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A more accurate multidimensional Hardy-Hilbert-type inequality

Bicheng Yang

Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 510303, PR China

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ABSTRACT

In this paper, by means of the weight coefficients, the transfer formula, Hermite-Hadamard's inequality and the technique of real analysis, a more accurate multidimensional Hardy-Hilbert-type inequality with a best possible constant factor is given, which is an extension of some published results. Moreover, the equivalent forms and the operator expressions are considered.

1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geqslant 0, a = \{a_m\}_{m=1}^{\infty} \in l^p, b = \{b_n\}_{n=1}^{\infty} \in l^q$, $\|a\|_p = (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0, \|b\|_q > 0$, then we have the following well-known Hardy-Hilbert's inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q, \tag{1}$$

and the following more accurate Hardy-Hilbert's inequality with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. Hardy et al., 1934, Theorem 315, 323):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-1} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{2}$$

We still have the following Hilbert-type inequality with the best possible constant factor $\left[\frac{\pi}{\sin(\pi/p)}\right]^2$ (cf. Hardy et al., 1934, Theorem 342):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\ln(m/n)a_m b_n}{m-n} < \left[\frac{\pi}{\sin(\pi/p)}\right]^2\|a\|_p\|b\|_q. \tag{3}$$

Inequalities (1), (2) and (3) are important in Analysis and its applications (cf. Hardy et al., 1934; Mitrinović et al., 1991; Yang, 2011).

Assuming that $\{\mu_m\}_{m=1}^{\infty}$ and $\{\upsilon_n\}_{n=1}^{\infty}$ are positive sequences with

$$U_m = \sum_{i=1}^{m}\mu_i, V_n = \sum_{j=1}^{n}\upsilon_j \ (m, n \in \mathbf{N} = \{1, 2, \ldots\}),$$

We have the following Hardy-Hilbert-type inequality (cf. Hardy et al., 1934, Theorem 321):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}\frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}}. \tag{4}$$

For $\mu_i = \upsilon_j = 1$ $(i, j \in \mathbf{N})$, inequality (4) reduces to (1).

In 2015, by using the transfer formula, Yang (2015) gave the following multidimensional Hilbert's inequality: For $i_0, j_0 \in \mathbf{N}, \alpha, \beta > 0$,

$$\|x\|_{\alpha} := \left(\sum_{k=1}^{i_0}|x^{(k)}|^{\alpha}\right)^{\frac{1}{\alpha}} \ (x = (x^{(1)}, \ldots, x^{(i_0)}) \in \mathbf{R}^{i_0}),$$

$$\|y\|_{\beta} := \left(\sum_{k=1}^{j_0}|y^{(k)}|^{\beta}\right)^{\frac{1}{\beta}} \ (y = (y^{(1)}, \ldots, y^{(j_0)}) \in \mathbf{R}^{j_0}),$$

$0 < \lambda_1 \leqslant i_0, 0 < \lambda_2 \leqslant j_0, \lambda_1 + \lambda_2 = \lambda, a_m, b_n \geqslant 0$, we have

E-mail address: bcyang@gdei.edu.cn

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$$\sum_n \sum_m \frac{1}{\|m\|_\alpha^\lambda + \|n\|_\beta^\lambda} a_m b_n < K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} \left[\sum_m \|m\|_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p\right]^{\frac{1}{p}} \left[\sum_n \|n\|_\beta^{q(j_0-\lambda_2)-j_0} b_n^q\right]^{\frac{1}{q}}, \tag{5}$$

where $\sum_m = \sum_{m_{i_0}=1}^\infty \cdots \sum_{m_1=1}^\infty$, $\sum_n = \sum_{n_{j_0}=1}^\infty \cdots \sum_{n_1=1}^\infty$, the series in the right-hand side of (5) are positive values, and the best possible constant factor $K_1^{\frac{1}{p}} K_2^{\frac{1}{q}}$ is indicated by

$$K_1^{\frac{1}{p}} K_2^{\frac{1}{q}} = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} \frac{\pi}{\lambda \sin\left(\frac{\pi\lambda_1}{\lambda}\right)}.$$

With regards to the above assumptions, we still have the following multidimensional Hilbert-type inequality (cf. Yang, 2014):

$$\sum_n \sum_m \frac{\ln(\|m\|_\alpha / \|n\|_\beta)}{\|m\|_\alpha^\lambda - \|n\|_\beta^\lambda} a_m b_n < K \left[\sum_m \|m\|_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p\right]^{\frac{1}{p}} \left[\sum_n \|n\|_\beta^{q(j_0-\lambda_2)-j_0} b_n^q\right]^{\frac{1}{q}}, \tag{6}$$

where,

$$K = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} \left[\frac{\pi}{\lambda \sin\left(\frac{\pi\lambda_1}{\lambda}\right)}\right]^2$$

is the best possible. For $i_0 = j_0 = \lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, inequality (5) ((6)) reduces to (1) ((3)). Some other results on this type of inequalities and multiple inequalities were provided by Hong (2005), Krnić et al. (2008), Krnić and Vuković (2012), Rassias and Yang (2014), Shi and Yang (2015), Hong (2006, 2010), Perić and Vuković (2011), He (2015), Adiyasuren et al. (2016).

Recently, by using the weight coefficients, Huang (2015) gave an extension of (3) as follows: For $0 < \lambda_1, \lambda_2 \leqslant 1, \lambda_1 + \lambda_2 = \lambda, a_m, b_n \geqslant 0$,

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\ln(U_m/V_n)}{U_m^\lambda - V_n^\lambda} a_m b_n < \left[\frac{\pi}{\lambda \sin\left(\frac{\pi\lambda_1}{\lambda}\right)}\right]^2 \left[\sum_{m=1}^\infty \frac{U_m^{p(1-\lambda_1)-1} a_m^p}{\mu_m^{p-1}}\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty \frac{V_n^{q(1-\lambda_2)-1} b_n^q}{\nu_n^{q-1}}\right]^{\frac{1}{q}}, \tag{7}$$

where, the constant factor $\left[\frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}\right]^2$ is the best possible (the series in the right-hand side of (7) are positive values). Another results on Hardy-Hilbert-type inequalities and Hilbert-type inequalities were given by Yang (2015), Shi and Yang (2015), Huang (2015), Wang et al. (2015), Yang and Chen (2016), Brnetić and Pečarić (2004), Krnić and Pečarić (2005), Krnic et al. (2005), Li et al. (2007), Laith (2008), Agarwal et al. (2015).

In this paper, by means of the weight coefficients, the transfer formula, Hermite-Hadamard's inequality and the technique of analysis, a more accurate multidimensional Hardy-Hilbert-type inequality with a best possible constant factor is given, which is an extension of (6) and (7). Meanwhile, the equivalent forms and the operator expressions are considered.

2. Some lemmas

If $\mu_i^{(k)} > 0, 0 \leqslant \widetilde{\mu}_i^{(k)} \leqslant \frac{1}{2}\mu_i^{(k)}$ $(k = 1, \ldots, i_0; i = 1, \ldots, m), \upsilon_j^{(l)} > 0, 0 \leqslant \widetilde{\upsilon}_j^{(l)} \leqslant \frac{1}{2}\upsilon_j^{(l)}$ $(l = 1, \ldots, j_0; j = 1, \ldots, n)$, then we set

$$U_m^{(k)} := \sum_{i=1}^m \mu_i^{(k)}, \widetilde{U}_m^{(k)} := U_m^{(k)} - \widetilde{\mu}_m^{(k)} \ (k = 1, \ldots, i_0),$$

$$V_n^{(l)} := \sum_{j=1}^n \upsilon_j^{(l)}, \widetilde{V}_n^{(l)} := V_n^{(l)} - \widetilde{\upsilon}_n^{(l)} \ (l = 1, \ldots, j_0),$$

$$U_m := (U_m^{(1)}, \ldots, U_m^{(i_0)}), \widetilde{\mu}_m := (\widetilde{\mu}_m^{(1)}, \ldots, \widetilde{\mu}_m^{(i_0)}),$$
$$\widetilde{U}_m := (\widetilde{U}_m^{(1)}, \ldots, \widetilde{U}_m^{(i_0)}) = U_m - \widetilde{\mu}_m,$$

$$V_n := (V_n^{(1)}, \ldots, V_n^{(j_0)}), \widetilde{\upsilon}_n := (\widetilde{\upsilon}_n^{(1)}, \ldots, \widetilde{\upsilon}_n^{(j_0)}),$$
$$\widetilde{V}_n := (\widetilde{V}_n^{(1)}, \ldots, \widetilde{V}_n^{(j_0)}) = V_n - \widetilde{\upsilon}_n \ (m, n \in \mathbf{N}). \tag{8}$$

We also set functions $\mu_k(t) := \mu_m^{(k)}, t \in (m - \frac{1}{2}, m + \frac{1}{2})$ $(m \in \mathbf{N})$; $\upsilon_l(t) := \upsilon_n^{(l)}, t \in (n - \frac{1}{2}, n + \frac{1}{2})$ $(n \in \mathbf{N})$, and

$$U_k(x) := \int_{\frac{1}{2}}^x \mu_k(t)dt \ (k = 1, \ldots, i_0),$$

$$V_l(y) := \int_{\frac{1}{2}}^y \upsilon_l(t)dt \ (l = 1, \ldots, j_0), \tag{9}$$

$$U(x) := (U_1(x), \ldots, U_{i_0}(x)),$$

$$V(y) := (V_1(y), \ldots, V_{j_0}(y)) \left(x, y \geqslant \frac{1}{2}\right). \tag{10}$$

It follows that

$$U_k(m) = \int_{\frac{1}{2}}^m \mu_k(t)dt = \int_{\frac{1}{2}}^{m+\frac{1}{2}} \mu_k(t)dt - \frac{1}{2}\mu_m^{(k)}$$

$$\leqslant \widetilde{U}_m^{(k)} \leqslant U_k\left(m + \frac{1}{2}\right) \ (k = 1, \ldots, i_0; m \in \mathbf{N}),$$

$$V_l(n) \leqslant \widetilde{V}_n^{(l)} \leqslant V_l\left(n + \frac{1}{2}\right) \ (l = 1, \ldots, j_0; n \in \mathbf{N}),$$

and for $x \in (m - \frac{1}{2}, m + \frac{1}{2}), U_k'(x) = \mu_k(x) = \mu_m^{(k)}$, $(k = 1, \ldots, i_0; m \in \mathbf{N})$; for $y \in (n - \frac{1}{2}, n + \frac{1}{2}), V_l'(y) = \upsilon_l(y) = \upsilon_n^{(l)}$, $(l = 1, \ldots, j_0; n \in \mathbf{N})$.

Lemma 1. (cf. Yang and Chen, 2016) *Suppose that $g(t)$ (> 0) is strictly decreasing and strictly convex in $(\frac{1}{2}, \infty)$, satisfying $\int_{\frac{1}{2}}^\infty g(t)dt \in \mathbf{R}_+$. We have the following Hermite-Hadamard's inequality*

$$\int_n^{n+1} g(t)dt < g(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} g(t)dt \ (n \in \mathbf{N}), \tag{11}$$

and then

$$\int_1^\infty g(t)dt < \sum_{n=1}^\infty g(n) < \int_{\frac{1}{2}}^\infty g(t)dt. \tag{12}$$

Lemma 2. *If $i_0 \in \mathbf{N}, \alpha, M > 0, \Psi(u)$ is a non-negative measurable function in $(0, 1]$, and*

$$D_M := \left\{x = (x_1, \ldots, x_{i_0}) \in \mathbf{R}_+^{i_0}; u = \sum_{i=1}^{i_0}\left(\frac{x_i}{M}\right)^\alpha \leqslant 1\right\}, \tag{13}$$

then we have the following transfer formula (cf. Hong, 2005):

$$\int \cdots \int_{D_M} \Psi\left(\sum_{i=1}^{i_0}\left(\frac{x_i}{M}\right)^\alpha\right) dx_1 \cdots dx_s = \frac{M^{i_0}\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_0^1 \Psi(u) u^{\frac{i_0}{\alpha}-1} du. \tag{14}$$

Lemma 3. *If $i_0, j_0 \in \mathbf{N}, \alpha, \beta, \varepsilon > 0, \mu_m^{(k)} \geqslant \mu_{m+1}^{(k)}$ $(m \in \mathbf{N}; k = 1, \ldots, i_0)$, $\upsilon_n^{(l)} \geqslant \upsilon_{n+1}^{(l)}$ $(n \in \mathbf{N}; l = 1, \ldots, i_0), b := \min_{1 \leqslant i \leqslant i_0, 1 \leqslant j \leqslant j_0}\{\mu_1^{(i)}, \upsilon_1^{(j)}\} (> 0)$, then we have*

$$\sum_m \|\widetilde{U}_m\|_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \leqslant \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\varepsilon b^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} + O(1), \tag{15}$$

$$\sum_n \|\widetilde{V}_n\|_\beta^{-j_0-\varepsilon} \prod_{k=1}^{j_0} \upsilon_n^{(k)} \leqslant \frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\varepsilon b^\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)} + \widetilde{O}(1) \ (\varepsilon \to 0^+). \tag{16}$$

Proof. For $M > bi_0^{1/\alpha}$, we set

$$\Psi(u) = \begin{cases} 0, & 0 < u < \frac{b^\alpha i_0}{M^\alpha}, \\ \frac{1}{(Mu^{1/\alpha})^{i_0+\varepsilon}}, & \frac{b^\alpha i_0}{M^\alpha} \leqslant u \leqslant 1. \end{cases}$$

By (14), it follows that

$$\int_{\{x\in\mathbf{R}_+^{i_0};x_i\geqslant b\}} \frac{dx}{\|x\|_\alpha^{i_0+\varepsilon}} = \lim_{M\to\infty} \int\cdots\int_{D_M} \Psi\left(\sum_{i=1}^{i_0}\left(\frac{x_i}{M}\right)^\alpha\right) dx_1\cdots dx_{i_0}$$
$$= \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0}\Gamma\left(\frac{i_0}{\alpha}\right)} \int_{b^\alpha i_0/M^\alpha}^1 \frac{u^{\frac{i_0}{\alpha}-1}}{(Mu^{1/\alpha})^{i_0+\varepsilon}} du = \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\varepsilon b^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}.$$

In view of (12) and the above result, since $U_k(m) \leqslant \widetilde{U}_m^{(k)}$, we find

$$0 < \sum_{\{m\in\mathbf{N}^{i_0};m_i\geqslant 2\}} \|\widetilde{U}_m\|_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)}$$
$$\leqslant \sum_{\{m\in\mathbf{N}^{i_0};m_i\geqslant 2\}} \int_{\{x\in\mathbf{R}_+^{i_0};m_i-\frac{1}{2}\leqslant x_i<m_i+\frac{1}{2}\}} \|U(m)\|_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} dx$$
$$< \sum_{\{m\in\mathbf{N}^{i_0};m_i\geqslant 2\}} \int_{\{x\in\mathbf{R}_+^{i_0};m_i-\frac{1}{2}\leqslant x_i<m_i+\frac{1}{2}\}} \|U(x)\|_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_k(x) dx$$
$$= \int_{\{x\in\mathbf{R}_+^{i_0};x_i\geqslant\frac{3}{2}\}} \|U(x)\|_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_k(x) dx$$
$$\overset{v=U(x)}{=} \int_{\{v\in\mathbf{R}_+^{i_0};v_i\geqslant\mu_1^{(i)}\}} \|v\|_\alpha^{-i_0-\varepsilon} dv \leqslant \int_{\{v\in\mathbf{R}_+^{i_0};v_i\geqslant b\}} \|v\|_\alpha^{-i_0-\varepsilon} dv$$
$$= \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\varepsilon b^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}.$$

For $i_0 = 1, 0 < \sum_{\{m\in\mathbf{N}^{i_0};m_i=1\}} \|\widetilde{U}_m\|_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \leqslant \left(\mu_1^{(1)}\right)^{-\varepsilon} < \infty$; for $i_0 \geqslant 2$, we set

$$H_i := \sum_{\{m\in\mathbf{N}^{i_0};m_i=1\}} \|\widetilde{U}_m\|_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \ (i = 1,\ldots,i_0).$$

Without lose of generality, we estimate H_{i_0} as follows:

$$H_{i_0} \leqslant \sum_{\{m\in\mathbf{N}^{i_0};m_{i_0}=1\}} \|U(m)\|_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)}$$
$$= \mu_1^{(i_0)} \sum_{m\in\mathbf{N}^{i_0-1}} \frac{\prod_{k=1}^{i_0-1} \mu_m^{(k)}}{\left[\sum_{i=1}^{i_0-1} U_i^\alpha(m) + \left(\frac{1}{2}\mu_1^{(i_0)}\right)^\alpha\right]^{\frac{1}{\alpha}(i_0+\varepsilon)}}$$
$$< \sum_{m\in\mathbf{N}^{i_0-1}} \int_{\{x\in\mathbf{R}_+^{i_0-1};m_i-\frac{1}{2}\leqslant x_i<m_i+\frac{1}{2}\}} \frac{\mu_1^{(i_0)} \prod_{k=1}^{i_0-1} \mu_k(x) dx}{\left[\sum_{i=1}^{i_0-1} U_i^\alpha(x) + \left(\frac{1}{2}\mu_1^{(i_0)}\right)^\alpha\right]^{\frac{1}{\alpha}(i_0+\varepsilon)}}$$
$$= \mu_1^{(i_0)} \int_{\{x\in\mathbf{R}^{i_0-1};x_i\geqslant\frac{1}{2}\}} \frac{\prod_{k=1}^{i_0-1} \mu_k(x)}{\left[\sum_{i=1}^{i_0-1} U_i^\alpha(x) + \left(\frac{1}{2}\mu_1^{(i_0)}\right)^\alpha\right]^{\frac{1}{\alpha}(i_0+\varepsilon)}} dx$$
$$\overset{v=U(x)}{\leqslant} \mu_1^{(i_0)} \int_{\mathbf{R}_+^{i_0-1}} \frac{1}{\left[M^\alpha \sum_{i=1}^{i_0-1}\left(\frac{v_i}{M}\right)^\alpha + \left(\frac{1}{2}\mu_1^{(i_0)}\right)^\alpha\right]^{\frac{1}{\alpha}(i_0+\varepsilon)}} dv.$$

By (14), we find

$$0 < H_{i_0} \leqslant \mu_1^{(i_0)} \lim_{M\to\infty} \frac{M^{i_0-1}\Gamma^{i_0-1}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0-1}{\alpha}\right)} \int_0^1 \frac{u^{\frac{i_0-1}{\alpha}-1} du}{\left[M^\alpha u + \left(\frac{1}{2}\mu_1^{(i_0)}\right)^\alpha\right]^{\frac{i_0+\varepsilon}{\alpha}}}$$
$$\overset{t=\frac{M^\alpha u}{\left(\frac{1}{2}\mu_1^{(i_0)}\right)^\alpha}}{=} \frac{2^{1+\varepsilon}}{\left(\mu_1^{(i_0)}\right)^\varepsilon} \frac{\Gamma^{i_0-1}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0-1}{\alpha}\right)} \int_0^\infty \frac{t^{\frac{i_0-1}{\alpha}-1} dt}{(t+1)^{\frac{i_0+\varepsilon}{\alpha}}}$$
$$= \frac{2^{1+\varepsilon}}{\left(\mu_1^{(i_0)}\right)^\varepsilon} \frac{\Gamma^{i_0-1}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0-1}{\alpha}\right)} B\left(\frac{i_0-1}{\alpha}, \frac{1+\varepsilon}{\alpha}\right) < \infty,$$

namely, $H_{i_0} = O_{i_0}(1)$. Hence, we have

$$\sum_m \|\widetilde{U}_m\|_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} \leqslant \sum_{\{m\in\mathbf{N}^{i_0};m_i\geqslant 2\}} \|\widetilde{U}_m\|_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)} + \sum_{i=1}^{i_0} H_i$$
$$\leqslant \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\varepsilon b^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)} + \sum_{i=1}^{i_0} O_i(1) \ (\varepsilon \to 0^+),$$

and then (15) follows. In the same way, we have (16). □

Definition 1. For $0 < \alpha, \beta, \lambda \leqslant 1, \lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = \lambda$, we define weight coefficients $w(\lambda_1, n)$ and $W(\lambda_2, m)$ as follows:

$$w(\lambda_1, n) := \sum_m \frac{\ln(\|\widetilde{U}_m\|_\alpha / \|\widetilde{V}_n\|_\beta)}{\|\widetilde{U}_m\|_\alpha^\lambda - \|\widetilde{V}_n\|_\beta^\lambda} \frac{\|\widetilde{V}_n\|_\beta^{\lambda_2}}{\|\widetilde{U}_m\|_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu_m^{(k)}, \quad (17)$$

$$W(\lambda_2, m) := \sum_n \frac{\ln(\|\widetilde{U}_m\|_\alpha / \|\widetilde{V}_n\|_\beta)}{\|\widetilde{U}_m\|_\alpha^\lambda - \|\widetilde{V}_n\|_\beta^\lambda} \frac{\|\widetilde{U}_m\|_\alpha^{\lambda_1}}{\|\widetilde{V}_n\|_\beta^{j_0-\lambda_2}} \prod_{l=1}^{j_0} \upsilon_n^{(l)}. \quad (18)$$

Example 1. Setting $g(t) := \frac{\ln t}{t-1} \ (t > 0), g(1) = \lim_{t\to 1} \frac{\ln t}{t-1} = 1$, we find

$$g(t) = \frac{\ln[1 + (t-1)]}{t-1} = \sum_{k=0}^\infty (-1)^k \frac{(t-1)^k}{k+1}$$
$$= \sum_{k=0}^\infty \frac{(-1)^k k!}{k+1} \frac{(t-1)^k}{k!} (-1 < t-1 \leqslant 1),$$

and then, $g^{(k)}(1) = \frac{(-1)^k k!}{k+1} (k = 0, 1, \ldots), g'(1) = -\frac{1}{2}, g''(1) = \frac{2}{3}$.

We put $h(t) := t - 1 - t\ln t$, and find $g'(t) = \frac{h(t)}{t(t-1)^2} \ (t \in \mathbf{R}_+ \setminus \{1\})$. Since $h'(t) = -\ln t > 0 (0 < t < 1); h'(t) = -\ln t < 0 (t > 1)$, we have $\max h(t) = h(1) = 0$ and $g'(t) < 0 (t > 0)$, with $g'(1) = -\frac{1}{2} < 0$.

We put $J(t) := -(t-1)^2 - 2t(t-1) + 2t^2 \ln t$, and find $g''(t) = \frac{J(t)}{t^2(t-1)^3} \ (t \in \mathbf{R}_+ \setminus \{1\})$. Since $J'(t) = -4(t-1) + 4t\ln t, J''(t) = 4\ln t < 0 (0 < t < 1); J''(t) = 4\ln t > 0 (t > 1)$, we have $\min J'(t) = J'(1) = 0, J'(t) > 0 (t \in \mathbf{R}_+ \setminus \{1\})$ and $J(t)$ is strict decreasing in $\mathbf{R}_+$. Since $J(1) = 0$, we have $J(t) < 0 (0 < t < 1); J(t) > 0 (t > 1)$ and $g''(t) = \frac{J(t)}{t^2(t-1)^3} > 0 (t > 0)$, with $g''(1) = \frac{2}{3} > 0$.

For $0 < \lambda \leqslant 1$, we set $G(u) = \frac{1}{\lambda} g(u^\lambda) = \frac{\ln u}{u^\lambda - 1} (u > 0)$. Then we obtain
$G'(u) = g'(u^\lambda) u^{\lambda-1} < 0; G''(u) = g''(u^\lambda) u^{2\lambda-2} + (\lambda - 1) g'(u^\lambda) u^{\lambda-2} > 0$.

With regards to the assumptions of Definition 1, we set $k_\lambda(x, y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda} \ (x, y > 0)$, and find

$$\frac{\partial}{\partial x} k_\lambda(x, y) = \frac{\partial}{\partial x} y^\lambda G\left(\frac{x}{y}\right) < 0,$$
$$\frac{\partial^2}{\partial x^2} k_\lambda(x, y) = \frac{\partial^2}{\partial x^2} y^\lambda G\left(\frac{x}{y}\right) > 0;$$

$$\frac{\partial}{\partial y}k_\lambda(x,y)=\frac{\partial}{\partial y}x^\lambda G\Big(\frac{y}{x}\Big)<0,$$
$$\frac{\partial^2}{\partial y^2}k_\lambda(x,y)=\frac{\partial^2}{\partial y^2}x^\lambda G\Big(\frac{y}{x}\Big)>0.$$

In the same way, since $0<\lambda_1<1\leqslant i_0, 0<\lambda_2<1\leqslant j_0$, we still can find that $k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}}$ $(k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}})$ is strictly decreasing and strictly convex in $x\in(0,\infty)$ $(y\in(0,\infty))$, satisfying

$$\frac{\partial}{\partial x}\left(k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}}\right)<0,\frac{\partial^2}{\partial x^2}\left(k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}}\right)>0;$$
$$\frac{\partial}{\partial y}\left(k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}}\right)<0,\frac{\partial^2}{\partial y^2}\left(k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}}\right)>0.$$

We obtain

$$k(\lambda_1):=\int_0^\infty k_\lambda(u,1)\frac{du}{u^{1-\lambda_1}}=\int_0^\infty\frac{u^{\lambda_1-1}\ln u}{u^\lambda-1}du$$
$$\overset{v=u^\lambda}{=}\frac{1}{\lambda^2}\int_0^\infty\frac{v^{(\lambda_1/\lambda)-1}dv}{v-1}=\left[\frac{\pi}{\lambda\sin\left(\frac{\pi\lambda_1}{\lambda}\right)}\right]^2\in\mathbf{R}_+. \quad (19)$$

(ii) If $(-1)^ih^{(i)}(t)>0$ $(t>0; i=0,1,2), A>0, 0<\alpha\leqslant 1$, then we have

$$\frac{d}{dx}h((A+x^\alpha)^{\frac{1}{\alpha}})=h'((A+x^\alpha)^{\frac{1}{\alpha}})(A+x^\alpha)^{\frac{1}{\alpha}-1}x^{\alpha-1}<0,$$

$$\begin{aligned}\frac{d^2}{dx^2}h((A+x^\alpha)^{\frac{1}{\alpha}})&=h''((A+x^\alpha)^{\frac{1}{\alpha}})(A+x^\alpha)^{\frac{2}{\alpha}-2}x^{2\alpha-2}\\&+(1-\alpha)h'((A+x^\alpha)^{\frac{1}{\alpha}})(A+x^\alpha)^{\frac{1}{\alpha}-2}x^{2\alpha-2}\\&+(\alpha-1)h'((A+x^\alpha)^{\frac{1}{\alpha}})(A+x^\alpha)^{\frac{1}{\alpha}-1}x^{\alpha-2}\\&=h''((A+x^\alpha)^{\frac{1}{\alpha}})(A+x^\alpha)^{\frac{2}{\alpha}-2}x^{2\alpha-2}\\&+A(\alpha-1)h'((A+x^\alpha)^{\frac{1}{\alpha}})(A+x^\alpha)^{\frac{1}{\alpha}-2}x^{\alpha-2}>0\ (x>0).\end{aligned}$$

Hence, by (11), for $m_i-\frac{1}{2}<x_i<m_i+\frac{1}{2}$ $(i=1,\dots,i_0; m\in\mathbf{N})$, we have $\prod_{k=1}^{i_0}\mu_m^{(k)}=\prod_{k=1}^{i_0}\mu_k(x)$ and

$$\frac{\ln(\|U(m)\|_\alpha/\|\widetilde{V}_n\|_\beta)}{\|U(m)\|_\alpha^\lambda-\|\widetilde{V}_n\|_\beta^\lambda}\|U(m)\|_\alpha^{\lambda_1-i_0}\prod_{k=1}^{i_0}\mu_m^{(k)}$$
$$<\int_{\{x\in\mathbf{R}_+^{i_0};m_i-\frac{1}{2}<x_i<m_i-\frac{1}{2}\}}\frac{\ln(\|U(x)\|_\alpha/\|\widetilde{V}_n\|_\beta)}{\|U(x)\|_\alpha^\lambda-\|\widetilde{V}_n\|_\beta^\lambda}\|U(x)\|_\alpha^{\lambda_1-i_0}\prod_{k=1}^{i_0}\mu_k(x)dx.$$

Lemma 4. *With regards to the assumptions of Definition 1, (i) we have*

$$w(\lambda_1,n)<K_\beta(\lambda_1)\ (n\in\mathbf{N}^{j_0}), \quad (20)$$
$$W(\lambda_2,m)<K_\alpha(\lambda_1)\ (m\in\mathbf{N}^{i_0}), \quad (21)$$

where,

$$K_\beta(\lambda_1)=\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}k(\lambda_1), K_\alpha(\lambda_1)=\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}k(\lambda_1); \quad (22)$$

(ii) for $\mu_m^{(k)}\geqslant\mu_{m+1}^{(k)}$ $(m\in\mathbf{N}), \upsilon_n^{(l)}\geqslant\upsilon_{n+1}^{(l)}$ $(n\in\mathbf{N}), U_\infty^{(k)}=V_\infty^{(l)}=\infty$ $(k=1,\dots,i_0, l=1,\dots,j_0)$, we have

$$0<K_\alpha(\lambda_1)(1-\theta_\lambda(n))<w(\lambda_1,n)\ (n\in\mathbf{N}^{j_0}), \quad (23)$$

where, for $c:=\max_{1\leqslant k\leqslant i_0}\{\mu_1^{(k)}\}$ (>0),

$$0<\theta_\lambda(n):=\frac{1}{\lambda^2k(\lambda_1)}\int_0^{c^\lambda i_0^{\lambda/\alpha}/\|\widetilde{V}_n\|_\beta^\lambda}\frac{\ln t}{t-1}t^{\frac{\lambda_1}{\lambda}-1}dt=O\left(\frac{1}{\|\widetilde{V}_n\|_\beta^{\lambda_1/2}}\right). \quad (24)$$

Proof. (i) Since $\|\widetilde{U}_m\|_\alpha\geqslant\|U(m)\|_\alpha$, by (12), (14) and Example 1 (ii), for $0<\lambda_1<1\leqslant i_0, \lambda>0$, it follows that

$$\begin{aligned}w(\lambda_1,n)&\leqslant\sum_m\frac{\ln(\|U(m)\|_\alpha/\|\widetilde{V}_n\|_\beta)}{\|U(m)\|_\alpha^\lambda-\|\widetilde{V}_n\|_\beta^\lambda}\frac{\|\widetilde{V}_n\|_\beta^{\lambda_2}}{\|U(m)\|_\alpha^{i_0-\lambda_1}}\prod_{k=1}^{i_0}\mu_m^{(k)}\\&<\sum_m\int_{\{x\in\mathbf{R}_+^{i_0};m_i-\frac{1}{2}<x_i\leqslant m_i+\frac{1}{2}\}}\frac{\ln(\|U(x)\|_\alpha/\|\widetilde{V}_n\|_\beta)}{\|U(x)\|_\alpha^\lambda-\|\widetilde{V}_n\|_\beta^\lambda}\\&\times\frac{\|\widetilde{V}_n\|_\beta^{\lambda_2}}{\|U(x)\|_\alpha^{i_0-\lambda_1}}\prod_{k=1}^{i_0}\mu_k(x)dx\end{aligned}$$

$$=\int_{\{x\in\mathbf{R}_+^{i_0};x_i>\frac{1}{2}\}}\frac{\ln(\|U(x)\|_\alpha/\|\widetilde{V}_n\|_\beta)}{\|U(x)\|_\alpha^\lambda-\|\widetilde{V}_n\|_\beta^\lambda}\frac{\|\widetilde{V}_n\|_\beta^{\lambda_2}}{\|U(x)\|_\alpha^{i_0-\lambda_1}}\prod_{k=1}^{i_0}\mu_k(x)dx$$

$$\overset{v=U(x)}{=}\int_{\mathbf{R}_+^{i_0}}\frac{\ln(\|v\|_\alpha/\|\widetilde{V}_n\|_\beta)}{\|v\|_\alpha^\lambda-\|\widetilde{V}_n\|_\beta^\lambda}\|v\|_\alpha^{\lambda_1-i_0}\|\widetilde{V}_n\|_\beta^{\lambda_2}dv$$

$$=\lim_{M\to\infty}\int_{\mathbf{D}_M}\frac{\ln(M\left[\sum_{i=1}^{i_0}\left(\frac{v_i}{M}\right)^\alpha\right]^{1/\alpha}/\|\widetilde{V}_n\|_\beta)\|\widetilde{V}_n\|_\beta^{\lambda_2}}{M^\lambda\left[\sum_{i=1}^{i_0}\left(\frac{v_i}{M}\right)^\alpha\right]^{\lambda/\alpha}-\|\widetilde{V}_n\|_\beta^\lambda}\left\{M\left[\sum_{i=1}^{j_0}\left(\frac{v_i}{M}\right)^\alpha\right]^{\frac{1}{\alpha}}\right\}^{\lambda_1-i_0}dv$$

$$=\lim_{M\to\infty}\frac{M^{i_0}\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0}\Gamma\left(\frac{i_0}{\alpha}\right)}\int_0^1\frac{\ln(Mu^{1/\alpha}/\|\widetilde{V}_n\|_\beta)\|\widetilde{V}_n\|_\beta^{\lambda_2}}{M^\lambda u^{\lambda/\alpha}-\|\widetilde{V}_n\|_\beta^\lambda}(Mu^{1/\alpha})^{\lambda_1-i_0}u^{\frac{i_0}{\alpha}-1}du$$

$$\overset{t=M^\lambda u^{\lambda/\alpha}/\|V_n\|_\beta^\lambda}{=}\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\lambda^2\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\int_0^\infty\frac{t^{\frac{\lambda_1}{\lambda}-1}\ln t}{t-1}dt=\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)k(\lambda_1)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}=K_\alpha(\lambda_1).$$

Hence, we have (20). In the same way, we have (21).

(ii) Since for $m_i\leqslant x_i<m_i+\frac{1}{2}, \mu_{m_i}^{(k)}\geqslant\mu_{m_i+1}^{(k)}=\mu_k(x+\frac{1}{2})$; for $m_i+\frac{1}{2}\leqslant x_i<m_i+1, \mu_m^{(k)}=\mu_k(x+\frac{1}{2})$, by (12) and in the same way, for $c=\max_{1\leqslant k\leqslant i_0}\{\mu_1^{(k)}\}$ (>0), we have

$$\begin{aligned}w(\lambda_1,n)&\geqslant\sum_m\frac{\ln\left(\|U(m+\frac{1}{2})\|_\alpha/\|\widetilde{V}_n\|_\beta\right)\|\widetilde{V}_n\|_\beta^{\lambda_2}}{\|U(m+\frac{1}{2})\|_\alpha^\lambda-\|\widetilde{V}_n\|_\beta^\lambda}\|U\left(m+\frac{1}{2}\right)\|_\alpha^{\lambda_1-i_0}\prod_{k=1}^{i_0}\mu_m^{(k)}\\&>\sum_m\int_{\{x\in\mathbf{R}_+^{i_0};m_i\leqslant x_i<m_i+1\}}\frac{\ln\left(\|U(x+\frac{1}{2})\|_\alpha/\|\widetilde{V}_n\|_\beta\right)\|\widetilde{V}_n\|_\beta^{\lambda_2}}{\|U(x+\frac{1}{2})\|_\alpha^\lambda-\|\widetilde{V}_n\|_\beta^\lambda}\\&\times\|U(x+\frac{1}{2})\|_\alpha^{\lambda_1-i_0}\prod_{k=1}^{i_0}\mu_k\left(x+\frac{1}{2}\right)dx\\&=\int_{[1,\infty)^{i_0}}\frac{\ln\left(\|U(x+\frac{1}{2})\|_\alpha/\|\widetilde{V}_n\|_\beta\right)\|\widetilde{V}_n\|_\beta^{\lambda_2}}{\|U(x+\frac{1}{2})\|_\alpha^\lambda-\|\widetilde{V}_n\|_\beta^\lambda}\\&\times\|U(x+\frac{1}{2})\|_\alpha^{\lambda_1-i_0}\prod_{k=1}^{i_0}\mu_k\left(x+\frac{1}{2}\right)dx\\&\overset{v=U(x+\frac{1}{2})}{\geqslant}\int_{[c,\infty)^{i_0}}\frac{\ln\left(\|v\|_\alpha/\|\widetilde{V}_n\|_\beta\right)\|\widetilde{V}_n\|_\beta^{\lambda_2}}{\|v\|_\alpha^\lambda-\|\widetilde{V}_n\|_\beta^\lambda}\|v\|_\alpha^{\lambda_1-i_0}dv.\end{aligned}$$

For $M>ci_0^{1/\alpha}$, we set

$$\Psi(u)=\begin{cases}0, & 0<u\leqslant \frac{c^{\alpha}i_0}{M^{\alpha}},\\ \frac{\ln(Mu^{1/\alpha}/\|\widetilde{V}_n\|_\beta)}{M^{\lambda}u^{\lambda/\alpha}-\|\widetilde{V}_n\|_\beta^{\lambda}}(Mu^{1/\alpha})^{\lambda_1-i_0}, & \frac{c^{\alpha}i_0}{M^{\alpha}}<u\leqslant 1.\end{cases}$$

By (14), it follows that

$$\int_{\{x\in\mathbf{R}_+^{i_0};x_i\geqslant c\}}\frac{\ln\left(\|x\|_\alpha/\|\widetilde{V}_n\|_\beta\right)\|\widetilde{V}_n\|_\beta^{\lambda_2}}{\|x\|_\alpha^{\lambda}-\|\widetilde{V}_n\|_\beta^{\lambda}}\|x\|_\alpha^{\lambda_1-i_0}dx$$
$$=\lim_{M\to\infty}\int\cdots\int_{D_M}\Psi\left(\sum_{i=1}^{i_0}\left(\frac{x_i}{M}\right)^{\alpha}\right)dx_1\cdots dx_{i_0}=\lim_{M\to\infty}\frac{M^{i_0}\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0}\Gamma\left(\frac{i_0}{\alpha}\right)}$$
$$\times\int_{c^{\alpha}i_0/M^{\alpha}}^{1}\frac{\ln(Mu^{1/\alpha}/\|\widetilde{V}_n\|_\beta)\|\widetilde{V}_n\|_\beta^{\lambda_2}}{M^{\lambda}u^{\lambda/\alpha}-\|\widetilde{V}_n\|_\beta^{\lambda}}(Mu^{1/\alpha})^{\lambda_1-i_0}u^{\frac{i_0}{\alpha}-1}du$$
$$\overset{t=\frac{M^{\lambda}u^{\lambda/\alpha}}{\|\widetilde{V}_n\|_\beta^{\lambda}}}{=}\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)\lambda^2}\int_{c^{\lambda}i_0^{\lambda/\alpha}/\|\widetilde{V}_n\|_\beta^{\lambda}}^{\infty}\frac{\ln t}{t-1}t^{\frac{\lambda_1}{\lambda}-1}dt.$$

Hence, we have

$$w(\lambda_1,n)>\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})\lambda^2}\int_{c^{\lambda}i_0^{\lambda/\alpha}/\|\widetilde{V}_n\|_\beta^{\lambda}}^{\infty}\frac{\ln t}{t-1}t^{\frac{\lambda_1}{\lambda}-1}dt$$
$$=K_\alpha(\lambda_1)(1-\theta_\lambda(n))>0.$$

Since $\frac{t^{\lambda_1/(2\lambda)}\ln t}{t-1}\to 0\,(t\to 0^+)$, there exists a constant $M>0$, such that $\frac{t^{\lambda_1/(2\lambda)}\ln t}{t-1}\leqslant M\left(t\in\left(0,c^{\lambda}i_0^{\lambda/\alpha}/\|\widetilde{V}_1\|_\beta^{\lambda}\right)\right.$. We obtain

$$0<\theta_\lambda(n)=\frac{1}{\lambda^2k(\lambda_1)}\int_0^{c^{\lambda}i_0^{\lambda/\alpha}/\|\widetilde{V}_n\|_\beta^{\lambda}}\frac{\ln t}{t-1}t^{\frac{\lambda_1}{\lambda}-1}dt$$
$$\leqslant\frac{M}{\lambda^2k(\lambda_1)}\int_0^{c^{\lambda}i_0^{\lambda/\alpha}/\|\widetilde{V}_n\|_\beta^{\lambda}}t^{\frac{\lambda_1}{2\lambda}-1}dt=\frac{2M}{\lambda\lambda_1k(\lambda_1)}\left(\frac{ci_0^{1/\alpha}}{\|\widetilde{V}_n\|_\beta}\right)^{\frac{\lambda_1}{2}},$$

and then (23) and (24) follow. □

3. Main results

Setting functions

$$\widetilde{\Phi}(m):=\frac{\|\widetilde{U}_m\|_\alpha^{p(i_0-\lambda_1)-i_0}}{\left(\prod_{k=1}^{i_0}\mu_m^{(k)}\right)^{p-1}},\ \widetilde{\Psi}(n):=\frac{\|\widetilde{V}_n\|_\beta^{q(j_0-\lambda_2)-j_0}}{\left(\prod_{l=1}^{j_0}\upsilon_n^{(l)}\right)^{q-1}}\ (m\in\mathbf{N}^{i_0},n\in\mathbf{N}^{j_0}),$$

and the following normed spaces

$$l_{p,\widetilde{\Phi}}:=\left\{a=\{a_m\};\|a\|_{p,\widetilde{\Phi}}:=\left(\sum_m\widetilde{\Phi}(m)|a_m|^p\right)^{\frac{1}{p}}<\infty\right\},$$
$$l_{q,\widetilde{\Psi}}:=\left\{b=\{b_n\};\|b\|_{q,\widetilde{\Psi}}:=\left(\sum_n\widetilde{\Psi}(n)|b_n|^q\right)^{\frac{1}{q}}<\infty\right\},$$
$$l_{p,\widetilde{\Psi}^{1-p}}:\left\{c=\{c_n\};\|c\|_{p,\widetilde{\Psi}^{1-p}}:=\left(\sum_n\widetilde{\Psi}^{1-p}(n)|c_n|^p\right)^{\frac{1}{p}}<\infty\right\},$$

we have

Theorem 1. *If $p>1,\frac{1}{p}+\frac{1}{q}=1,0<\alpha,\beta,\lambda\leqslant 1,\lambda_1,\lambda_2>0,\lambda_1+\lambda_2=\lambda$, then for $a_m,b_n\geqslant 0,a=\{a_m\}\in l_{p,\widetilde{\Phi}},b=\{b_n\}\in l_{q,\widetilde{\Psi}},\|a\|_{p,\widetilde{\Phi}},\|b\|_{q,\widetilde{\Psi}}>0$, we have the following equivalent inequalities*

$$I:=\sum_n\sum_m\frac{\ln(\|\widetilde{U}_m\|_\alpha/\|\widetilde{V}_n\|_\beta)a_mb_n}{\|\widetilde{U}_m\|_\alpha^{\lambda}-\|\widetilde{V}_n\|_\beta^{\lambda}}<K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)\|a\|_{p,\widetilde{\Phi}}\|b\|_{q,\widetilde{\Psi}},\tag{25}$$

$$J:=\left\{\sum_n\frac{\prod_{k=1}^{j_0}\upsilon_n^{(k)}}{\|\widetilde{V}_n\|_\beta^{j_0-p\lambda_2}}\left[\sum_m\frac{\ln(\|\widetilde{U}_m\|_\alpha/\|\widetilde{V}_n\|_\beta)a_m}{\|\widetilde{U}_m\|_\alpha^{\lambda}-\|\widetilde{V}_n\|_\beta^{\lambda}}\right]^p\right\}^{\frac{1}{p}}$$
$$<K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)\|a\|_{p,\widetilde{\Phi}}.\tag{26}$$

where,

$$K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)=\left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1}\Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1}\Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}}\left[\frac{\pi}{\lambda\sin\left(\frac{\pi\lambda_1}{\lambda}\right)}\right]^2.\tag{27}$$

Proof. By Hölder's inequality with weight (cf. Kuang, 2004), we have

$$I=\sum_n\sum_m\frac{\ln(\|\widetilde{U}_m\|_\alpha/\|\widetilde{V}_n\|_\beta)}{\|\widetilde{U}_m\|_\alpha^{\lambda}-\|\widetilde{V}_n\|_\beta^{\lambda}}\left[\frac{\|\widetilde{U}_m\|_\alpha^{\frac{i_0-\lambda_1}{q}}}{\|\widetilde{V}_n\|_\beta^{\frac{j_0-\lambda_2}{p}}}\frac{\left(\prod_{l=1}^{j_0}\upsilon_n^{(l)}\right)^{\frac{1}{p}}a_m}{\left(\prod_{k=1}^{i_0}\mu_m^{(k)}\right)^{\frac{1}{q}}}\right]$$
$$\times\left[\frac{\|\widetilde{V}_n\|_\beta^{\frac{j_0-\lambda_2}{p}}}{\|\widetilde{U}_m\|_\alpha^{\frac{i_0-\lambda_1}{q}}}\frac{\left(\prod_{k=1}^{i_0}\mu_m^{(k)}\right)^{\frac{1}{q}}}{\left(\prod_{l=1}^{j_0}\upsilon_n^{(l)}\right)^{\frac{1}{p}}}b_n\right]$$
$$\leqslant\left[\sum_m W(\lambda_2,m)\frac{\|\widetilde{U}_m\|_\alpha^{p(i_0-\lambda_1)-i_0}a_m^p}{\left(\prod_{k=1}^{i_0}\mu_m^{(k)}\right)^{p-1}}\right]^{\frac{1}{p}}\left[\sum_n w(\lambda_1,n)\frac{\|\widetilde{V}_n\|_\beta^{q(j_0-\lambda_2)-j_0}b_n^q}{\left(\prod_{l=1}^{j_0}\upsilon_n^{(l)}\right)^{q-1}}\right]^{\frac{1}{q}}.$$

Then by (20) and (21), we have (25). We set

$$b_n:=\frac{\prod_{l=1}^{j_0}\upsilon_n^{(l)}}{\|\widetilde{V}_n\|_\beta^{j_0-p\lambda_2}}\left[\sum_m\frac{\ln(\|\widetilde{U}_m\|_\alpha/\|\widetilde{V}_n\|_\beta)a_m}{\|\widetilde{U}_m\|_\alpha^{\lambda}-\|\widetilde{V}_n\|_\beta^{\lambda}}\right]^{p-1},n\in\mathbf{N}^{j_0}.$$

Then we have $J=\|b\|_{q,\widetilde{\Psi}}^{q-1}$. Since the right-hand side of (26) is finite, it follows that $J<\infty$. If $J=0$, then (26) is trivially valid; if $J>0$, then by (25), we have

$$\|b\|_{q,\widetilde{\Psi}}^{q}=J^p=I<K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)\|a\|_{p,\widetilde{\Phi}}\|b\|_{q,\widetilde{\Psi}},$$
$$\|b\|_{q,\widetilde{\Psi}}^{q-1}=J<K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)\|a\|_{p,\widetilde{\Phi}},$$

namely, (26) follows. On the other hand, assuming that (26) is valid, by Hölder's inequality (cf. Kuang, 2004), we have

$$I=\sum_n\frac{\left(\prod_{l=1}^{j_0}\upsilon_n^{(l)}\right)^{1/p}}{\|\widetilde{V}_n\|_\beta^{(j_0/p)-\lambda_2}}\sum_m\frac{\ln(\|\widetilde{U}_m\|_\alpha/\|\widetilde{V}_n\|_\beta)a_m}{\|\widetilde{U}_m\|_\alpha^{\lambda}-\|\widetilde{V}_n\|_\beta^{\lambda}}\times\frac{\|\widetilde{V}_n\|_\beta^{(j_0/p)-\lambda_2}}{\left(\prod_{l=1}^{j_0}\upsilon_n^{(l)}\right)^{1/p}}$$
$$\times b_n\leqslant J\|b\|_{q,\widetilde{\Psi}}.\tag{28}$$

Then by (26), we have (25), which is equivalent to (26). □

Theorem 2. *With regards to the assumptions of Theorem 1, if $\mu_m^{(k)}\geqslant\mu_{m+1}^{(k)}\ (m\in\mathbf{N}),\upsilon_n^{(l)}\geqslant\upsilon_{n+1}^{(l)}\ (n\in\mathbf{N}),U_\infty^{(k)}=V_\infty^{(l)}=\infty\ (k=1,\ldots,i_0,l=1,\ldots,j_0)$, then the constant factor $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ in (25) and (26) is the best possible.*

Proof. For $0<\varepsilon<\min\{p\lambda_1,p(1-\lambda_2)\},\widetilde{\lambda}_1=\lambda_1-\frac{\varepsilon}{p}\ (\in(0,1)),\widetilde{\lambda}_2=\lambda_2+\frac{\varepsilon}{p}\ (<1)$, we set

$$\widetilde{a} = \{\widetilde{a}_m\}, \widetilde{a}_m := \|\widetilde{U}_m\|_\alpha^{-i_0+\widetilde{\lambda}_1} \prod_{k=1}^{i_0} \mu_m^{(k)} \ (m \in \mathbf{N}^{i_0}),$$

$$\widetilde{b} = \{\widetilde{b}_n\}, \widetilde{b}_n := \|\widetilde{V}_n\|_\beta^{-j_0+\widetilde{\lambda}_2-\varepsilon} \prod_{l=1}^{j_0} \upsilon_n^{(l)} \ (n \in \mathbf{N}^{j_0}).$$

Then by (15) and (16), we obtain

$$\|\widetilde{a}\|_{p,\widetilde{\Phi}} \|\widetilde{b}\|_{q,\widetilde{\Phi}} = \left[\sum_m \frac{\|\widetilde{U}_m\|_\alpha^{p(i_0-\lambda_1)-i_0} \widetilde{a}_m^p}{\left(\prod_{k=1}^{i_0} \mu_m^{(k)}\right)^{p-1}}\right]^{\frac{1}{p}} \left[\sum_n \frac{\|\widetilde{V}_n\|_\beta^{q(j_0-\lambda_2)-j_0} \widetilde{b}_n^q}{\left(\prod_{l=1}^{j_0} \upsilon_n^{(l)}\right)^{q-1}}\right]^{\frac{1}{q}}$$
$$= \left(\sum_m \|\widetilde{U}_m\|_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_m^{(k)}\right)^{\frac{1}{p}} \left(\sum_n \|\widetilde{V}_n\|_\beta^{-j_0-\varepsilon} \prod_{l=1}^{j_0} \upsilon_n^{(j)}\right)^{\frac{1}{q}}$$
$$\leqslant \frac{1}{\varepsilon}\left(\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{b^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1} \Gamma\left(\frac{i_0}{\alpha}\right)} + \varepsilon O(1)\right)^{\frac{1}{p}} \left(\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{b^\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma\left(\frac{j_0}{\beta}\right)} + \varepsilon \widetilde{O}(1)\right)^{\frac{1}{q}}.$$

By (23) and (24), we find

$$\widetilde{I} := \sum_n \left[\sum_m \frac{\ln(\|\widetilde{U}_m\|_\alpha / \|\widetilde{V}_n\|_\beta) \widetilde{a}_m}{\|\widetilde{U}_m\|_\alpha^\lambda - \|\widetilde{V}_n\|_\beta^\lambda}\right] \widetilde{b}_n = \sum_n \frac{w(\widetilde{\lambda}_1, n)}{\|\widetilde{V}_n\|_\beta^{j_0+\varepsilon}} \prod_{l=1}^{j_0} \upsilon_n^{(l)}$$
$$> K_\alpha(\widetilde{\lambda}_1) \sum_n \left(1 - O\left(\frac{1}{\|\widetilde{V}_n\|_\beta^{\widetilde{\lambda}_1/2}}\right)\right) \frac{1}{\|\widetilde{V}_n\|_\beta^{j_0+\varepsilon}} \prod_{l=1}^{j_0} \upsilon_n^{(l)}$$
$$= K_\alpha(\widetilde{\lambda}_1) \left(\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\varepsilon b^\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma\left(\frac{j_0}{\beta}\right)} + \widetilde{O}(1) - O_1(1)\right).$$

If there exists a constant $K \leqslant K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$, such that (25) is valid when replacing $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$ by K, then we have $\varepsilon \widetilde{I} < \varepsilon K \|\widetilde{a}\|_{p,\widetilde{\Phi}} \|\widetilde{b}\|_{q,\widetilde{\Phi}}$, namely,

$$K_\alpha\left(\lambda_1 - \frac{\varepsilon}{p}\right)\left(\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{b^\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma\left(\frac{j_0}{\beta}\right)} + \varepsilon \widetilde{O}(1) - \varepsilon O_1(1)\right)$$
$$< K\left(\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{b^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1} \Gamma\left(\frac{i_0}{\alpha}\right)} + \varepsilon O(1)\right)^{\frac{1}{p}} \left(\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{b^\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma\left(\frac{j_0}{\beta}\right)} + \varepsilon \widetilde{O}(1)\right)^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, it follows that

$$\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1} \Gamma\left(\frac{j_0}{\beta}\right)} \frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right) k(\lambda_1)}{\alpha^{i_0-1} \Gamma\left(\frac{i_0}{\alpha}\right)} \leqslant K\left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1} \Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1} \Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{q}},$$

and then $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \leqslant K$. Hence, $K = K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$ is the best possible constant factor of (25). The constant factor in (26) is still the best possible. Otherwise, we would reach a contradiction by (28) that the constant factor in (25) is not the best possible. □

4. Operator expressions

With regards to the assumptions of Theorem 1, in view of

$$c_n := \frac{\prod_{k=1}^{j_0} \upsilon_n^{(k)}}{\|\widetilde{V}_n\|_\beta^{j_0-p\lambda_2}} \left[\sum_m \frac{\ln(\|\widetilde{U}_m\|_\alpha / \|\widetilde{V}_n\|_\beta)}{\|\widetilde{U}_m\|_\alpha^\lambda - \|\widetilde{V}_n\|_\beta^\lambda} a_m\right]^{p-1}, n \in \mathbf{N}^{j_0}$$

$$c = \{c_n\}, \|c\|_{p,\widetilde{\Psi}^{1-p}} = J < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\widetilde{\Phi}} < \infty,$$

we can set the following definition:

Definition 2. Define a multidimensional Hilbert's operator $T: l_{p,\widetilde{\Phi}} \to l_{p,\widetilde{\Psi}^{1-p}}$ as follows: For any $a \in l_{p,\widetilde{\Phi}}$, there exists a unique representation $Ta = c \in l_{p,\widetilde{\Psi}^{1-p}}$, satisfying

$$Ta(n) := \sum_m \frac{\ln(\|\widetilde{U}_m\|_\alpha / \|\widetilde{V}_n\|_\beta)}{\|\widetilde{U}_m\|_\alpha^\lambda - \|\widetilde{V}_n\|_\beta^\lambda} a_m \ (n \in \mathbf{N}^{j_0}). \quad (29)$$

For $b \in l_{q,\widetilde{\Psi}}$, we define the following formal inner product of Ta and b as follows:

$$(Ta, b) := \sum_n \left[\sum_m \frac{\ln(\|\widetilde{U}_m\|_\alpha / \|\widetilde{V}_n\|_\beta)}{\|\widetilde{U}_m\|_\alpha^\lambda - \|\widetilde{V}_n\|_\beta^\lambda} a_m\right] b_n. \quad (30)$$

Then by Theorem 1, we have the following equivalent inequalities:

$$(Ta, b) < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\widetilde{\Phi}} \|b\|_{q,\widetilde{\Psi}}, \quad (31)$$

$$\|Ta\|_{p,\widetilde{\Psi}^{1-p}} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\widetilde{\Phi}}. \quad (32)$$

It follows that T is bounded with

$$\|T\| := \sup_{a(\neq\theta) \in l_{p,\widetilde{\Phi}}} \frac{\|Ta\|_{p,\widetilde{\Psi}^{1-p}}}{\|a\|_{p,\widetilde{\Phi}}} \leqslant K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1). \quad (33)$$

Since by Theorem 2, the constant factor $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$ in (32) is the best possible, we have

$$\|T\| = K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) = \left[\frac{\Gamma^{j_0}\left(\frac{1}{\beta}\right)}{\beta^{j_0-1} \Gamma\left(\frac{j_0}{\beta}\right)}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}\left(\frac{1}{\alpha}\right)}{\alpha^{i_0-1} \Gamma\left(\frac{i_0}{\alpha}\right)}\right]^{\frac{1}{q}} \left[\frac{\pi}{\lambda \sin\left(\frac{\pi \lambda_1}{\lambda}\right)}\right]^2. \quad (34)$$

Remark 1. (i) For $\widetilde{\mu}_i^{(k)} = 0 \ (k = 1, \ldots, i_0; i = 1, \ldots, m), \widetilde{\upsilon}_j^{(l)} = 0 \ (l = 1, \ldots, j_0; j = 1, \ldots, n)$, setting

$$\Phi(m) := \frac{\|U_m\|_\alpha^{p(i_0-\lambda_1)-i_0}}{\left(\prod_{k=1}^{i_0} \mu_m^{(k)}\right)^{p-1}} \ (m \in \mathbf{N}^{i_0}),$$
$$\Psi(n) := \frac{\|V_n\|_\beta^{q(j_0-\lambda_2)-j_0}}{\left(\prod_{l=1}^{j_0} \upsilon_n^{(l)}\right)^{q-1}} \ (n \in \mathbf{N}^{j_0}),$$

then (25) and (26) reduce the following equivalent inequalities with the same best possible constant factor $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$:

$$\sum_n \sum_m \frac{\ln(\|U_m\|_\alpha / \|V_n\|_\beta) a_m b_n}{\|U_m\|_\alpha^\lambda - \|V_n\|_\beta^\lambda} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\Phi} \|b\|_{q,\Psi}, \quad (35)$$

$$\left\{\sum_n \frac{\prod_{k=1}^{j_0} \upsilon_n^{(k)}}{\|V_n\|_\beta^{j_0-p\lambda_2}} \left[\sum_m \frac{\ln(\|U_m\|_\alpha / \|V_n\|_\beta) a_m}{\|U_m\|_\alpha^\lambda - \|V_n\|_\beta^\lambda}\right]^p\right\}^{\frac{1}{p}} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|a\|_{p,\Phi}. \quad (36)$$

Hence, (25) and (26) are more accurate extensions of (35) and (36).

(ii) For $\mu_i^{(k)} = 1 \ (k = 1, \ldots, i_0; i = 1, \ldots, m), \upsilon_j^{(l)} = 1 \ (l = 1, \ldots, j_0; j = 1, \ldots, n)$, (35) reduces to (5); for $i_0 = j_0 = 1$, (35) reduces to (6). Hence, (35) is an extension of (5) and (6); so is (25).

5. Conclusions

In this paper, by means of the weight coefficients, the transfer formula, Hermite-Hadamard's inequality and the technique of real analysis, a more accurate multidimensional Hardy-Hilbert-type inequality with a best possible constant factor is given by Theorems 1 and 2, which is an extension of some published results. Moreover, the equivalent forms with the best possible constant factor are obtained by Theorems 1 and 2, and the operator expressions are also considered. The method of weight coefficients is very important, which helps us to prove the main inequalities with the best possible constant factor. The lemmas and theorems of this paper provide an extensive account of this type of inequalities.

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