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ORIGINAL ARTICLE

A new fractional derivative and its application to explanation of polar bear hairs

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Abstract A new fractional derivative is defined through the variational iteration method, and its application in explaining the excellent thermal protection of polar bear hairs is elucidated. The fractal porosity of its inner structure makes a polar bear mathematically adapted for living in a harsh Arctic region.

1. Introduction

There are many definitions on fractional derivatives. The most used ones are (Yang, 2012):

1) Caputo's definition:

$$D_x^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}\frac{d^n f(t)}{dt^n}dt \tag{1}$$

2) Riemann–Liouville definition

$$D_x^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}f(t)dt \tag{2}$$

3) Jumarie's definition (Jumarie, 2006)

$$D_x^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}[f(t)-f(0)]dt \tag{3}$$

4) Xiao-Jun Yang's definition

$$D_x^{(\alpha)}f(x_0) = f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}|_{x=x_0} = \lim_{x\to x_0}\frac{\Delta^\alpha(f(x)-f(x_0))}{(x-x_0)^\alpha}, \tag{4}$$

where $\Delta^\alpha(f(x)-f(x_0)) \cong \Gamma(1+\alpha)\Delta(f(x)-f(x_0))$.

5) Chen's fractal derivative (Chen et al., 2010; Chen, 2006):

$$\frac{df}{dx^\alpha} = \lim_{s\to x}\frac{f(x)-f(s)}{x^\alpha - s^\alpha}, \tag{5}$$

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6) Ji-Huan He's fractal derivative (He, 2011, 2014)

$$\frac{Du}{Dx^{\alpha}} = \Gamma(1+\alpha) \lim_{\Delta x = x_1 - x_2 \to L} \frac{u(x_1) - u(x_2)}{(x_1 - x_2)^{\alpha}} \tag{6}$$

where Δx does not tend to zero, it can be the thickness (L) of a porous medium. Applications of the fractal derivative to fractal media have attracted much attention, for example it can model heat transfer and water permeation in multi-scale fabrics and wool fibers (Fan and He, 2012; Fan and Shang, 2013a,b).

The definition, Eq. (6), is similar to Leibniz's calculus. Leibniz did not take the limit in his infinitesimal calculus. The derivative of $f(x)$ with respect to x, in the sense of Leibniz's notation, is the standard part of the infinitesimal ratio:

$$f'(x) = st\left(\frac{\Delta f}{\Delta x}\right) = st\left(\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right) \tag{7}$$

instead of Newton's notation $f'(x) = \lim_{\Delta x \to 0} \Delta y / \Delta x$.

In a fractal medium, the distance between x_1 and x_2 tends to infinity ($\Delta x \to \infty$) even when $x_1 \to x_2$, and therefore Leibniz's work was nearer to fractal and Cantor sets which are the basis for fractional calculus (He, 2014).

2. Definition on fractional derivative through the variational iteration method

The variational iteration method was first used to solve fractional differential equations in 1998 (He, 1998), and it has been shown to solve a large class of nonlinear differential problems effectively, easily, and accurately with the approximations converging rapidly to accurate solutions, and now it has matured into a relatively fledged theory for various nonlinear problems, especially for fractional calculus (He, 1998, 2011, 2012; Wu, 2012). A complete review on its development and its application is available in Refs. (He, 2006, 2008).

We consider the following linear equation of n-th order

$$u^{(n)} = f(t) \tag{8}$$

By the variational iteration method (He, 1998), we have the following variational iteration algorithm

$$u_{m+1}(t) = u_m(t) + (-1)^n \int_{t_0}^{t} \frac{1}{(n-1)!}(s-t)^{n-1}[u_m^{(n)}(s) - f_m(s)]ds. \tag{9}$$

We introduce an integration operator I^n defined by He, 2014

$$I^n f = \int_{t_0}^{t} \frac{1}{(n-1)!}(s-t)^{n-1}[u_0^{(n)}(s) - f(s)]ds = \frac{1}{\Gamma(n)} \int_{t_0}^{t} (s-t)^{n-1}[f_0(s) - f(s)]ds \tag{10}$$

where $f_0(t) = u_0^{(n)}(t)$.

We can define a fractional derivative in the form

$$D_t^{\alpha} f = D_t^{\alpha} \frac{d^n}{dt^n}(I^n f) = \frac{d^n}{dt^n}(I^{n-\alpha} f) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1}[f_0(s) - f(s)]ds \tag{11}$$

where $f_0(t)$ is a known function, its physical explanation will be given in the next section.

3. An application

As an application of the new fractional derivative, we consider the fractal-like porous hairs of polar bear (He et al., 2011; Wang et al., 2012). Hairs of a polar bear (*Ursus maritimus*) are of superior properties such as the excellent thermal protection. How can polar bears resist such cold environment? Its fractal porosity plays an important role.

Using Fourier's Law of thermal conduction in fractal porosity of polar bear hairs (Yang, 2012), we obtain the following fractional differential equation

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(D \frac{\partial^{\alpha} T}{\partial x^{\alpha}}\right) = 0 \tag{12}$$

with boundary conditions

$$T(0) = T_0, T(L) = T_L \tag{13}$$

where T is the temperature, D is the thermal conductivity of heat flux in the fractal medium, α is the fractional dimensions of the fractal medium, $\partial^{\alpha}/\partial x^{\alpha}$ is the fractional derivative defined as (He, 2014)

$$\frac{\partial T^{\alpha}}{\partial x^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{t_0}^{t} (s-x)^{n-\alpha-1}[T_0(s) - T(s)]ds \tag{14}$$

where $T_0(x)$ can be the solution of its continuous partner of the problem with the same boundary/initial conditions of the fractal partner.

By the fractional complex transform (Li and He, 2010; He and Li, 2012; Li et al., 2012)

$$s = \frac{x^{\alpha}}{\Gamma(1+\alpha)} \tag{15}$$

Eq. (12) is converted to a partial differential equation, which reads

$$\frac{\partial}{\partial s}\left(D \frac{\partial T}{\partial s}\right) = 0 \tag{16}$$

Eq. (16) has the solution

$$T = a + bs = a + \frac{b x^{\alpha}}{\Gamma(1+\alpha)} \tag{17}$$

After incorporating the boundary conditions of Eq. (13), we have

$$T = T_0 + \frac{(T_L - T_0)}{L^{\alpha}} x^{\alpha} \tag{18}$$

It is obvious that the solution has the following remarkable property:

$$\frac{dT}{dx}(x=0) = \begin{cases} 0, \alpha > 1 \\ \frac{(T_L - T_0)}{L}, \alpha = 1 \\ \infty, \alpha < 1 \end{cases} \tag{19}$$

The slope at $x = 0$ depends strongly upon the value of the fractional order, or value of the fractal dimensions (He et al., 2012). For a polar bear the temperature of its body surface should be changed as smooth as possible, it requires $\alpha > 1$.

A hollow hair with a labyrinth-like fractal porosity in polar bear hairs was found in Ref. (He et al., 2011), this special structure guarantees $\alpha > 1$.

4. Conclusions

Using the variational iteration method, we can easily derive a more generalized fractional derivative. A simple example is given to illustrate how to solve fractional differential equations in the new fractional notation. The slope at the boundary depends strongly upon the fractal structure of porosity. The polar bear has evolved in a perfect mathematical way, and its mechanism can be used for biomimic design for various functional textiles.

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