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Boundedness and asymptotic stability of nonlinear Volterra integro-differential equations using Lyapunov functional

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ABSTRACT

In this paper, I consider Lyapunov functionals combined with the Laplace transform to obtain boundedness results regarding the solutions of the nonlinear Volterra integro-differential equations

$$x'(t) = A(t)x(t) + B(t) + \int_0^t C(t,s)f(x(s))ds + g(x(t)).$$

Asymptotic stability results regarding the zero solution are carried out for the case where $B(t)$ is identically zero. Numerical examples are proposed to perform the given results.

1. Introduction

An integro-differential equation is an ordinary differential equation of which one of the variables is an integral. They are used in a large number of physical domains. Maxwell's equations are probably their most famous representatives. They appear in problems of radiative energy transfers and problems of oscillations of a rope, a membrane or an axis. Many publication emphasizes also advantages of the integro-differential equations in various branches of technology, with special attention paid to large sense of power engineering. The famous Volterra integro-differential equations are largely used as models for a large class of semiconductor devices with abrupt *pn*-junctions (Unterreiter, 1996) which lead directly to both integro-equations types or integro-differential equations. Endemic infectious diseases where infection confers permanent immunity are also modelled by systems of nonlinear Volterra integro-differential equations type (Hethcote and Tudor, 1980). These models can take into account of distributed infectious period, immunity, birth and death dynamics.

The approximation by linear models made it possible to obtain practical results that were considered sufficient; it is not the same today where the concern for performance is more and more demanding. The study of nonlinear systems is added in order to get closer to physical reality. To all this we can add, the study of disturbances - elements external to the studied system, not only undesirable, but also mostly unpredictable. It is interesting in these conditions to define other forms of stability and to analyze the different theoretical implications for increasingly complex systems.

Continuous models are to be preferred (El Hajji, 2018, 2017, 2015; Sari et al., 2012), on grounds of realism, over discrete models, the scientifically faithful forms of such continuous models rarely have closed-form solutions. Where practically useful insights into solutions are sought, one may turn to numerical methods, applied to realistic models, to provide approximate values. In this situation, one seeks 'appropriate' numerical formulae; we are once more led to consider discrete equations.

When we observe the (deterministic) evolution of a quantity varying over time, we usually have discrete data, that is to say, values measured at regular (or sometimes irregular) time intervals, but rarely data continuously recorded. This naturally leads to the choice of models for difference equations (or recurrences). But these discrete sequences are sometimes easier to understand and to study if they are seen as the sampled values of a continuous (and even derivable) function of time but whose values would have been considered only at certain moments.

Continuous models are often preferred to discrete models by mathematicians (El Hajji, 2018, 2017, 2015; Sari et al., 2012) because

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the arsenal of tools they have developed to study them makes them generally easier to manipulate. For the biologist, there are cases where some will be more relevant than the others but most often there is the choice. On the other hand it is always useful to know how one passes from one to the other, by "smoothing" data to model them more simply continuously on the one hand or, conversely, by discretizing a model to study it with a computer on the other hand.

In this paper, I am interested in the qualitative analysis of solutions for the nonlinear Volterra integro-differential equations given by:

$$x'(t) = A(t)x(t) + B(t) + \int_0^t C(t,s)f(x(s))ds + g(x(t)) \tag{1}$$

for $t \in \mathbb{R}^+$. $x(t), A(t)$ and $B(t)$ are continuous scalar functions defined on $\mathbb{R}^+$. $f(x)$ and $g(x)$ are continuous scalar functions defined on $\mathbb{R}$. $C(t,s)$ is a scalar function defined on $\mathbb{R}^+ \times \mathbb{R}^+$.

I am mainly interested by boundedness results concerning solutions of Eq. (1). Throughout this paper, I make the following assumptions: there exist positive constants λ_1, λ_2 and M such that functions f, g and B satisfy:

$$|g(x)| \leqslant \lambda_1 |x|, \tag{2}$$

$$|f(x)| \leqslant \lambda_2 |x|, \tag{3}$$

and

$$|B| \leqslant M. \tag{4}$$

Several works have been devoted to the study of the qualitative analysis of solutions for different forms of the nonlinear Volterra integral differential equations. Recently, several authors have studied the behaviour of solutions of variant forms of (1). In particular, Medina (Medina, 2001, 1997, 1996) obtained stability and boundedness results of the solutions of the homogeneous part of (1) by means of representing the solution in terms of the resolvent matrix. Elaydi and Murakami (1998), used the notion of total stability and established results on the asymptotic behaviour of the zero solution of (1). Their work heavily depended on showing or assuming the summability of the resolvent matrix. However, a major limitation of this procedure is that the resolvent matrix is an abstract term. When $g(x) = x$ and $|f(x(t))| \leqslant \lambda_2(t)|x(t)|$, it was shown that the zero solution of (1) is uniformly asymptotically stable provided that $\int_0^\infty \lambda_2(t)dt < \infty$. More results regarding the stability of the zero solution of Volterra-like integral differential equations can be found in the literature, for example, Crisci et al. (1997), Elaydi (1994) and Agarwal and Pang (1997) and the references therein.

In this paper, I do not impose any condition on λ_2 other than it is simply a positive constant. I look in a first step to use a Lyupanov functional $V(t)$ coupled with the Laplace transform to obtain boundedness results concerning the solutions of Eq. (1). In a second step, stability and boundedness results are given for a particular case of Eq. (1). Examples are given as applications to the obtained results.

2. Main results

Definition 1. A function $y(t)$ is exponentially bounded for $t \geqslant 0$ if there exist two constants $m \geqslant 0$ and c such that

$$|y| \leqslant me^{ct}, \quad \forall\, t \geqslant 0.$$

Definition 2. If $y(t)$ is a piecewise continuous function defined for $t \geqslant 0$ of exponential order, then the Laplace transform $L(y)(s)$ of $y(t)$ is defined by the following integral expression:

$$L(y)(s) = \widetilde{Y}(s) = \int_0^{+\infty} e^{-st} y(t)dt,$$

where s is a real number.

For the first part, I assume that there exist a positive decreasing continuous function $\varphi(t) \in L^1([0,\infty))$

Lemma 1. *Consider a positive uniformly continuous scalar function $\beta(t)$ and a positive continuous scalar function $H(t)$ such that*

$$H(t) = \beta(t) + \lambda_3 \int_0^t \varphi(t-s)\beta(s)ds, \quad \lambda_3 > 0 \tag{5}$$

$$H'(t) = -\alpha\beta(t), \qquad \alpha > 0, \quad \beta(0) = 1, \tag{6}$$

then one has

$$\beta(t) + \int_0^t (\lambda_3\varphi(t-s) + \alpha)\beta(s)ds = 1, \tag{7}$$

$$\beta(t) \in L^1([0,\infty)), \tag{8}$$

and

$$\lim_{t\to+\infty} \beta(t) = 0 \tag{9}$$

Proof. By integrating (6) on $(0,t)$, one obtains

$$H(t) = H(0) - \alpha \int_0^t \beta(s)ds. \tag{10}$$

Now by using (5), one obtains $H(0) = \beta(0)$=1. By equating Eq. (5) to Eq. (10), we get

$$\beta(t) + \lambda_3 \int_0^t \varphi(t-s)\beta(s)ds = H(0) - \alpha \int_0^t \beta(s)ds.$$

Now simplifying the obtained equation to get

$$\beta(t) + \int_0^t (\lambda_3\varphi(t-s) + \alpha)\beta(s)ds = 1,$$

then expression (7) is verified.

From (10) we have,

$$\alpha \int_0^t \beta(s)ds = H(0) - H(t) \leqslant H(0) = 1$$

Since $\beta(t) \geqslant 0\ \forall t \geqslant 0$, and by the assumption $H'(t) = -\alpha\beta(t), \alpha > 0$, one deduces that H is a monotonically decreasing function. Therefore

$$\int_0^t \beta(s)ds \leqslant \frac{1}{\alpha}, \quad \forall\, t \geqslant 0.$$

In addition when $t \to \infty$, one obtains

$$\int_0^\infty \beta(s)ds \leqslant \frac{1}{\alpha}.$$

which means that $\beta(t) \in L^1([0,\infty))$ and (8) is then fulfilled.

In order to prove that $\beta(t) \to 0$ when $t \to +\infty$, I use a classical prove.

Let $\varepsilon > 0$, as $\beta(t)$ is uniformly continuous then

$$\exists \eta > 0,\ \forall x, y > 0,\ |x-y| < \eta, \quad |\beta(x) - \beta(y)| < \varepsilon.$$

The fact that $\beta(t)$ is positive and $\beta(t) \in L^1([0,\infty))$ derive

$$\exists A > 0,\ \forall x, y > A,\ |\int_x^y \beta(t)dt| < \eta\varepsilon \to \int_x^y \beta(t)dt < \eta\varepsilon.$$

Let $x > A$, one has

$$\begin{aligned}\eta\beta(x) = \int_x^{x+\eta} \beta(x)ds \leqslant &\ \int_x^{x+\eta} |\beta(x) - \beta(s)|ds + \int_x^{x+\eta} |\beta(s)|ds \\ \leqslant &\ \int_x^{x+\eta} \varepsilon ds \quad + \eta\varepsilon \\ = &\ 2\eta\varepsilon.\end{aligned}$$

Then

$\beta(x) < 2\varepsilon, \quad \forall x > A,$

which implies that $\lim_{t\to+\infty}\beta(t) = 0$ and the property (9) is then fulfilled. This completes the proof. □

2.1. Boundedness result

In the next theorem, I state and prove my first main result.

Theorem 1. *In addition to assumptions (2)–(4), assume that there exist a positive scalar function $\varphi : [0,\infty) \to [0,\infty)$ satisfying*

$$\varphi'(t) \le 0, \ \forall\, t \geqslant 0 \quad \text{and} \quad \varphi \in L^1([0,\infty))$$

Suppose that there exists a constant $\lambda_3 > 0$ such that

$$\lambda_2|C(t,s)| + \lambda_3\varphi'(t-s) \leqslant 0, \quad \forall\, 0 \leqslant s \leqslant t < \infty, \tag{11}$$

and there exist a negative scalar function $A(t)$ defined on $[0,\infty)$ and a positive constant α such that

$$A(t) + \lambda_1 + \lambda_3\varphi(0) \leqslant -\alpha, \tag{12}$$

then all solutions of Eq. (1) are bounded.

Proof. Define the Lyapunov functional V by

$$V(t) = |x| + \lambda_3\int_0^t \varphi(t-s)|x(s)|ds, \qquad t \geqslant 0. \tag{13}$$

By differentiating $V(t)$, one obtains

$$V'(t) = x'(t)\frac{x}{|x|} + \lambda_3\varphi(0)|x| + \lambda_3\int_0^t \varphi'(t-s)|x(s)|ds, \qquad t \geqslant 0.$$

Then, substituting (1) into the expression of $V'(t)$, we get

$$V'(t) = \frac{x(t)}{|x|}\left(A(t)x(t) + B(t) + \int_0^t C(t,s)f(x(s))ds + g(x(t))\right) + \lambda_3\varphi(0)|x| + \lambda_3\int_0^t \varphi'(t-s)|x(s)|ds,$$

By developing, we get

$$\begin{aligned} V'(t) = \ & A(t)\tfrac{x^2}{|x|} + \tfrac{x(t)}{|x|}B(t) + \tfrac{x(t)}{|x|}\int_0^t C(t,s)f(x(s))ds + \tfrac{x(t)}{|x|}g(x(t)) + \lambda_3\varphi(0)|x| + \lambda_3\int_0^t \varphi'(t-s)|x(s)|ds \\ = \ & A(t)|x| + \tfrac{x(t)}{|x|}B(t) + \tfrac{x(t)}{|x|}\int_0^t C(t,s)f(x(s))ds + \tfrac{x(t)}{|x|}g(x(t)) + \lambda_3\varphi(0)|x| + \lambda_3\int_0^t \varphi'(t-s)|x(s)|ds \\ \leqslant \ & A(t)|x| + |B| + \int_0^t |C(t,s)||f(x(s))|ds + |g(x(t))| + \lambda_3\varphi(0)|x| + \lambda_3\int_0^t \varphi'(t-s)|x(s)|ds. \end{aligned}$$

Using assumptions (2)–(4), we get

$$\begin{aligned} V'(t) &\leqslant A(t)|x| + M + \lambda_2\int_0^t |C(t,s)||x(s)|ds + \lambda_1|x| + \lambda_3\varphi(0)|x| + \lambda_3\int_0^t \varphi'(t-s)|x(s)|ds \\ &= (A(t) + \lambda_1 + \lambda_3\varphi(0))|x| + \int_0^t (\lambda_2|C(t,s)| + \lambda_3\varphi'(t-s))|x(s)|ds + M. \end{aligned}$$

Now using assumptions (11) and (12), one deduces that

$$V'(t) \leqslant -\alpha|x| + M, \qquad M > 0 \tag{14}$$

Applying the Laplace transform to the Eq. (7), we obtain

$$L(\beta) + L\left(\int_0^t \lambda_3\varphi(t-s)\beta(s)ds\right) + L\left(\int_0^t \alpha\beta(s)ds\right) = L(1),$$

which means that

$$L(\beta) + \lambda_3 L(\varphi * \beta) + \alpha L(1 * \beta) = L(\beta) + \lambda_3 L(\varphi)L(\beta) + \alpha L(1)L(\beta) = \frac{1}{s}.$$

Now by solving for $L(\beta)$, one obtains

$$L(\beta) = \frac{1}{(1 + \lambda_3 L(\varphi) + \alpha\frac{1}{s})s} \tag{15}$$

From the inequality (14), there exists a non-negative function $\eta : [0,\infty) \to [0,\infty)$ such that

$$V'(t) = -\alpha|x| + M - \eta(t)$$

Since η is a linear combination of functions of exponential order, η is also of exponential order and so we can apply the Laplace transform and we obtain

$$sL(V) - V(0) = -\alpha L(|x|) + \frac{M}{s} - L(\eta),$$

which gives

$$L(V) = \left[V(0) - \alpha L(|x|) + \frac{M}{s} - L(\eta)\right]\frac{1}{s}.$$

Now applying the Laplace transform to Eq. (13), we have

$$L(V) = L(|x|) + \lambda_3 L(\varphi)L(|x|).$$

Then we obtain

$$\left[V(0) - \alpha L(|x|) + \frac{M}{s} - L(\eta)\right]\frac{1}{s} = L(|x|) + \lambda_3 L(\varphi)L(|x|).$$

Now, by solving for $L(|x|)$, we get

$$L(|x|) = \frac{V(0) + \frac{M}{s} - L(\eta)}{(1 + \lambda_3 L(\varphi) + \frac{\alpha}{s})s}$$

or also

$$L(|x|) = \left[V(0) + \frac{M}{s} - L(\eta)\right]\frac{1}{(1 + \lambda_3 L(\varphi) + \frac{\alpha}{s})s}$$

Using Eq. (15), we get

$$L(|x|) = \left[V(0) + \frac{M}{s} - L(\eta)\right]L(\beta).$$

Then, we obtain

$$L(|x|) = L(\beta)V(0) + L(\beta)\frac{M}{s} - L(\beta)L(\eta).$$

using into account properties of the Laplace transform, one obtains

$$L(|x|) = L(\beta)V(0) + L(M * \beta) - L(\eta * \beta).$$

and using the convolation properties, one can deduce that

$$L(|x|) = L(\beta)V(0) + L\left(\int_0^t M\beta(s)ds\right) - L\left(\int_0^t \eta(t-s)\beta(s)ds\right).$$

Now, we apply the inverse Laplace Transform to get

$$\begin{aligned}|x| &= \beta(t)V(0) + M\int_0^t \beta(s)ds - \int_0^t \eta(t-s)\beta(s)ds\\ &\leqslant \beta(t)V(0) + M\int_0^t \beta(s)ds\end{aligned}$$

Since $\beta(t) \in L^1([0,\infty))$ and $\lim_{t\to+\infty}\beta(t) = 0$, there exists a positive constant $k > 0$ such that

$$|x(t)| \leqslant k, \quad \forall t \geqslant 0$$

This means that the solution y is bounded and this completes the proof. □

2.2. Asymptotic stability results

For the rest, suppose that $B(t) = 0$. Consider the nonlinear Volterra integro-differential equation

$$x'(t) = A(t)x(t) + \int_0^t C(t,s)f(x(s))ds + g(x(t)) \tag{16}$$

In Theorem 2, I give stability and boundedness results concerning solutions of (16).

Theorem 2. *Assume that (2) and (3) are satisfied and assume that $A(t) \leq 0$ such that there exist $\lambda_1 \geqslant 0$ and $\lambda_2 \geqslant 0$ satisfying*

$$|A(s)| \geqslant \lambda_1 + \lambda_2\int_s^t |C(u,s)|du, \quad \forall\, 0 \leqslant s \leqslant t \leqslant \infty \tag{17}$$

Then the zero solution of (16) is stable.

If, in addition, there exist two positive constants $t_2 \geqslant 0$ and $\alpha > 0$ such that

$$|A(s)| \geqslant \lambda_1 + \lambda_2\int_s^t |C(u,s)|du + \alpha, \quad \forall\, 0 \leqslant t_2 \leqslant s \leqslant t \leqslant \infty,$$

and if both $|A(s)|$ and $\int_s^t |c(u,s)|du$are bounded, then the zero solution of (16) is asymptotically stable.

Proof. By integrating (16) on $(0,t)$, one obtains

$$\begin{aligned}x(t) - x(0) &= \int_0^t x'(s)ds\\ &= \int_0^t A(s)x(s)ds + \int_0^t\int_0^u C(u,s)f(x(s))dsdu\\ &\quad + \int_0^t g(x(s))ds\end{aligned}$$

Then, by interchanging the order of the integration, one deduces

$$\begin{aligned}x(t) = &\ x(0) + \int_0^t A(s)x(s)ds + \int_0^t\int_0^u C(u,s)f(x(s))dsdu + \int_0^t g(x(s))ds\\ = &\ x(0) + \int_0^t A(s)x(s)ds + \int_0^t\int_s^t C(u,s)f(x(s))duds + \int_0^t g(x(s))ds\end{aligned}$$

Define the Lyapunov functional $H(t,x(.))$ by

$$H(t,x(.)) = |x| + \int_0^t \left(|A(s)| - \lambda_1 - \lambda_2\int_s^t |C(u,s)|du\right)|x(s)|ds.$$

which is positive due to the fact that $|A(s)| \geqslant \lambda_1 + \lambda_2\int_s^t |C(u,s)|du,\ \forall\, 0 \leqslant s \leqslant t < \infty$.

Now by deriving $H(t,x(.))$, one obtains

$$\begin{aligned}H'(t,x(.)) = &\ \tfrac{x}{|x|}x' + \left(|A| - \lambda_1 - \lambda_2\int_t^t |C(u,t)|du\right)|x(t)| + \int_0^t \tfrac{d}{dt}\left(\left(|A(s)| - \lambda_1 - \lambda_2\int_s^t |C(u,s)|du\right)|x(s)|\right)ds\\ = &\ \tfrac{x}{|x|}\left(A(t)x(t) + \int_0^t C(t,s)(f(x(s))ds + g(x(t))) + |A||x| - \lambda_1|x| + \int_0^t (0 - \lambda_2|C(t,s)||x(s)|\right)ds\\ = &\ A(t)|x| + \tfrac{x}{|x|}\int_0^t C(t,s)f(x(s))ds + \tfrac{x}{|x|}g(x(t)) + |A||x| - \lambda_1|x| - \lambda_2\int_0^t |C(t,s)||x(s)|ds\\ \leqslant &\ A(t)|x| + \tfrac{|x|}{|x|}\int_0^t |C(t,s)||f(x(s))|ds + \tfrac{|x|}{|x|}|g(x)| + |A||x| - \lambda_1|x| - \lambda_2\int_0^t |C(t,s)||(x(s))|ds\\ \leqslant &\ A(t)|x| + \lambda_2\int_0^t |C(t,s)||x(s)|ds + \lambda_1|x| + |A||x| - \lambda_1|y| - \lambda_2\int_0^t |C(t,s)||x(s)|ds = 0\end{aligned}$$

Hence, $H(t,x(.))$ is a decreasing Lyapunov function. Recall that the aim is to prove that the zero solution of (16) is stable. For given constants $\varepsilon > 0$ and $t_0 \geqslant 0$, define a continuous function $\phi : [0,t_0] \to R$ satisfying $|\phi| < \delta$ where $\delta > 0$ is a positive constant to be determined.

Since $H(t)$ is a decreasing function then

$$\begin{aligned}|y| &\leqslant H(t,x(.))\\ &\leqslant H(t_0,\phi(.))\\ &\leqslant H(t_0,\delta)\\ &= \delta + \delta\int_0^{t_0}\left[|A(s)| - \lambda_1 - \lambda_2\int_s^{t_0} |C(u,s)|du\right]ds\\ &= \delta\left(1 + \int_0^{t_0}\left[|A(s)| - \lambda_1 - \lambda_2\int_s^{t_0} |C(u,s)|du\right]ds\right)\end{aligned}$$

In order to obtain $|x| < \epsilon$, one can choose

$$\delta = \frac{\varepsilon}{\left(1 + \int_0^{t_0}\left[|A(s)| - \lambda_1 - \lambda_2\int_s^{t_0} |C(u,s)|du\right]ds\right)}, \quad \forall \epsilon > 0$$

Therefore, the zero solution of (16) is stable.

If there exist $t_2 \geqslant 0$ and $\alpha > 0$ such that $|A(s)| \geqslant \lambda_1 + \lambda_2\int_s^{t_0} |C(u,s)|du + \alpha,\ \forall\, t_2 \leqslant s \leqslant t < \infty$ then

$$\begin{aligned}|x| + \alpha\int_{t_2}^t |x(s)|ds &\leqslant |x| + \int_{t_2}^t \left[|A(s)| - \lambda_1 - \lambda_2\int_s^t |C(u,s)|du\right]|x(s)|ds\\ &\leqslant |x| + \int_0^t \left[|A(s)| - \lambda_1 - \lambda_2\int_s^t |C(u,s)|du\right]|x(s)|ds\\ &= H(t,x(.))\\ &\leqslant H(t_0,\phi(.))\\ &\leqslant H(t_0,\delta) = N, \quad (N > 0)\end{aligned}$$

Therefore

$$\alpha\int_{t_0}^t |x(s)|ds < N$$

and thus

$$\begin{aligned}&0 \leqslant \int_{t_0}^t |x(s)|ds < R = \frac{N}{\alpha}, \quad (R > 0) \Rightarrow \int_0^t |x(s)|ds < R\\ &\Rightarrow \int_0^\infty |x(s)|ds < \infty\end{aligned}$$

which means that $y \in L^1([0,\infty))$. Note that $y'(t)$ is bounded then $x(t) \to 0$ when $t \to \infty$ which means that the solution $x(t)$ is asymptotically stable.

3. Explicit examples

Consider the nonlinear Volterra integro-differential equation

$$x'(t) = -x(t) + \frac{1}{16}x(t)\sin(x(t)) + \int_0^t e^{-t-4+s}x(s)\sin(x(s))ds + \cos(t) \tag{18}$$

In this example, we have $A(t) = -1, f(x) = xsin(x),\ g(x) = \frac{1}{16}xsin(x),\ C(t,s) = e^{-t-4+s}$ and $B(t) = cos(t)$. These functions satisfy

$$|g(x)| \leqslant \frac{1}{16}|x| = \lambda_1|x|,\ |f(x)| \leqslant |x| = \lambda_2|x|,\ |C(t,s)| = e^{-t-4+s},\ |B| \leqslant 1 = M$$

Consider $\phi(t) = e^{-(t+3)}$ and define the Lyapunov functional V by

$$V(t) = |\, x(t) \,| + \lambda_3 \int_0^t \varphi(t-s) \,|\, x(s) \,|\, ds$$

Note that $\varphi'(t) = -e^{-(t+3)} \Rightarrow \phi'(t) \leqslant 0$. In addition

$$\lambda_2 \,|\, c(t,s) \,| + \lambda_3\varphi'(t-s) = e^{-(t+4-s)} - e^{-(t+3-s)} \leqslant 0,\ \forall 0 \leqslant s \leqslant t < \infty.$$

Moreover, condition (12) is satisfied because of the fact that $A(t) + \frac{1}{16} + \varphi(0) = -1 + \frac{1}{16} + e^{-3} \leqslant -\alpha$ where $\alpha \approx 0.889$, is a positive constant. Thus, by Theorem 1, all solutions of (18) are bounded.

Now by choosing $B(t) = 0$, the system (18) becomes

$$x'(t) = -x(t) + \tfrac{1}{16}x(t)\sin(x(t)) + \textstyle\int_0^t e^{-t-4+s}x(s)\sin(x(s))ds \tag{19}$$

Define the Lyapunov functional $H(t, x(.))$ by

$$H(t,x(.)) = |x(t)| + \int_0^t \left(|A(s)| - \lambda_1 - \lambda_2 \int_s^t |C(t,s)|du\right)|x(s)|ds.$$

Note that

$$\begin{aligned} |A(s)| - \lambda_1 - \lambda_2 \textstyle\int_s^t |C(u,s)|du = &\ 1 - \tfrac{1}{16} - \textstyle\int_s^t e^{-(u+4-s)}du \\ = &\ \tfrac{15}{16} + e^{-(t+4-s)} - e^{-4},\ \forall 0 \leqslant s \leqslant t < \infty \\ \geqslant &\ \tfrac{15}{16} - e^{-4} \approx 0.92 \\ > &\ 0 \end{aligned}$$

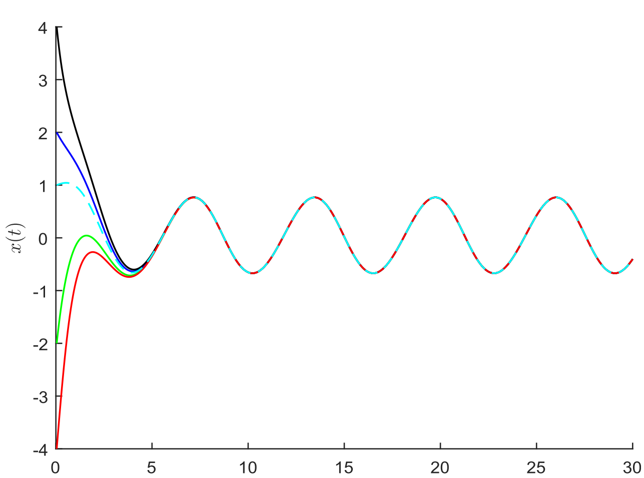

Fig. 1. $A(t) = -1, f(x) = xsin(x),\ g(x) = \frac{1}{16}xsin(x),\ C(t,s) = e^{-t-4+s}$ and $B(t) = cos(t)$. All solutions of (18) are bounded and converge to a periodic solution for all initial conditions.

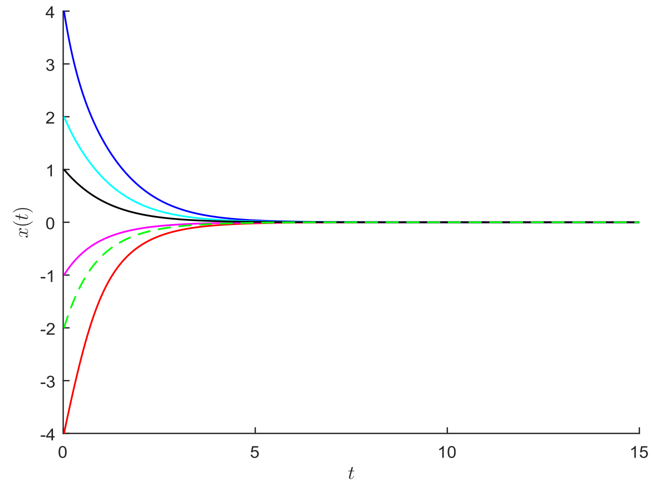

Fig. 2. $A(t) = -1, f(x) = xsin(x),\ g(x) = \frac{1}{16}xsin(x),\ C(t,s) = e^{-t-4+s}$ and $B(t) = 0$. All solutions of (19) converge to the zero solution of (19), which confirms the given results concerning the stability of the zero solution of (19).

Hence, condition (17) is satisfied, then the zero solution of (19) is stable.

Since $\int_s^t |C(u,s)|du = \int_s^t e^{-(u+4-s)}du = -e^{-(t+4-s)} + e^{-4} \leqslant e^{-4}$ then $A(t)$ and $\int_s^t |C(u,s)|du$ are bounded. Moreover, one can easily verify that $|A(s)| \geqslant \lambda_1 + \lambda_2 \int_s^t |C(u,s)|du + \alpha\ \forall\, 0 \leqslant t_2 \leqslant s \leqslant t < \infty$ and $\alpha \approx 0.92$, then from Theorem 2, the zero solution of (19) is asymptotically stable.

4. Numerical simulations

Consider a subdivision of the time interval $[0, T]$ as follows

$$[0,T] = \bigcup_{n=0}^{N_T-1}[t_n, t_{n+1}], \quad t_n = n\Delta t, \quad \Delta t = T/N_T$$

Let $x^{(n)}$ be an approximation of $x(t_n)$. By using the Euler implicit scheme, one obtains an analogous discrete system of the nonlinear Volterra integral differential Eqs. (1), for $t \in [0, T]$, given by:

$$\begin{cases} x^{(n+1)} = x^{(n)} + \Delta t\left[A^{(n)}x^{(n)} + B^{(n)} + \displaystyle\sum_{s=0}^{n} C(n,t_s)f\left(x^{(s)}\right) + g\left(x^{(n)}\right)\right] \\ x^{(0)} = x(0) \end{cases} \tag{20}$$

where $A^{(n)} = A(t_n)$ et $B^{(n)} = B(t_n)$. By reconsidering the explicit example (18), I obtain a periodic solution for different initial conditions (see Fig. 1) which confirms the fact that all solutions of (18) are bounded.

Now consider the explicit example (19), I obtain the asymptotic stability of the zero solution of (19) as it can be seen in Fig. 2.

5. Conclusion

In this paper, I used Lyapunov functionals combined with the Laplace transform to obtain boundedness results regarding the solutions of the nonlinear Volterra integral differential equations of the form

$$x'(t) = A(t)x(t) + B(t) + \int_0^t C(t,s)f(x(s))ds + g(x(t)).$$

Asymptotic stability results regarding the zero solution are carried out for a particular situation of this kind of equations. Numerical examples are proposed to perform the given results.

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