventory literature as 'partial backordering'. Taking into consideration a fixed backlogging rate, Taleizadeh and Pentico (2014) analyzed the best inventory policy for a company under an order-based rebate scheme. In this investigation line, some mentionable studies are as follows: Taleizadeh et al. (2015), Lin (2018), Gao et al. (2019), Taleizadeh et al. (2020), and Rahman et al. (2021). On another hand, by adopting a variable backlogging range with clients' waiting times, Khan et al. (2019) examined the best inventory policy for a company dealing with maximum lifespan-related deteriorating items under an order based rebate scheme. With variable backlogging rate, some mentionable studies are as follows: Panda et al. (2019), Pakhira et al. (2020), Xu et al. (2021), Alshanbari et al. (2021), and Manna et al. (2022).

Though a good number of research works have investigated the effects of order-based rebate facilities on inventory planning, only two studies (Shaikh et al., 2019; Khan et al., 2019) among the research works described above have explored optimal inventory planning for a deteriorating item. It is noteworthy that neither of these (Shaikh et al., 2019 ; Khan et al., 2019) considered the significance of storage duration on item consumption rates, even though consumption of seasonal food items is highly dependent on storage duration. This is a huge research gap in inventory planning for seasonal foods because seasonal food suppliers often allow discounts on unit acquisition costs based on order volume. The main contributions of the study are:

- The first contribution is that this research addresses a research gap in the field of inventory planning for deteriorating items, specifically focusing on seasonal perishable food items.
- The study explores the significance of storage duration on item consumption rates, which is often overlooked in previous research.
- It investigates the impacts of a rebate scheme on unit acquisition price for non-immediately deteriorating items, considering the interplay between stocking period and consumption rate.
- The research provides decision makers with proper inventory planning strategies for seasonal perishable food items under an order-based rebate scheme.
- The introduction of a real example of mango businesses in Bangladesh adds practical relevance and allows for verification of the proposed inventory procedure.

The rest of the study is outlined as follows: Section 2 states the inventory procedure briefly, along with the required notations and assumptions. In Section 3, models are formulated for both with and without stock-out scenarios, whereas Section 4 exposes all the necessary analytical outcomes for ensuring the existence of the optimal solution for both models. Integrating all the analytical outcomes, Section 5 presents two solution algorithms. Section 6 represents a real case study and two additional numerical examples. The sensitivity analysis of the best inventory planning is exposed in Section 7, while several policy directions to manage inventories economically are pointed out in Section 8. Finally, Section 9 highlights the conclusion along with some salient insights for the inventory planner.

2. Problem description

An inventory planning problem for a company is analyzed in this paper, where the company deals with a non-immediately deteriorating item, particularly a seasonal food, under an all-unit rebate mechanism. An all-unit rebate is a concession method on the unit acquisition price based upon the purchased quantity provided by the supplier of the item. As the item is a seasonal food, the company stores very fresh and mature items, and hence, the deterioration of the stored items does not commence immediately. The consumption rate of the stored foods increases over time as they ripen and does not increase when deterioration appears. As the all-unit discount system motivates to increase the purchased quantity to a large extent, the supplier sometimes does not deliver the items to the company on time due to scarcity, and therefore, the company may face a stock-out situation. Consequently, both the stock-out and zero-ending scenarios are delineated in this study. The foremost objective of this work is to help the company reduce total cost under both stock-out and zero-ending situations by providing accurate inventory planning guidelines for seasonal foods when an all-unit rebate is available.

To outline the inventory planning problem mentioned above mathematically, the following notation and hypotheses are adopted for the rest of the study.

2.1. Notation

μ:	Deterioration commencing time (unit of time)
T:	Duration of each cycle (unit of time) (a decision variable for both models)
t_1:	Period for stock available inventory (unit of time) (a decision variable for the model with shortages)
θ:	Deterioration rate on the positive stock in the warehouse ($0 < \theta << 1$)
σ:	Parameter related to backordering ($\sigma \geqslant 0$)
Q:	Purchased quantity by the company for each cycle (units)
R:	Backordering quantity (units)
S:	Stock amount at the starting moment of each cycle after backordering (units)
S_1:	Stock amount at time $t = \mu$ (units)
c_0:	Company's ordering cost ($/order)
h_c:	Unit carrying cost per unit of time ($/unit/unit of time)
c_j:	Unit acquisition price ($/unit)
c_s:	Shortage cost ($/unit/unit of time)
c_l:	Opportunity cost per unit ($c_1 > c_j$) ($/unit/unit of time)
$D(t)$:	Time-varying demand function (units/unit of time)
$B(t)$:	Backlogging rate (units/unit of time)
$I_1(t)$:	Inventory amount in the storage before commencing deterioration at any time $t \in [0, \mu]$
$TVC(T)$:	Total cost for the company under a zero-ending inventory situation ($/unit of time)
$TVC(t_1, T)$:	Total cost for the company under a stock-out inventory situation ($/unit of time)

2.2. Hypothesis

(i) The consumption rate $D(t)$ of the stored foods increases over time as they ripen and does not increase when deterioration appears. The demand function is given by

$$D(t) = \begin{cases} a + bt\,, & 0 \leqslant t \leqslant \mu \\ a + b\mu\,, & \mu \leqslant t \leqslant T \end{cases},$$

where $a > 0$, $b \geqslant 0$, $\mu \geqslant 0$.

When the demand parameter b and μ are zero, the demand becomes constant over the cycle.

(ii) The number of deteriorated items depends upon the current stock volume, and the rate of deterioration is θ where $0 < \theta << 1$.

(iii) Lead time is too insignificant to consider while the replenishment rate is infinite.

(iv) The problem is investigated for an infinite planning horizon.

(v) When the company faces the stock-out condition, the backlogging rate is interrelated with clients' waiting times and is expressed as $B(t) = e^{-\sigma \ (T-t)}$, where $\sigma \geqslant 0$ and $t_1 \leqslant t \leqslant T$. If $\sigma = 0$, then a complete backordering case raises.

(vi) Using $n-$ number of quantity breaks (q_i where $i = 1, 2, 3, \ldots\ldots, n$ and $q_1 < q_2 < \ldots\ldots < q_n < \infty$), a rebate scheme on the unit acquisition price based on the purchased quantity is offered by the supplier where the unit acquisition price (c_j) decreases stepwise based upon the purchased quantity (Q) such that $c_1 > c_2 > \cdots \cdots > c_n$. This graphic illustration of this rebate scheme with four quantity breaks is provided in Fig. 2.

3. Model formulation

Taking into consideration the assumptions mentioned in the above section, the model for the zero-end inventory situation is first formulated, and then the model for the inventory procedure with a stock-out situation is formulated.

3.1. Model for the zero-end inventory situation

Suppose a company's stocking procedure begins with Q units of seasonal food, and the complete length of the cycle is T, where Fig. 3 reveals the graphic illustration of the inventory cycle. In Fig. 3, the point μ specifies the time duration for deterioration-free inventory volume while T indicates the termination of the cycle. Consequently, the inventory volume drops due to the consumption of the items only over $[0, \mu]$, and then it decreases due to both deterioration and consumption of the items over $[\mu, T]$. Let $I_1(t)$ and $I_2(t)$ represent the inventory volume in the storage at any time t during $[0, \mu]$ and $[\mu, T]$, respectively.

Now, the differential equation for the inventory volume $I_1(t)$ over $[0, \mu]$ is

$$\frac{dI_1(t)}{dt} = -(a + bt), \quad 0 \leqslant t \leqslant \mu \tag{1}$$

with $I_1(0) = Q$ and $I_1(\mu) = S_1$.

Again, the differential equation for the inventory volume $I_2(t)$ over $[\mu, T]$ is

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -(a + b\mu), \quad \mu \leqslant t \leqslant T \tag{2}$$

with $I_2(\mu) = S_1$ and $I_2(T) = 0$.

Taking into account $I_1(0) = Q$, the solution of Eq. (1) is

$$I_1(t) = Q - at - \frac{1}{2}bt^2, \quad 0 \leqslant t \leqslant \mu \tag{3}$$

Applying another boundary condition $I_1(\mu) = S_1$ in (3), one has

$$I_1(\mu) = S_1 = Q - a\mu - \frac{1}{2}b\mu^2 \tag{4}$$

and

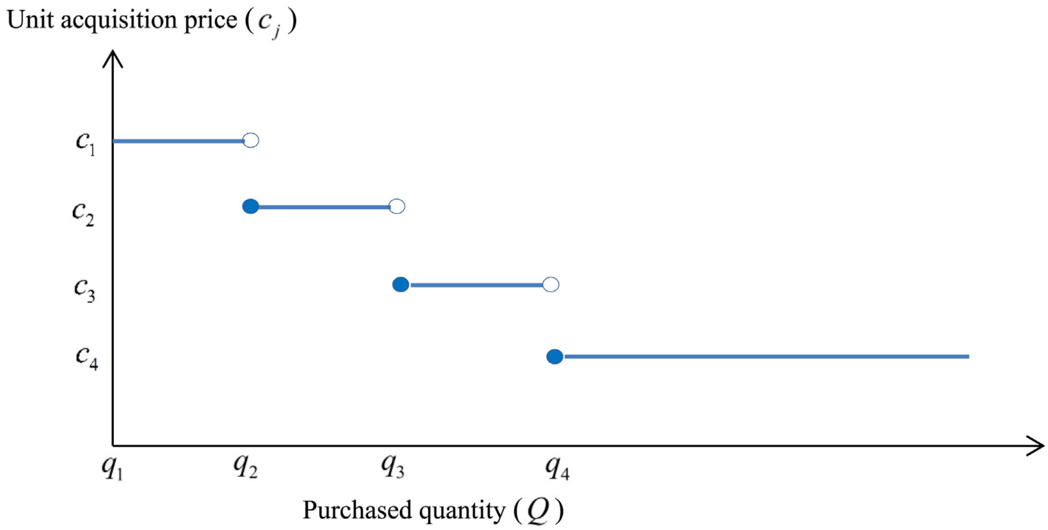

Fig. 2. Graphic illustration of an all-unit discount scheme with four quantity breaks.

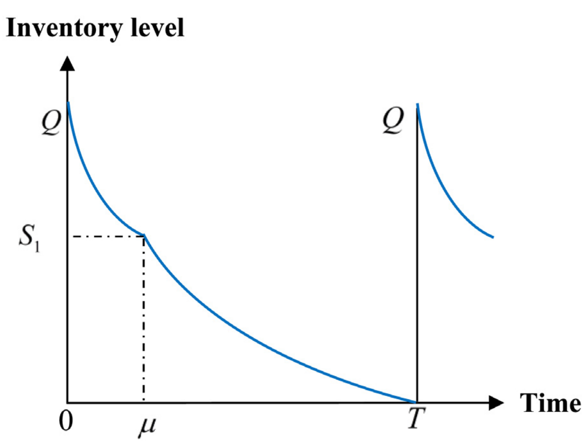

Fig. 3. Graphical illustration of the inventory framework without shortage.

$$Q = S_1 + a\mu + \frac{1}{2}b\mu^2 \tag{5}$$

On solving Eq. (2) with the auxiliary condition $I_2(T) = 0$, one has

$$I_2(t) = \frac{k_0}{\theta}\left[e^{\theta(T-t)} - 1\right], \quad \text{where } k_0 = a + b\mu. \tag{6}$$

Again, adopting $I_2(\mu) = S_1$ in (6), one finds

$$I_2(\mu) = S_1 = \frac{k_0}{\theta}\left[e^{\theta(T-\mu)} - 1\right], \text{ that is, } S_1 = \frac{k_0}{\theta}\left[e^{\theta(T-\mu)} - 1\right]. \tag{7}$$

Combining Eqs. (5) and (7), one has

$$Q = \frac{k_0}{\theta}\left[e^{\theta(T-\mu)} - 1\right] + a\mu + \frac{1}{2}b\mu^2 \tag{8}$$

and

$$T = \mu + \frac{1}{\theta}\ln\left|\frac{\theta}{k_0}\left(Q - a\mu - \frac{1}{2}b\mu^2\right) + 1\right|. \tag{9}$$

Cost Components:

The total cost of this inventory system per replenishment cycle consists of the following cost components:

(i) **Ordering cost** (*OC*): c_o
(ii) **Purchasing cost** (*PC*): c_jQ
(iii) **Holding cost** (*HC*): The cumulative holding cost for the periods $[0, \mu]$ and $[\mu, T]$ is

$$h_c\left[\int_0^{\mu} I_1(t)\,dt + \int_{\mu}^{T} I_2(t)\,dt\right] = h_c\left[\frac{1}{2}a\mu^2 + \frac{1}{3}b\mu^3 + \frac{k_0(1+\theta\mu)}{\theta^2}\left(e^{\theta(T-\mu)} - 1\right) - \frac{k_0}{\theta}(T-\mu)\right]$$

Hence, the company's total cost per unit of time is:

$$TVC(T) = \frac{1}{T}[OC + PC + HC]$$

$$= \frac{1}{T}\left[\begin{array}{c} c_o + c_j\left(a\mu + \frac{1}{2}b\mu^2\right) + h_c\left(\frac{1}{2}a\mu^2 + \frac{1}{3}b\mu^3\right) + \frac{k_0}{\theta}\left(e^{\theta(T-\mu)} - 1\right)\left\{c_j + \frac{(1+\theta\mu)}{\theta}h_c\right\} \\ -k_0\left(\frac{h_c}{\theta}\right)(T-\mu) \end{array}\right] \tag{10}$$

The company's goal is to figure out the optimum cycle length T^*, which ensures the minimum total cost per unit of time.

3.2. Model for the inventory procedure with a stock-out situation

Suppose a company places a requisition for $Q = S + R$ units of a decaying item early in a business cycle consisting of three different phases, such as $[0, \mu]$, $[\mu, t_1]$, and $[t_1, T]$. When the company receives the lot, R units are consumed to satisfy the accumulated backlogging demands; thus, the remaining available inventory level falls to S units. At this stage, the stock level falls just for satisfying clients' desires over the interval $[0, \mu]$. At time $t = \mu$, the inventory level becomes S_1. Because of the consolidated impacts of deterioration and client desire, the stock declines throughout the next phase $[\mu, t_1]$ of the cycle and eventually, falls to a zero level at $t = t_1$. After this phase, a stock-out situation is faced by the company when shortages are amassed depending on the clients' waiting time during the period $[t_1, T]$. At this stage, another order is submitted, and the entire stocking procedure is repeated. The complete inventory procedure is illustrated in Fig. 4.

The differential equations to illustrate the behavior of the present stock at any moment of the business cycle are:

$$\frac{dI_1(t)}{dt} = -(a + bt),\ 0 \leqslant t \leqslant \mu \tag{11}$$

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -(a + b\mu),\ \mu \leqslant t \leqslant t_1 \tag{12}$$

and

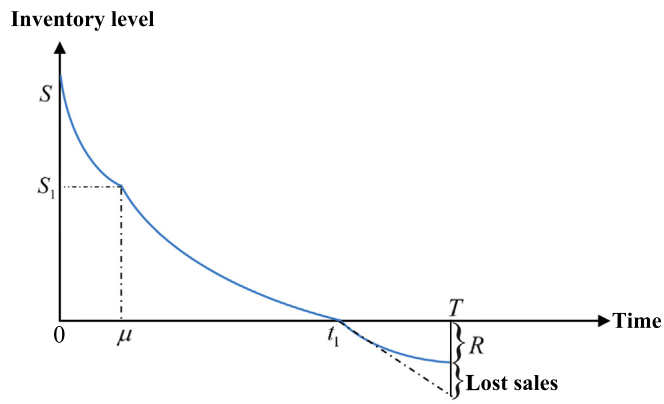

Fig. 4. Graphical illustration of the inventory framework with shortages.

$$\frac{dI_3(t)}{dt} = -(a+b\mu)\quad e^{-\sigma(T-t)} = k_0 e^{-\sigma(T-t)},\ t_1 \leqslant t \leqslant T \tag{13}$$

with $I_1(0) = S,\ I_1(\mu) = S_1,\ I_2(\mu) = S_1,\ I_2(t_1) = 0,\ I_3(t_1) = 0$ and $I_3(T) = -R$, where $k_0 = a + b\mu$.

Exploiting $I_1(0) = S$, the solution of Eq. (11) is given by

$$I_1(t) = S - at - \frac{1}{2}bt^2 \tag{14}$$

Applying $I_1(\mu) = S_1$ in Eq. (14), one has $I_1(\mu) = S_1 = S - a\mu - \frac{1}{2}b\mu^2$, i.e.,

$$S = S_1 + a\mu + \frac{1}{2}b\mu^2 \tag{15}$$

On solving Eq. (12) with $I_2(t_1) = 0$, one finds

$$I_2(t) = \frac{k_0}{\theta}\left[e^{\theta(t_1-\ t)} - 1\right],\ \text{ where }\ k_0 = a + b\mu \tag{16}$$

Again, applying $I_2(\mu) = S_1$ in Eq. (15), one has

$$I_2(\mu) = S_1 = \frac{k_0}{\theta}\left[e^{\theta(t_1-\ \mu)} - 1\right]$$

$$\text{Therefore,}\quad S_1 = \frac{k_0}{\theta}\left[e^{\theta(t_1-\ \mu)} - 1\right] \tag{17}$$

From Eqs. (15) and (17), one has

$$S = a\ \mu + \frac{1}{2}b\mu^2 + \frac{k_0}{\theta}\left[e^{\theta(t_1-\ \mu)} - 1\right] \tag{18}$$

Solving Eq. (13) with $I_3(t_1) = 0$, one has

$$I_3(t) = \frac{k_0}{\sigma}\left[e^{-\sigma(T-t_1)} - e^{-\sigma(T-t)}\right] \tag{19}$$

Applying $I_3(T) = -R$ in Eq. (19), one finds

$$R = \frac{k_0}{\sigma}\left[1 - e^{-\sigma(T-t_1)}\right] \tag{20}$$

Consequently, the total number of requisition amounts is

$$Q = S + R = a\mu + \frac{1}{2}b\mu^2 + \frac{k_0}{\theta}\left[e^{\theta(t_1-\ \mu)} - 1\right] + \frac{k_0}{\sigma}\left[1 - e^{-\sigma(T-t_1)}\right] \tag{21}$$

From Eq. (21), the business cycle duration is expressed as

$$T = t_1 - \frac{1}{\sigma}\ln\left|1 - \frac{\sigma}{k_0}\left\{Q - a\mu - \frac{1}{2}b\mu^2 - \frac{k_0}{\theta}\left(e^{\theta(t_1-\ \mu)} - 1\right)\right\}\right| \tag{22}$$

The total cost of the entire inventory framework per renewal cycle consists of the following components:

(1) **Ordering cost** (OC): c_0
(2) **Purchasing cost** (PC): $c_j\ Q = c_j(S+R)$
(3) **Holding cost** (HC): The total holding cost for the period $[0, \mu]$ and $[\ \mu, t_1]$ is

$$h_c\left[\int_0^{\mu} I_1(t)\,dt + \int_{\mu}^{t_1} I_2(t)\,dt\right]$$
$$= h_c\left[\frac{1}{2}a\mu^2 + \frac{1}{3}b\mu^3 + \frac{k_0(1+\theta\mu)}{\theta^2}\left(e^{\theta(t_1-\mu)} - 1\right) - \frac{k_0}{\theta}(t_1-\mu)\right]$$

(4) **Shortage cost** (SC): $-\ c_s\ \int_{t_1}^{T} I_3(t)\,dt = \ c_s\frac{k_0}{\sigma}\left[\frac{1}{\sigma} - \frac{e^{-\sigma(T-t_1)}}{\sigma} - (T-t_1)\,e^{-\sigma(T-t_1)}\right]$
(5) **Lost Sale cost** (LC): $c_l\ \int_{t_1}^{T}\ \left[1 - e^{-\sigma(T-t)}\right] k_0\,dt = c_l k_0\left[(T-t_1) - \frac{1}{\sigma} + \frac{e^{-\sigma(T-t_1)}}{\sigma}\right]$

Consequently, the company's total cost per unit of time is:

$$TVC(t_1,T) = \frac{1}{T}[OC + PC + HC + SC + LC]$$
$$= \frac{1}{T}\left[\begin{array}{c} c_o + c_j\left(a\mu + \frac{1}{2}b\mu^2\right) + h_c\left(\frac{1}{2}a\mu^2 + \frac{1}{3}b\mu^3\right) \\ + \frac{k_0}{\theta}\left(e^{\theta(t_1-\mu)} - 1\right)\left\{c_j + \frac{(1+\theta\mu)}{\theta}h_c\right\} \\ -k_0\left(\frac{h_c}{\theta}\right)\quad(t_1-\mu) + \frac{k_0}{\sigma}\left(1 - e^{-\sigma(T-t_1)}\right)\ \left(c_j + \frac{c_s}{\sigma} - c_l\right) \\ -\frac{k_0 c_s}{\sigma}(T-t_1)\quad e^{-\sigma(T-t_1)} + k_0 c_l(T-t_1) \end{array}\right] \tag{23}$$

Now the goal is to bring out the optimal period for stock available inventory t_1^*, and the optimal length of the fill-up cycle T^* to minimize the company's total cost per unit of time. For that, the following section deals with the theoretical findings for achieving the company's best inventory planning.

4. Theoretical results

First, the curvature of the company's total cost function (10) for the zero-end inventory situation is examined, and the curvature of the company's total cost (23) for the inventory system with a stock-out situation is investigated. From Khan et al. (2019), any function in the following structure

$$\varphi(t) = \frac{\Delta_1(t)}{\Delta_2(t)}t = (t_1, t_1, ..., t_n) \in R^n \tag{24}$$

is pseudo-convex, when both $\Delta_1(t) \geqslant 0$ and $\Delta_2(t) > 0$ are differentiable and the convexity condition is satisfied by $\Delta_1(t) \geqslant 0$ while the concavity condition is maintained by $\Delta_2(t) > 0$. Exploiting this important outcome, the pseudo-convexity of the company's total cost function (10) for the zero-end inventory situation with respect to T is examined first, and the joint pseudo-convex of the company's total cost (23) for the inventory system with stock-out situation with respect to t_1 and T is investigated.

4.1. Model for the zero-end inventory situation

Utilizing the important result mentioned above, the following theorem is proposed to examine the curvature of the cost function in Eq. (10).

Theorem 1. *The company's total cost TVC (T) is pseudo-convex in T, and therefore, TVC (T) holds the global minimum value at $T = T^*$. Furthermore, the point T^* is unique.*

Proof: See Appendix A in the Supplementary material file.

Now, to calculate the point T^*, it is necessary to compute the first-order derivative of $TVC(\ T\ )$ and set the result of the derivative as zero. After some simplifications, the condition for achieving T^* is:

$$c_o + \frac{k_0}{\theta}\left\{c_j + \left(\frac{1+\theta\mu}{\theta}\right)\mathrm{h_c}\right\}\ \left\{\ (1-\theta T)\, e^{\theta(T-\mu)} - 1\right\} + c_j\left(a\mu + \frac{1}{2}b\mu^2\right) + h_c\left(\frac{1}{2}a\mu^2 + \frac{1}{3}b\mu^3\right) + k_0\mu\left(\frac{h_c}{\theta}\right) = 0 \tag{25}$$

On solving Eq. (25), the value of T^* is found, which guarantees the global minimum cost $TVC(\ T^*\ )$ for the company. Next, our aim is to show the joint pseudo-convex of the company's total cost function (23) for the inventory system in a stock-out situation.

4.2. Model for the inventory procedure with stock-out situation

For achieving the best inventory planning for minimizing the company's total cost $TVC(t_1, T)$ when a stock-out situation appears, compute the first-order partial derivatives of $TVC(t_1, T)$ with respect to t_1 and T, then set them equal to zero.

$$\frac{\partial}{\partial\ t_1}[TVC(t_1, T)] = 0$$

i.e.,

$$\left\{c_j + \frac{(1+\theta\mu)}{\theta}\mathrm{h_c}\right\}\left(e^{\theta(t_1-\mu)}\right) + \left\{c_l - c_j - c_s(T-t_1)\ \right\}\ e^{-\sigma(T-t_1)} - \left(\frac{h_c}{\theta} + c_l\right) = 0 \tag{26}$$

and $\frac{\partial}{\partial T}[TVC(t_1, T)] = 0$
i.e.,

$$\frac{k_0}{\theta}\left\{c_j + \frac{(1+\theta\mu)}{\theta}\mathrm{h_c}\right\}\left(e^{\theta(t_1-\mu)} - 1\right) + k_0\left\{(c_l - c_j)T - \frac{1}{\sigma}\left(c_j + \frac{c_s}{\sigma} - c_l\right) - c_s\left(T + \frac{1}{\sigma}\right)(T - t_1)\right\}e^{-\sigma(T-t_1)} - k_0\left(\frac{h_c}{\theta}\right)(t_1-\mu) - k_0 c_l t_1 + \mathrm{c_o} + \mathrm{c_j}\left(\mathrm{a}\mu + \frac{1}{2}\mathrm{b}\mu^2\right) + \mathrm{h_c}\left(\frac{1}{2}\mathrm{a}\mu^2 + \frac{1}{3}\mathrm{b}\mu^3\right) + \frac{\mathrm{k_0}}{\sigma}\left(\mathrm{c_j} + \frac{\mathrm{c_s}}{\sigma} - \mathrm{c_l}\right)\Bigg] = 0 \tag{27}$$

On solving Eqs. (26) and (27) simultaneously, the optimal period for stock available inventory t_1^* and the optimal length of the fill-up cycle T^* to minimize the company's cumulative cost per unit of time are achieved. Now, the existence of the point (t_1^*, T^*) and the global optimality of the company's total cost $TVC(t_1, T)$ at the point (t_1^*, T^*) are both examined in the following theorem.

Theorem 2. *The company's total cost $TVC(t_1, T)$ is jointly pseudo-convex in t_1 and T, and therefore, $TVC(t_1, T)$ holds the global minimum value at (t_1^*, T^*). Furthermore, the point (t_1^*, T^*) is unique.*

Proof: See Appendix B in the Supplementary material file.

5. Solution algorithm

Inspired by Shaikh et al. (2019), this section deals with two different algorithms to achieve the best inventory plan for the company in zero-end inventory and stock-out situations. Suppose the supplier of the item offers a rebate scheme on the unit acquisition price based on the purchased quantity by using $n-$ number of quantity breaks (q_i where $i = 1, 2, 3, \ldots\ldots, n$ and $q_1 < q_2 < \ldots\ldots < q_n < \infty$) where the unit acquisition price (c_j) decreases stepwise based on the purchased quantity (Q) such that $c_1 > c_2 > \cdots\ \cdots > c_n$. Under this offered rebate scheme, the purchased quantity of the company may stand at any quantity break $Q = q_i$ or lie between any two successive quantity breaks. When the optimum requisition amount Q lies between any two successive quantity breaks, the best Q^* for the zero-end inventory situation is achieved from Eq. (8) with the assistance of T^* calculated by solving Eq. (25), while the best Q^* for the stock-out situation is achieved from Eq. (21) with the assistance of t_1^* and T^* calculated by solving Eqs. (26) and (27) simultaneously. On the other hand, when the purchased quantity of the company Q^* stands at any quantity break, then the duration of each cycle of the company under the zero-end inventory situation is straightforwardly computed from Eq. (9). However, under the stock-out situation, the duration of each cycle of the company is expressed as a function of Q (see Eq. (22)). As a result, the Eqs. (26) and (27) are not compatible to compute the best inventory plan for the company under this circumstance, and it is necessary to derive new conditions to achieve the best inventory plan.

5.1. If the purchased quantity of the company Q^* stands at any quantity break under a stock-out situation

This subsection determines the necessary condition for obtaining the best inventory plan for the company when the best Q^* stands at any quantity break under a stock-out situation. After substituting the expression of T with the assistance of Eq. (22), the company's total cost TVC in Eq. (23) involves only one decision variable, t_1. Thus, it is necessary to determine the condition for computing t_1^* under which the company's total cost $TVC(t_1)$ is optimized.

From Eq. (21), one has

$$\frac{dT}{dt_1} = \left\{\ \ 1 - e^{\theta\ (t_1-\mu)\ +\ \sigma(T-t_1)}\right\} \tag{28}$$

In this circumstance, the condition for the best t_1^* is found by setting $\frac{d}{dt_1}\{TVC(t_1)\} = 0$ with the assistance of Eq. (28). After performing some simplifications, the condition is given by

$$\left[\begin{array}{c} \left\{\ 1 - e^{\theta\ (t_1-\mu)\ +\ \sigma(T-t_1)}\right\}\ \left\{\frac{c_0}{k_0} + \frac{c_j}{k_0}\left(a\mu + \frac{1}{2}b\mu^2\right)\right. \\ + \frac{h_c}{k_0}\left(\frac{a\mu^2}{2} + \frac{b\mu^3}{3}\right) + \frac{1}{\theta}\left(e^{\theta(t_1-\mu)} - 1\right)\ \left(c_j + \frac{1+\theta\mu}{\theta}h_c\right) \\ -\left(\frac{h_c}{\theta}\right)(t_1-\mu) + \frac{1}{\sigma}\ \left(c_j + \frac{c_s}{\sigma} - c_l\right)\ \left(1 - e^{-\sigma(T-t_1)}\right) - \frac{c_s}{\sigma}(T-t_1)\ \ e^{-\sigma(T-t_1)} - c_l t_1\Big\} \end{array}\right] - T\ \left[\left\{\mu h_c + \left(\frac{h_c}{\theta} + c_l\right) - c_s(T-t_1)\ \ \right\}e^{\theta(t_1-\mu)} - \left(\frac{h_c}{\theta} + c_l\right)\right] = 0, \tag{29}$$

where $T = t_1 - \frac{1}{\sigma}\ln\left|1 - \frac{\sigma}{k_0}\left\{Q - a\mu - \frac{1}{2}b\mu^2 - \frac{k_0}{\theta}\left(e^{\theta(t_1-\ \mu)} - 1\right)\right\}\right|$

Now we present two algorithms: one for achieving the best inventory plan under zero-end inventory, and another for achieving the best inventory plan under a stock-out situation, considering the rebate scheme on unit acquisition price.

5.2. Algorithm for the company's best inventory plan under the zero-end inventory situation

Step 1	:	Initialize $TVC_{\min}(T) = \infty$ and $i = n$.
Step 2	:	Exploiting the values of $a, b, c_0, k_0, \theta, \mu, h_c$ and c_i, compute T by solving Eq. (25). Utilizing the derived value of T^*, determine the optimum requisition amount Q for the company from Eq. (8). (i) If $Q \in (q_i, q_{i+1})$, then determine the company's cost $TVC_i(T)$ adopting this feasible solution. If $TVC_i(T) < TVC_{\min}(T)$, then $TVC_{\min}(T) = TVC_i(T)$. Move to Step 5. (ii) If $Q \notin (q_i, q_{i+1})$, move to Step 3.
Step 3	:	Compute the duration of each cycle of the company from Eq. (9) by setting $Q = q_i$. Substituting the value of T^* into Eq. (10), calculate the company's cost $TVC_i(T)$. If $TVC_i(T) < TVC_{\min}(T)$, then $TVC_{\min}(T) = TVC_i(T)$. Move to Step 4
Step 4	:	If $i \geqslant 2$, move to Step 2 with $i = i - 1$; else, move to Step 5.
Step 5	:	The company's minimum cost per unit time is $TVC_{\min}(T)$ with the optimal T^*.

5.3. Algorithm for the company's best inventory plan under the stock-out situation

Step 1	:	Initialize $TVC_{\min}(t_1, T) = \infty$ and $i = n$.
Step 2	:	Exploiting the values of $a, b, c_0, k_0, \theta, \mu, h_c, d_c, \sigma, c_s, c_l$ and c_i, compute t_1^* and T^* by solving Eqs. (26) and (27). Utilizing the derived value of t_1^* and T^*, determine the optimum requisition amount Q from Eq. (21). (i) If $Q \in (q_i, q_{i+1})$, then determine the company's cost $TVC_i(t_1^*, T^*)$ adopting this feasible solution. If $TVC_i(t_1^*, T^*) < TVC_{\min}(t_1, T)$, then $TVC_{\min}(t_1, T) = TVC_i(t_1^*, T^*)$. Move to Step 5. (ii) If $Q \notin (q_i, q_{i+1})$, move to Step 3.
Step 3	:	Obtain the expression of T in terms of t_1 after setting $Q = q_i$ in Eq. (22). Compute t_1^* by solving Eq. (29) adopting the derived expression of T. Exploiting the value of t_1^* , find T^* from Eq. (22). Adopting t_1^* and T^*, determine $TVC_i(t_1^*, T^*)$ from Eq. (23). If $TVC_i(t_1^*, T^*) < TVC_{\min}(t_1, T)$, then $TVC_{\min}(t_1, T) = TVC_i(t_1^*, T^*)$. Move to Step 4.
Step 4	:	If $i \geqslant 2$, move to Step 2 with $i = i - 1$; else, move to Step 5.
Step 5	:	The company's minimum cost per unit time is $TVC_{\min}(t_1, T)$ with the optimal t_1^* and T^*.

6. Numerical analysis

In the subsequent subsections, we conduct numerical analysis for the proposed inventory procedures. Section 6.1 examines a real-life example of mango businesses in Bangladesh, considering the rebate scheme based on purchased quantity. Section 6.2 presents two numerical examples to verify the outcomes of the proposed inventory procedures.

6.1. Case study

Mahim Natural Mango (MNM) Company, a wholesale business of fresh mangoes in Bangladesh, operates with its head office in Rajshahi and 20 wholesale centers nationwide. During the May to August harvesting period, the centers procure mangoes directly from farms and sell them to local retailers. We conducted a case study on Manihari Mango Shop, a prominent retailer in Dhaka, gathering information from historical records and interviews with the shop's manager.

Mahim Natural Mango (MNM) Company, a wholesaler of mature mangoes with a short shelf life, offers a concession on unit acquisition cost based on order quantity for retailers. The cost per mango box (containing 10 kg of mangoes) is \$6 for quantities [0, 1350), \$5.5 for [1350, 1500), and \$5 for 1500 boxes or more. The goal of the Monihari Mango Shop manager is cost-effective inventory planning. The mangoes are non-instantaneous deteriorating items, with deterioration starting 2 days after storage. Demand increases with ripening, starting at 350 boxes/day and increasing by 3.5 boxes/day. Ordering cost is \$300/order, deteriorating rate is 0.02 box/day, and holding cost is \$0.01/box/day. In a stock-out situation, backlogging follows a negative exponential function with a waiting time sensitivity coefficient of 0.03. Shortage cost is \$3/box/day, and opportunity cost is \$7/box/day. All aforementioned information is expressed with notation as follows: $a = 350$, $b = 3.5$, $c_{\circ} = 300$, $k_{\circ} = 357$, $\theta = 0.02$, $\mu = 2$, $h_c = 0.01$, $\sigma = 0.03$, $c_s = 2$, $c_l = 7$ and the discount scheme as

Number of boxes	$Q \in [q_1, q_2) = [0, 1350)$	$Q \in [q_1, q_2) = [1350, 1500)$	$Q \in [q_1, q_2) = [1500, \infty)$
Acquisition cost (\$/box)	$c_1 = 6$	$c_2 = 5.5$	$c_3 = 5$

Now, the optimal inventory planning for the Monihari Mango Shop under the inventory procedure with zero-end is: $T^* = 4.173$ days, $Q^* = 1500$ boxes and $TVC_{\min}(T) = 1875.5$.

Again, the best inventory planning for the Monihari Mango Shop under a stock-out situation is: $t_1^* = 4.063$ days, $T^* = 4.186$ days,$Q^* = 1503$ boxes and $TVC_{\min}(t_1^*, T^*) = 1875.2$. The detailed calculations of the optimal inventory planning for both situations are supplier in Appendices C and D in the Supplementary material file.

Since $TVC_{\min}(t_1^*, T^*) = 1875.2 < TVC_{\min}(T)$ =1875.5, adopting our model with partial backlogged shortages is better for the Monihari Mango Shop to run the mango business with a total minimum cost.

6.2. Numerical illustration

To examine the working performance of the algorithms 5.2 and 5.3 derived in the previous section, two numerical examples are illustrated under the zero-end inventory and stock-out situations.

Example 1. Inventory procedure with zero-end

Suppose the following rebate scheme on the unit acquisition price based upon the purchased amount is presented by the supplier.

Amount (units)	$Q \in [q_1, q_2) = [0, 400)$	$Q \in [q_2, q_3) = [400, 460)$	$Q \in [q_3, q_4) = [460, \infty)$
Acquisition cost ($/unit)	$c_1 = 4.75$	$c_2 = 4.5$	$c_3 = 4$

The values of the other known parameters are: $a = 100$, $b = 1.5$, $\mu = 2$, $\theta = 0.05$, $k_0 = 103$, $c_0 = 200$ and $h_c = 0.03$. For this example, the company's minimum cost per unit time is $TVC_{\min}(T) = 475.81$ with the optimal $T^* = 4.351$ and $Q^* = 460$. As seen from Fig. 5, the company's cost per unit of time is convex in T at $Q^* = 460$. In addition, the detailed calculation of the optimal solution is provided in Appendix E in the Supplementary material file.

Example 2. Inventory procedure with a stock-out situation

This example adopts the same rebate scheme on unit acquisition price and other data in Example 1 along with $\sigma = 0.04, c_s = 3, c_l = 5$ to adopt the model under a stock-out situation. The company's minimum cost per unit time is $TVC_{\min}(t_1^*, T^*) = 474.32$ with the optimal $t_1^* = 4.212$, $T^* = 4.413$ and $Q^* = 465$ (the solution process is provided in Appendix F in the Supplementary material file). Fig. 6 exhibits the behavior of the company's cost $TVC(t_1, T)$ under stock-out situation against variables t_1 and T. As seen from Fig. 6, the company's cost $TVC(t_1, T)$ is jointly convex in t_1 and T, and therefore, $TVC(t_1, T)$ holds the global minimum value at the unique point (t_1^*, T^*). To examine the behavior of the company's total cost $TVC(t_1, T)$ more explicitly, two additional graphs (Figs. 7 and 8) are provided against each variable separately.

7. Sensitivity investigation

This section examines the impact of altering system parameters on the company's best inventory strategy. Important observations and cost-minimization guidelines are derived. Computational results, showcasing parameter variations from −20% to +20%, are presented in Table 1 (see Appendix G in the Supplementary material file). Furthermore, Table 1 includes numerical data from Example 2 for reference.

The following observations are seen in Table 1:

(i) Though the company's optimal purchased quantity (Q^*) remains fixed, it increases significantly when the value of $c_\circ$, a and μ increase. On the other hand, the optimal Q^* declines for the increasing values of the remaining parameters b, h_c, θ, c_j, c_s and c_l. When the cost of creating each order increases, the company wants to enlarge the best-purchased quantity to shrink the average ordering cost per unit. Similarly, if the initial market demand increases, the total demand also rises, and subsequently, the company purchases a higher order size. It is also observed that the optimal Q^* does not change when the backlog parameter σ varies.

(ii) Maximum backordering level (R^*) rises when $c_\circ, h_c, \theta$ and $c_j (j = 1, 2, 3)$ increase, while R^* shrinks concerning the positive variations of the parameters b, μ, c_l and c_s. It is worth mentioning that the parameter σ does not influence the optimal value of backordering level.

(iii) The company's initial best positive stock level (S^*) rises for the variation of the values of c_o, μ and c_s in positive way, while S^* reduces concerning the change in the parameters b, h_c, θ and $c_j (j = 1, 2, 3)$ in positive way. It is noticed that S^* decreases when the value of the initial market demand a changes from −20% to −10% and S^* increases when the initial market demand a changes from +10% to +20%. Another

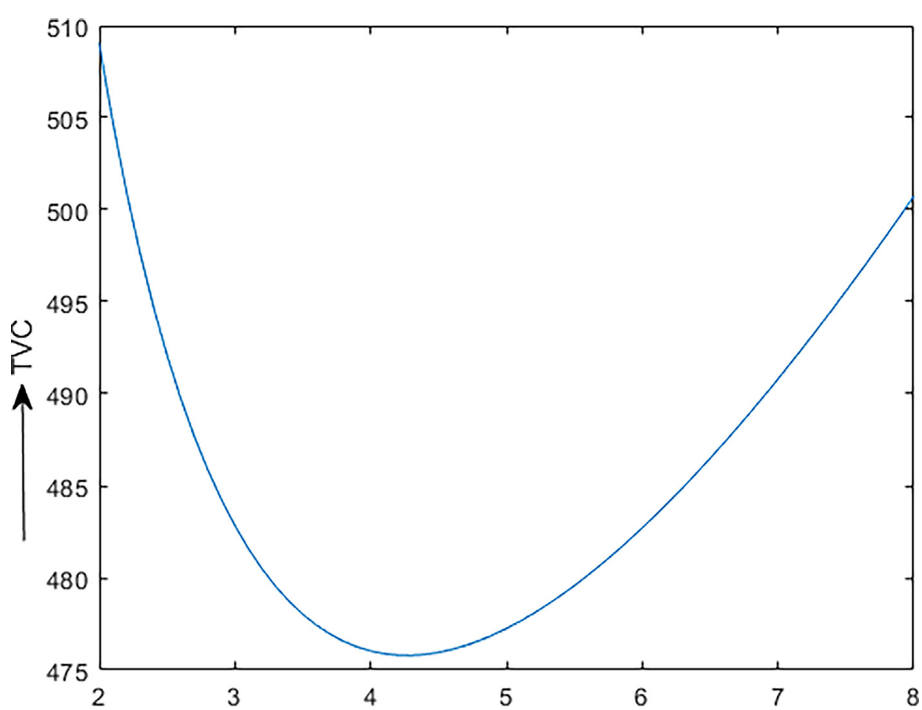

Fig. 5. Behavior of the company's total cost in Example 1.

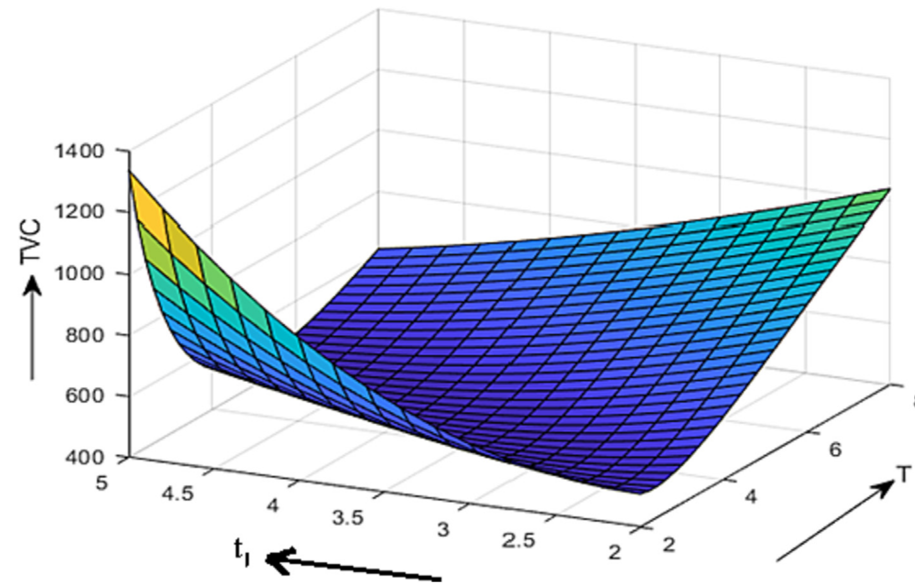

Fig. 6. Behavior of the company's total cost in Example 2.

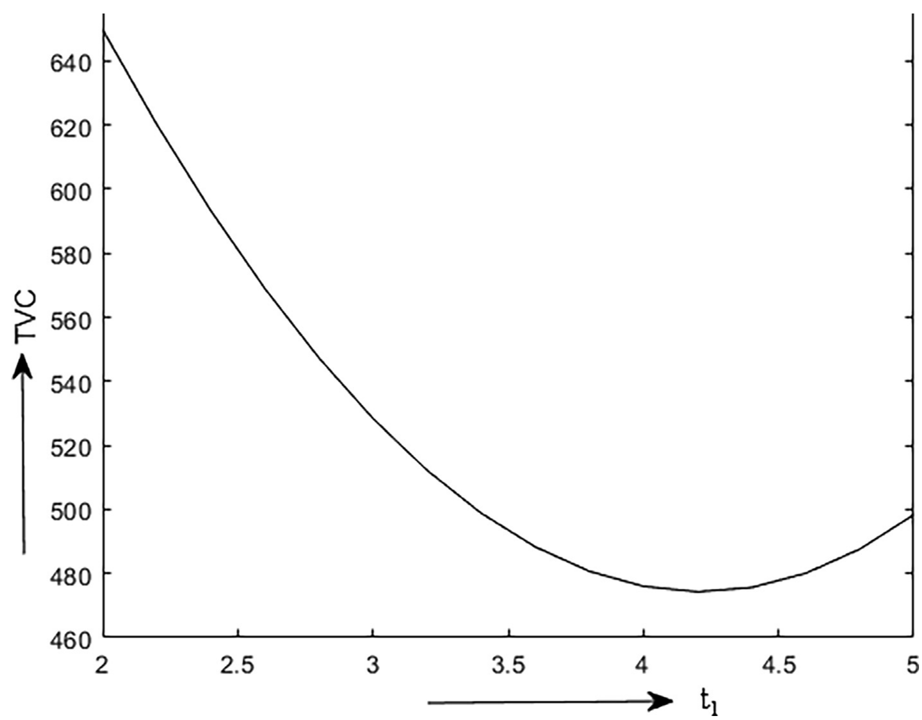

Fig. 7. Behavior of the company's total cost with respect to t_1 in Example 2.

interesting observation is that the optimal value of the company's initial best positive stock level is not affected by the changes of the parameters σ and c_l.

(iv) The optimal period for stock available inventory (t_1^*) decreases initially and then increases for the positive changes in ordering cost $c_\circ$. In addition, it is noticed that the best t_1^* increases when the parameters μ, c_l and c_s increase, while t_1^* decreases with respect to a, b, h_c, θ and c_j. Another interesting observation is that optimal period for stock available inventory (t_1^*) is not affected by the changes of the parameter σ.

(v) The company's best cycle length (T^*) for each business cycle declines noticeably when $a, \theta, b, h_c, c_l, c_s$ and $c_j (j = 1, 2, 3)$ increase, while the optimal T^* for the company increase when the parameters c_o and μ increase. However, the optimal T^* is independent from the backlog parameter σ.

(vi) The company's minimum cost (TVC^*) rises for increasing the value of all parameters except the parameter μ, which determines the time period after which deterioration starts. The company's minimum cost (TVC^*) decreases as the parameter μ increases. Parameters a and c_j have high effects on the company's minimum cost (TVC^*).

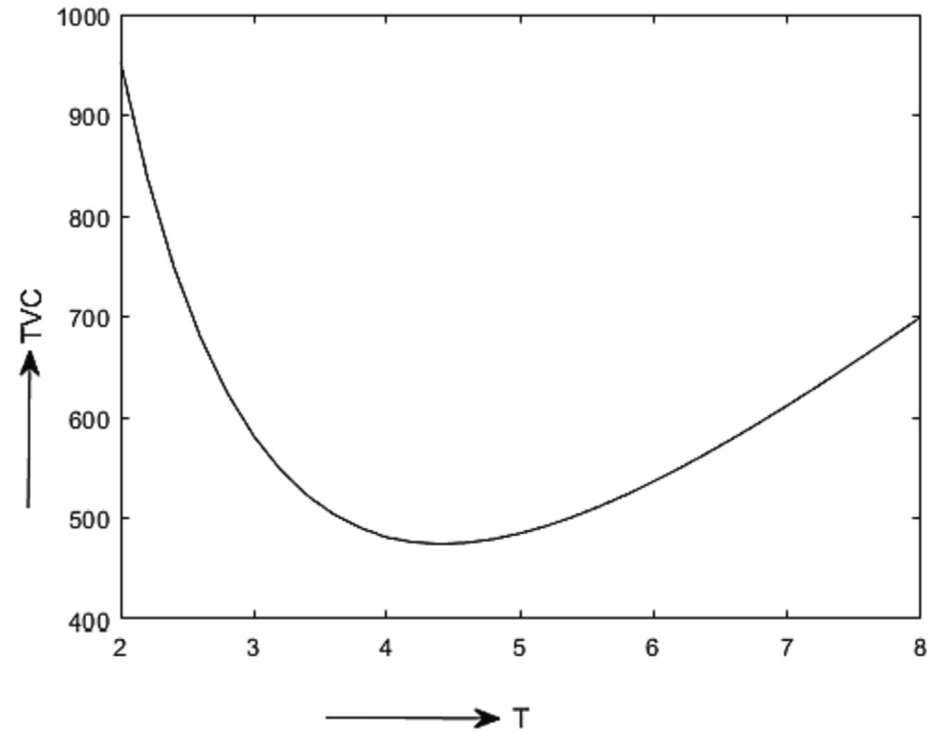

Fig. 8. Behavior of the company's total cost with respect to T in Example 2.

8. Management insights

Based on the accomplished sensitivity analyses for both models, the following guidelines are made for the company:

- If the cost of making each order is high, then the company should try to implement some measures to reduce this cost. Additionally, to mitigate the impact of high ordering costs, the company should consider increasing the quantity purchased, which would help reduce the average ordering cost per unit. By implementing these strategies, the company can optimize its inventory management and minimize the costs associated with ordering.
- The higher value of the parameter μ indicates a longer lifetime of the items, which is an index of better quality. From the analyses performed above, it is evident that the optimal cost decreases for increasing values of μ. Thus, the company should ensure the supply of the best quality items from the supplier so as to reduce the cost with higher value of μ. Further, due to item deterioration, some items are damaged at a fixed rate, and the company cannot collect any income from these items. Accordingly, if the company can reduce the number of damaged items, the total cost will decrease to a great extent. Thus, the company is recommended to reduce the rate of deterioration of items by providing a good storage environment and ensuring the supply of best quality items from the supplier.
- The unit acquisition price has the most adverse effect on the company's total cost per unit of time. A small increase in unit acquisition price results in a large increase in total cost. Consequently, the company should take careful steps to minimize unit acquisition price by selecting the supplier offering the lowest unit acquisition price. Besides, the company can make a higher purchase quantity under the order-based rebate scheme to enjoy a lower acquisition price.

9. Conclusion

This work investigates the effects of a quantity-based discount frame on the best inventory planning of a company dealing with a non-instantaneous decay item under both zero-end inventory and stock-out situations where customers' consumption behavior depends upon time. For both inventory procedures, the ordering cost, holding cost, and deterioration rate are assumed to be constant, while the unit acquisition cost is adopted as dependent on order size under the offered rebate scheme. Due to the offered rebate scheme and item deterioration, it becomes much more complicated to make the best inventory plan for the company. The utmost goal of this study is to make a decision framework for the best inventory planning of a company under a quantity based-based rebate scheme, taking into consideration all the possibilities of the purchased quantity amount. For that, the curvature of the company's total cost under both zero-end inventory and stock-out situations is examined analytically, and then, to achieve the best inventory plan for the company under both situations, two algorithms are proposed under the offered rebate scheme on unit acquisition price based on the purchased quantity. A real company dealing with seasonal mangoes in Bangladesh is considered, and the best inventory planning for the mango company is determined by exploiting the proposed model. To extract salient guidelines for minimizing the company's cost under a rebate scheme on unit acquisition price based on the purchased quantity, the changing trend of the company's best inventory strategy by altering the value of each system parameter is examined by accomplishing sensitivity analyses. For instance, when the cost of making each order is high, the company should increase the quantity purchased to shrink the average ordering cost per unit. Moreover, the company is recommended to reduce the rate of deterioration of items by providing a good storage environment and ensuring the supply of the best quality items from the supplier. Besides, the company should make a higher purchase quantity under the order-based rebate scheme to enjoy a lower acquisition price.

No consequence of the unit selling price on the customers' consumption behavior is considered during model formulations. Therefore, considering the consequences of price along with time on demand would be a worthy future research line. Moreover, adopting uncertainty on demand would be another important immediate extension of the proposed inventory procedures.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Supplementary material

Supplementary data to this article can be found online at https://doi.org/10.1016/j.jksus.2023.102840.

References

Ai, X.Y., Zhang, J.L., Wang, L., 2017. Optimal joint replenishment policy for multiple non-instantaneous deteriorating items. Int. J. Prod. Res. 55 (16), 4625–4642.

Alfares, H.K., Ghaithan, A.M., 2016. Inventory and pricing model with price-dependent demand, time-varying holding cost, and quantity discounts. Comput. Ind. Eng. 94, 170–177.

Alshanbari, H.M., El-Bagoury, A.A.A.H., Khan, M., Mondal, S., Shaikh, A.A., Rashid, A., 2021. Economic order quantity model with weibull distributed deterioration under a mixed cash and prepayment scheme. Comput. Intell. Neurosci. 2021.

Cárdenas-Barrón, L.E., Mandal, B., Sicilia, J., San-José, L.A., Abdul-Jalbar, B., 2021. Optimizing price, order quantity, and backordering level using a nonlinear holding cost and a power demand pattern. Comput. Oper. Res. 133, 105339.

Cunha, L.R.A., Delfino, A.P.S., dos Reis, K.A., Leiras, A., 2018. Economic production quantity (EPQ) model with partial backordering and a discount for imperfect quality batches. Int. J. Prod. Res. 56 (18), 6279–6293.

De-la-Cruz-Márquez, C.G., Cárdenas-Barrón, L.E., Mandal, B., 2021. An inventory model for growing items with imperfect quality when the demand is price sensitive under carbon emissions and shortages. Mathe. Probl. Eng. 2021.

Dobson, G., Pinker, E.J., Yildiz, O., 2017. An EOQ model for perishable goods with age-dependent demand rate. Eur. J. Oper. Res. 257 (1), 84–88.

Duary, A., Khan, M.A.A., Pani, S., Shaikh, A.A., Hezam, I.M., Alraheedi, A.F., Gwak, J., 2022. Inventory model with nonlinear price-dependent demand for non-instantaneous decaying items via advance payment and installment facility. AIMS Mathe. 7 (11), 19794–19821.

Gao, H., Wang, D., Santibanez Gonzalez, E.D., Ju, Y., 2019. Optimal stocking strategies for inventory mechanism with a stochastic short-term price discount and partial backordering. Int. J. Prod. Res. 57 (24), 7471–7500.

Khan, M.A.A., Shaikh, A.A., Panda, G.C., Konstantaras, I., Taleizadeh, A.A., 2019. Inventory system with expiration date: Pricing and replenishment decisions. Comput. Ind. Eng. 132, 232–247.

Khan, M., Shaikh, A.A., Panda, G.C., Bhunia, A.K., Konstantaras, I., 2020. Non-instantaneous deterioration effect in ordering decisions for a two-warehouse inventory system under advance payment and backlogging. Ann. Oper. Res. 289 (2), 243–275.

Khan, M.A.A., Shaikh, A.A., Cárdenas-Barrón, L.E., 2021. An inventory model under linked-to-order hybrid partial advance payment, partial credit policy, all-units discount and partial backlogging with capacity constraint. Omega 103, 102418.

Khan, M.A.A., Shaikh, A.A., Cárdenas-Barrón, L.E., Mashud, A.H.M., Treviño-Garza, G., Céspedes-Mota, A., 2022a. An inventory model for non-instantaneously deteriorating items with nonlinear stock-dependent demand, hybrid payment scheme and partially backlogged shortages. Mathematics 10 (3), 434.

Khan, M.A.A., Halim, M.A., AlArjani, A., Shaikh, A.A., Uddin, M.S., 2022b. Inventory management with hybrid cash-advance payment for time-dependent demand, time-varying holding cost and non-instantaneous deterioration under backordering and non-terminating situations. Alex. Eng. J. 61 (11), 8469–8486.

Khan, M.A.A., Ahmed, S., Babu, M.S., Sultana, N., 2022c. Optimal lot-size decision for deteriorating items with price-sensitive demand, linearly time-dependent holding cost under all-units discount environment. Int. J. Syst. Sci.: Oper. Logist. 9 (1), 61–74.

Kumar, A., Santra, P.K., Mahapatra, G.S., 2023. Fractional order inventory system for time-dependent demand influenced by reliability and memory effect of promotional efforts. Comput. Ind. Eng. 179, 109191.

Kumar, S., Singh, A.K., Patel, M.K., 2016. Optimization of Weibull deteriorating items inventory model under the effect of price and time dependent demand with partial backlogging. Sādhanā 41 (9), 977–984.

Lesmono, D., Limansyah, T., Loedy, N., 2020. A joint return policy for a multi-item perishable inventory model with deterministic demands, return and all-units discount. Int. J. Mathe. Eng. Manage. Sci. 5 (3), 416.

Li, G., He, X., Zhou, J., Wu, H., 2019. Pricing, replenishment and preservation technology investment decisions for non-instantaneous deteriorating items. Omega 84, 114–126.

Li, R., Teng, J.T., Chang, C.T., 2021. Lot-sizing and pricing decisions for perishable products under three-echelon supply chains when demand depends on price and stock-age. Ann. Oper. Res. 307 (1), 303–328.

Lin, H.J., 2018. Investing in transportation emission cost reduction on environmentally sustainable EOQ models with partial backordering. J. Appl. Sci. Eng. 21 (3), 291–303.

Lin, H.J., 2021. An economic production quantity model with backlogging and imperfect rework process for uncertain demand. Int. J. Prod. Res. 59 (2), 467–482.

Manna, A.K., Khan, M.A.A., Rahman, M.S., Shaikh, A.A., Bhunia, A.K., 2022. Interval valued demand and prepayment-based inventory model for perishable items via parametric approach of interval and meta-heuristic algorithms. Knowl.-Based Syst. 242, 108343.

Pakhira, R., Sarkar, S., Ghosh, U., 2020. Study of memory effect in an inventory model for deteriorating items with partial backlogging. Comput. Ind. Eng. 148, 106705.

Panda, G.C., Khan, M., Shaikh, A.A., 2019. A credit policy approach in a two-warehouse inventory model for deteriorating items with price-and stock-dependent demand under partial backlogging. J. Ind. Eng. Int. 15 (1), 147–170.

Pervin, M., Roy, S.K., Weber, G.W., 2018. Analysis of inventory control model with shortage under time-dependent demand and time-varying holding cost including stochastic deterioration. Ann. Oper. Res. 260 (1), 437–460.

Prasad, K., Mukherjee, B., 2016. Optimal inventory model under stock and time dependent demand for time varying deterioration rate with shortages. Ann. Oper. Res. 243 (1), 323–334.

Qiu, X., Lee, C.Y., 2019. Quantity discount pricing for rail transport in a dry port system. Transport. Res. Part E: Logist. Transport. Rev. 122, 563–580.

Rabbani, M., Zia, N.P., Rafiei, H., 2015. Coordinated replenishment and marketing policies for non-instantaneous stock deterioration problem. Comput. Ind. Eng. 88, 49–62.

Rahman, M.S., Khan, M.A.A., Halim, M.A., Nofal, T.A., Shaikh, A.A., Mahmoud, E.E., 2021. Hybrid price and stock dependent inventory model for perishable goods with advance payment related discount facilities under preservation technology. Alex. Eng. J. 60 (3), 3455–3465.

Rahman, M.S., Duary, A., Khan, M., Shaikh, A.A., Bhunia, A.K., 2022. Interval valued demand related inventory model under all units discount facility and deterioration via parametric approach. Artif. Intell. Rev. 55 (3), 2455–2494.

San-José, L.A., Sicilia, J., Alcaide-López-de-Pablo, D., 2018. An inventory system with demand dependent on both time and price assuming backlogged shortages. Eur. J. Oper. Res. 270 (3), 889–897.

San-José, L.A., Sicilia, J., Cárdenas-Barrón, L.E., Gutiérrez, J.M., 2019. Optimal price and quantity under power demand pattern and non-linear holding cost. Comput. Ind. Eng. 129, 426–434.

San-José, L.A., Sicilia, J., González-De-la-Rosa, M., Febles-Acosta, J., 2020. Best pricing and optimal policy for an inventory system under time-and-price-dependent demand and backordering. Ann. Oper. Res. 286 (1), 351–369.

Shaikh, A.A., Khan, M.A.A., Panda, G.C., Konstantaras, I., 2019. Price discount facility in an EOQ model for deteriorating items with stock-dependent demand and partial backlogging. Int. Trans. Oper. Res. 26 (4), 1365–1395.

Taleizadeh, A.A., Pentico, D.W., 2014. An economic order quantity model with partial backordering and all-units discount. Int. J. Prod. Econ. 155, 172–184.

Taleizadeh, A.A., Stojkovska, I., Pentico, D.W., 2015. An economic order quantity model with partial backordering and incremental discount. Comput. Ind. Eng. 82, 21–32.

Taleizadeh, A.A., Hazarkhani, B., Moon, I., 2020. Joint pricing and inventory decisions with carbon emission considerations, partial backordering and planned discounts. Ann. Oper. Res. 290 (1), 95–113.

Wu, K.S., Ouyang, L.Y., Yang, C.T., 2006. An optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand and partial backlogging. Int. J. Prod. Econ. 101 (2), 369–384.

Xu, C., Zhao, D., Min, J., Hao, J., 2021. An inventory model for nonperishable items with warehouse mode selection and partial backlogging under trapezoidal-type demand. J. Oper. Res. Soc. 72 (4), 744–763.