Journal of King Saud University – Science 32 (2020) 858–861

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

A new algorithm for computing the differential transform in nonlinear two-dimensional partial differential equations

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ARTICLE INFO

Article history:
Received 27 November 2018
Accepted 11 April 2019
Available online 12 April 2019

Keywords:
Differential transform method
Nonlinear partial differential equations
Multivariable Faa di Bruno formula

ABSTRACT

In this work, a new algorithm is proposed for computing the differential transform of two-dimensional nonlinear functions. This algorithm overcomes the drawbacks of previous algorithms as it is straightforward for any form of analytic nonlinearities and does not require any intermediate calculations or algebraic manipulations. This is accomplished by defining a new form for two dimensional polynomials that generalize the differential transform of the corresponding one-dimensional function to higher dimensions. The correctness of this algorithm is proved via the multivariable Faa di Bruno formula. Several examples with different types of nonlinearities are solved to verify the efficiency of the proposed algorithm.

1. Introduction

The differential transform method (DTM) (Zhou, 1986) is designed to obtain Taylor series solution of different types of equations via an iterative scheme. The advantage of this method is reducing the size of computations and it is applicable to many types of applied problems (Jang et al., 2001). The DTM has been generalized to solve partial differential equations (PDEs) in higher dimensions (Kurnaz et al., 2005; Islam et al., 2009). The two-dimensional differential transform method (2D-DTM) (Chen and Ho, 1999) and its modifications are widely utilized to solve PDEs that arise in many areas of science and engineering (Kadkhoda et al., 2018; Chang and Chang, 2009; Ayaz, 2003).

Although it has been shown that the 2D-DTM is an efficient tool for handling nonlinear PDEs in two dimensions, the reported researches only handled the polynomial type nonlinearity in the unknown function and its derivatives. The classical way to calculate the differential transform for other types of nonlinearities is by expanding the nonlinear function as an infinite power series then compute the differential transform of the infinite series terms (Zhou, 1986). The obvious drawback of this technique is the computational cost and complexities.

An algorithm to compute the differential transform of nonlinear function in two dimensions is presented in Chang and Chang, 2009. This algorithm uses calculus along with the properties of DTM to formulate an equation which is solved for the required differential transform. But for different types of nonlinear function, the algorithm performs a different series of algebraic manipulations. Also, in many cases, it requires the computation of differential transforms of some other functions to be used in computing the differential transform of the originally required function. This is more difficult for composite nonlinearities.

Another approach considered by some authors is using the projected, or reduced, DTM (Jang, 2010; Yua et al., 2016). In this technique, the Taylor series of the unknown function is considered only with respect to some of the independent variables instead of considering the Taylor series for all variables. But this approach only works if the unknown function is in a separable form.

Recently, an algorithm which uses the Adomian polynomials has been successfully applied to integer and fractional order nonlinear differential equations (Elsaid, 2012a,b). The advantage of this algorithm is that the differential transform of any analytic nonlinear function is computed with less computational work. The algorithm also benefits from the availability of several fast algorithms that have been developed for computing Adomian polynomials (Duan, 2011, 2010). But due to the form of the Adomian polynomials, this algorithm is limited to one-dimensional problems.

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2019.04.004
1018-3647/

In this article, a new algorithm is proposed to generalize the work in Elsaid (2012a,b) to two-dimensional problems. First, two-dimensional polynomials are composed by modifying the form of the classical Adomian polynomials. Then, the differential transform of nonlinear function is directly computed by replacing the dependent variable component in these polynomials by the corresponding differential transform component having the same index. This algorithm computes the differential transform for any analytic nonlinearity in separable and nonseparable forms without dealing with infinite series or using mathematical manipulations. We illustrate the idea of the algorithm in two-dimensional problems, but it is easily generalized to higher dimensions.

2. Differential transform in two dimensions

The basic idea of the DTM is to overcome the symbolic calculations for the derivatives of functions needed for calculating the Taylor series coefficients. These coefficients are determined by solving the recursive algebraic equation induced from the given differential equation. For two dimensional case, let $v(x,y)$ be the solution function which is analytic at a point (x_0,y_0). Then, it can be represented by the two-dimensional Taylor series (also referred to as the inverse differential transform) of the form

$$v(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}V(i,j)(x-x_0)^i(y-y_0)^j, \tag{1}$$

where $V(i,j)$ is the differential transform of $v(x,y)$ and is defined for non-negative integers i and j by Chen and Ho (1999)

$$V(i,j)=\frac{1}{i!j!}\left[\frac{\partial^{i+j}v(x,y)}{\partial x^i\partial y^j}\right]_{(x_0,y_0)}. \tag{2}$$

Consider the uncorrelated functions $v(x,y), p(x,y)$, and $q(x,y)$ and their corresponding differential transforms $V(i,j), P(i,j)$, and $Q(i,j)$, respectively. Then Chen and Ho (1999):

1- If $v(x,y)=p(x,y)\pm q(x,y)$, then $V(i,j)=P(i,j)\pm Q(i,j)$.
2- If $v(x,y)=ap(x,y)$, then $V(i,j)=aP(i,j)$, where a is a constant.
3- If $v(x,y)=p(x,y)q(x,y)$, then $V(i,j)=\sum_{m=0}^{i}\sum_{n=0}^{j}P(m,j-n)Q(i-m,n)$.
4- If $v(x,y)=x^ry^s$, then $V(i,j)=\delta(i-r)\delta(j-s)$ where $\delta(j-k)=\begin{cases}1 & j=k\\ 0 & j\neq k\end{cases}$.
5- If $v(x,y)=\frac{\partial^{r+s}p(x,y)}{\partial x^r\partial y^s}$, then $V(i,j)=\frac{(i+r)!(j+s)!}{i!j!}P(i+r,j+s)$.

It is obvious that there are no forms for obtaining the differential transform of nonlinear functions other than polynomials. Even for a power function, the differential transform is obtained by property (3) and requires nested summations that lead to computational complexities for high values of i and j.

3. The proposed algorithm

In the standard Adomian method, the solution v of a functional equation is defined by the series

$$v=\sum_{n=0}^{\infty}v_n, \tag{3}$$

and a nonlinear function $f(v)$ is represented by the series

$$f(v)=\sum_{m=0}^{\infty}A_m, \tag{4}$$

where A_m are the Adomian polynomials which are computed by the relation Adomian (1994)

$$A_m=\frac{1}{m!}\left[\frac{d^m}{d\lambda^m}\left[f\left(\sum_{i=0}^{\infty}\lambda^i v_i\right)\right]\right]_{\lambda=0},\quad m=0,1,\cdots. \tag{5}$$

A rearrangement of Adomian polynomials yields the following form Adomian (1988) and Behiry et al. (2007)

$$\begin{aligned}A_0&=f(v_0)\\ A_1&=v_1f^{(1)}(v_0)\\ A_2&=v_2f^{(1)}(v_0)+\frac{1}{2!}v_1^2f^{(2)}(v_0)\\ &\vdots\end{aligned} \tag{6}$$

The algorithm presented in Elsaid (2012a,b) shows how the differential transform of nonlinear functions in one dimension problems can be obtained by simply replacing the dependent variable components in the Adomian polynomials form by their corresponding differential transform component of the same index. We generalize that algorithm by defining two-dimensional polynomials as follows. In the rearranged Adomian polynomials given by (6), A_m is replaced by the two-dimensional polynomial $P_{i,j}$ according to the following relation

$$A_m\Rightarrow\sum_{i+j=m}P_{i,j}, \tag{7}$$

and in a similar manner, v_m is replaced by the two-dimensional variable $v_{i,j}$ as follows

$$v_m\Rightarrow\sum_{i+j=m}v_{i,j}. \tag{8}$$

The one dimensional polynomials and variables in the Eqs. (6) are replaced by the two dimensional ones to obtain

$$\begin{aligned}&P_{0,0}=f(v_{0,0})\\ &P_{1,0}+P_{0,1}=(v_{1,0}+v_{0,1})f^{(1)}(v_{0,0})\\ &P_{2,0}+P_{1,1}+P_{0,2}=(v_{2,0}+v_{1,1}+v_{0,2})f^{(1)}(v_{0,0})\\ &+\frac{1}{2!}(v_{1,0}+v_{0,1})^2f^{(2)}(v_{0,0})\\ &\vdots\end{aligned}$$

By equating the terms having the same indicial sum on both sides of each equation, we propose the following definition for two-dimensional polynomials $P_{i,j}$

$$\begin{aligned}P_{0,0}&=f(v_{0,0})\\ P_{1,0}&=v_{1,0}f^{(1)}(v_{0,0})\\ P_{0,1}&=v_{0,1}f^{(1)}(v_{0,0})\\ P_{2,0}&=v_{2,0}f^{(1)}(v_{0,0})+\frac{1}{2!}(v_{1,0})^2f^{(2)}(v_{0,0})\\ P_{1,1}&=v_{1,1}f^{(1)}(v_{0,0})+v_{1,0}v_{0,1}f^{(2)}(v_{0,0})\\ P_{0,2}&=v_{0,2}f^{(1)}(v_{0,0})+\frac{1}{2!}(v_{0,1})^2f^{(2)}(v_{0,0})\\ &\vdots\end{aligned} \tag{9}$$

In the following theorem, we show how the differential transform $F(i,j)$ of the nonlinear function $f(v(x,y))$ is directly obtained from the two-dimensional polynomials $P_{i,j}$.

Theorem 1. *Let $f(v(x,y))$ be a nonlinear function where both functions $f(v)$ and $v(x,y)$ are differentiable to order $i+j$ at the neighborhood of a point (x_0,y_0). Then, the differential transform $F(i,j)$*

of $f(v(x,y))$ is obtained by replacing each $v_{k,h}$ and $\frac{\partial^{r+s} v_{k,h}}{\partial x^r \partial y^s}$ in the two-dimensional polynomials $P_{i,j}$ defined by (9) by $V(k,h)$ and $\frac{(k+r)!(h+s)!}{k!h!} V(k+r,h+s)$, respectively.

Proof. As $f(v)$ and $v(x,y)$ are differentiable to order $i+j$ at a neighborhood of point (x_0,y_0), then we have

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} f|_{(v(x_0,y_0))} = i!j! \sum_{\sigma=1}^{i+j} \frac{d^\sigma}{du^\sigma} f|_{(v(x_0,y_0))} \sum_{E_\sigma} \prod_{\Lambda_{i,j}} \frac{1}{c_{\mu_n}!} \left(\frac{v_{|\mu_n|}}{\mu_n!} \right)^{c_{\mu_n}}, \tag{10}$$

where μ_n denotes an ordered pair (μ_{1_n}, μ_{2_n}) with the standard multi-index notations $|\mu_n| = \mu_{1_n} + \mu_{2_n}, \mu_n! = \mu_{1_n}! \mu_{2_n}!$, and $v_{|\mu_n|} = \frac{\partial^{|\mu_n|}}{\partial x^{\mu_{1n}} \partial y^{\mu_{2n}}} v$. The index set E_σ is defined for non-negative integers c_{μ_n} by

$$E_\sigma = \left\{ \left(c_{\mu_1}, c_{\mu_2}, \ldots, c_{\mu_{i+j}} \right) : c_{\mu_n} \in \mathbb{N}, 1 \leqslant |\mu_n| \leqslant i+j, \sum_{|\mu_n|=1}^{i+j} c_{\mu_n} = \sigma \right\} \tag{11}$$

and the set of ordered pairs $\Lambda_{i,j}$ is defined by

$$\Lambda_{i,j} = \left\{ \left(\mu_1, \mu_2, \ldots, \mu_{i+j} \right) : 1 \leqslant |\mu_n| \leqslant i+j, \sum_{|\mu_n|=1}^{i+j} c_{\mu_n} \mu_n = (i,j) \right\} \tag{12}$$

This follows from Faa di Bruno formula for multivariable case (Encinas and Masque, 2003). From this formula we can write the differential transform $F(i,j)$ for nonlinear function $f(v(x,y))$ as

$$F(i,j) = \sum_{\sigma=1}^{i+j} f^{(\sigma)}(V(0,0)) \sum_{E_\sigma} \prod_{\Lambda_{i,j}} \frac{1}{c_{\mu_n}!} \left(V(\mu_{1_n}, \mu_{2_n}) \right)^{c_{\mu_n}}, \tag{13}$$

which is the series form for the two-dimensional polynomials $P_{i,j}$ with each $v_{k,h}$ replaced by $V(k,h)$.

Next, consider the case where the nonlinear function is given by $f(v_{r,s}(x,y))$ where $v_{r,s}(x,y)$ denotes the partial derivative $\frac{\partial^{r+s} v}{\partial x^r \partial y^s}$. Then, formula (10) becomes

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} f|_{(v_{r,s}(x_0,y_0))} = i!j! \sum_{\sigma=1}^{i+j} \frac{d^\sigma}{dv^\sigma} f|_{(v_{r,s}(x_0,y_0))} \sum_{E_\sigma} \prod_{\Lambda_{i,j}} \frac{1}{c_{\mu_n}!} \left(\frac{(v_{r,s})_{|\mu_n|}}{\mu_n!} \right)^{c_{\mu n}}. \tag{14}$$

The constants are adjusted to coincide with the definition of the differential transform as

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} f|_{(v_{r,s}(x_0,y_0))} = i!j! \sum_{\sigma=1}^{i+j} \frac{d^\sigma}{du^\sigma} f|_{\left(r!s! \frac{v_{r,s}(x_0,y_0)}{r!s!}\right)} \times \sum_{E_\sigma} \prod_{\Lambda_{i,j}} \frac{1}{c_{\mu_n}!} \left(\frac{(r+\mu_{1_n})!(s+\mu_{2_n})!(v_{r,s})_{|\mu_n|}}{\mu_n!(r+\mu_{1_n})!(s+\mu_{2n})!} \right)^{c_{\mu_n}}, \tag{15}$$

and the differential transform $F(i,j)$ for nonlinear function $f(v_{r,s}(x,y))$ is given by

$$F(i,j) = \sum_{\sigma=1}^{i+j} f^{(\sigma)}(r!s!V(r,s)) \times \sum_{E_\sigma} \prod_{\Lambda_{i,j}} \frac{1}{c_{\mu_n}!} \left(\frac{(r+\mu_{1_n})!(s+\mu_{2n})!V(r+\mu_{1_n}, s+\mu_{2_n})}{\mu_n!} \right)^{c_{\mu n}}, \tag{16}$$

which is the series form of $P_{i,j}$ with each $\frac{\partial^{r+s} v_{k,h}}{\partial x^r \partial y^s}$ replaced by $\frac{(k+r)!(h+s)!}{k!h!} V(k+r,h+s)$. □

4. Examples

In this section, three PDEs with different types of nonlinearities are solved using the proposed algorithm. These nonlinearities include rational functions, composite nonlinearity and with non-separable solution function, and system of coupled nonlinear PDEs. In all problems, i and j are non-negative integers.

Example 1. Consider the nonlinear Schrödinger equation with saturable nonlinearity Pankov and Rothos (2008)

$$\imath q_t + q_{xx} + \frac{b|q|^2 q}{1+|q|^2} = 0, \tag{17}$$

where $\imath = \sqrt{-1}$. We consider this problem for an initial condition $q(x,0) = e^{\imath x}$. Then the recurrence scheme is given by

$$\begin{cases} \imath(j+1)Q(i,j+1) + (i+1)(i+2)Q(i+2,j) + bF(i,j) = 0, \\ Q(i,0) = \frac{\imath^i}{i!}, \end{cases} \tag{18}$$

where $F(i,j)$ denotes the differential transform of the nonlinear term $\frac{q^2 \tilde{q}}{1+q\tilde{q}}$ where $\tilde{q}$ denotes the complex conjugate of v. By forming the two-dimensional polynomial $P_{i,j}$ as in (9), $F(i,j)$ is directly obtained as

$$F(0,0) = \frac{(Q(0,0))^2 \tilde{Q}(0,0)}{1+\tilde{Q}(0,0)Q(0,0)},$$

$$F(0,1) = \frac{Q(0,0)\left(\tilde{Q}(0,0)Q(0,1)\left(2+\tilde{Q}(0,0)Q(0,0)\right) + \tilde{Q}(0,1)Q(0,0)\right)}{\left(1+\tilde{Q}(0,0)Q(0,0)\right)^2},$$

$$F(1,0) = \frac{Q(0,0)\left(\tilde{Q}(0,0)Q(1,0)\left(2+\tilde{Q}(0,0)Q(0,0)\right) + \tilde{Q}(1,0)Q(0,0)\right)}{\left(1+\tilde{Q}(0,0)Q(0,0)\right)^2},$$

$$F(0,2) = \frac{1}{\left(1+\tilde{Q}(0,0)Q(0,0)\right)^3} \Big(\left(\tilde{Q}(0,0)Q(0,1)\right)^2 + 2Q(0,0)\tilde{Q}(0,1)Q(0,1) + Q(0,0)\tilde{Q}(0,0)\Big(Q(0,2)\Big(2+3\tilde{Q}(0,0)Q(0,0) + (Q(0,0))^2\left(\tilde{Q}(0,0)\right)^2\Big) + Q(0,0)\Big(\tilde{Q}(0,2) + Q(0,0)\Big(\tilde{Q}(0,0)\tilde{Q}(0,2) - \left(\tilde{Q}(0,1)\right)^2\Big)\Big)\Big)\Big),$$

⋮

and so on. Substituting these terms in the inverse differential transform yields Taylor series of the exact solution to problem (17) given by $q(x,t) = e^{\imath\left(x + \left(\frac{b}{2}-1\right)t\right)}$.

Example 2. Consider the PDE with composite nonlinearity

$$\begin{cases} v_t - v_x + e^{x+t} = e^{\tan v}, \\ v(x,0) = \tan^{-1} x, \; v(0,t) = \tan^{-1} t. \end{cases} \tag{19}$$

The recurrence scheme of this problem is given by

$$\begin{cases} (i+1)V(i+1,j) - (j+1)V(i,j+1) + \frac{1}{i!j!} = F(i,j), \\ V(i,0) = V(0,j) = 0, \quad \text{for even } i \text{ and } j, \\ V(i,0) = \frac{(-1)^{\lfloor i/2 \rfloor}}{i}, V(0,j) = \frac{(-1)^{\lfloor j/2 \rfloor}}{j}, \text{otherwise} \end{cases} \tag{20}$$

where $\lfloor z \rfloor$ denotes the floor function of z and $F(i,j)$ denotes the differential transform of the nonlinear term $e^{\tan v}$ obtained from the two-dimensional polynomial $P_{i,j}$ as follows

$$F(0,0)=e^{\tan(V(0,0))},$$
$$F(0,1)=V(0,1)\sec^2(V(0,0))e^{\tan(V(0,0))},$$
$$F(1,0)=V(1,0)\sec^2(V(0,0))e^{\tan(V(0,0))},$$
$$F(0,2)=\sec^2(V(0,0))e^{\tan(V(0,0))}\left((V(0,1))^2\left(\frac{\sec^2(V(0,0))}{2}\right.\right.$$
$$\left.\left.+\tan(V(0,0))+V(0,2)\right)\right),$$
$$\vdots$$

By solving Eq. (20), the following differential transform components are obtained: $V(i,i)=0, V(1,2)=V(2,1)=-1, V(1,3)=V(3,1)=0, V(2,3)=V(3,2)=2,\ldots$. When these values are substituted in the inverse differential transform series, we obtain

$$v(x,t)=x+t-\frac{x^3+t^3}{3}+\frac{x^5+t^5}{5}-\frac{x^7+t^7}{7}+\ldots-\left(x^2t+xt^2\right)+\left(x^4t+xt^4\right)+\ldots+2\left(x^3t^2+x^2t^3\right)-3\left(x^5t^2+x^2t^5\right)+\ldots. \quad (21)$$

This is Taylor series of the exact solution of this problem given by $v(x,t)=\tan^{-1}(x+t)$.

Example 3. Consider the coupled Burger's equations Abdou and Soliman (2005)

$$\begin{cases} w_t-w_{xx}+(wv)_x-2ww_x=0\\ v_t-v_{xx}+(wv)_x-2vv_x=0\\ w(x,0)=\sin x,\ v(x,0)=\sin x\end{cases} \quad (22)$$

The recurrence scheme of this problem is

$$\begin{cases}(j+1)W(i,j+1)=(i+2)(i+1)W(i+2,j)+2F(i,j)-G(i,j),\\ (j+1)V(i,j+1)=(i+2)(i+1)V(i+2,j)+2D(k,h)-G(k,h),\\ W(k,0)=V(0,h)=0,\ \text{for even } i \text{ and } j\\ ,W(k,0)=\frac{1}{i!},V(0,h)=\frac{1}{j!}\ \text{otherwise.}\end{cases} \quad (23)$$

where $F(i,j), D(i,j)$, and $G(i,j)$ denote the differential transforms of ww_x, vv_x, and $(wv)_x$, respectively. The computations of the differential transform $G(i,j)$ yields

$$G(0,0)=W(0,0)V(1,0)+W(1,0)V(0,0),$$
$$G(0,1)=W(1,0)V(0,1)+W(1,1)V(0,0)+W(0,1)V(1,0)+V(1,1)W(0,0),$$
$$G(1,0)=2W(2,0)V(0,0)+2W(1,0)V(1,0)+2V(2,0)W(0,0),$$
$$G(0,2)=W(1,0)V(0,2)+W(1,1)V(0,1)+W(1,2)V(0,0)+V(1,0)W(0,2)+V(1,1)W(0,1)+V(1,2)W(0,0),$$
$$G(2,0)=3W(1,0)V(2,0)+3W(2,0)V(1,0)+3W(3,0)V(0,0)+3V(3,0)W(0,0),$$
$$G(1,1)=2W(1,0)V(1,1)+2W(2,0)V(0,1)+2W(1,1)V(1,0)+2W(2,1)V(0,0)+2V(2,0)W(0,1)+2V(2,1)W(0,0),$$
$$\vdots$$

The differential transforms $F(i,j)$ and $D(i,j)$ are obtained in a similar manner. Then, by solving the system of Eqs. (23), the following components are obtained: $W(0,1)=V(0,1)=0$, $W(1,1)=V(1,1)=-1, W(2,1)=V(2,1)=0$, $W(3,1)=V(3,1)=\frac{1}{3!},\ldots$. By substituting the obtained values of $W(k,h)$ and $V(k,h)$ in the inverse differential transform series, the solution of this problem takes the form

$$w(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(2i+1)!j!}(-t)^jx^{2i+1}, \quad (24)$$

$$v(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(2i+1)!j!}(-t)^jx^{2i+1}. \quad (25)$$

These are the Taylor series of the exact solution to the considered coupled Burger's equations given in Abdou and Soliman (2005) by $w(x,t)=e^{-t}\sin x$ and $v(x,t)=e^{-t}\sin x$.

5. Conclusion

A new algorithm for computing differential transforms of nonlinear functions when using 2D-DTM is presented. A definition is developed for two-dimensional polynomials which are utilized to directly obtain the differential transform for two-dimensional nonlinear function from the corresponding differential transform in one dimension. The algorithm is applicable for analytic nonlinear functions and does not require any calculus or algebraic manipulations. The correctness of the proposed algorithm is proved using the multivariable Faa di Bruno formula.

The numerical simulations carried out illustrate the advantages of the proposed algorithm over the reported algorithms in literature. The differential transform of the nonlinear rational function and of the composite nonlinear term are obtained in a straightforward manner without intermediate computations or algebraic manipulations and for a nonseparable solution function. Also the differential transforms of coupled nonlinearity that involves derivatives of the unknown functions are easily computed. The algorithm enhances the applicability of the 2D-DTM to solve real-world nonlinear PDEs. Higher dimension polynomials can be defined in a similar manner and the idea of the algorithm can be extended to higher order PDEs.

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