



Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Frictional self-contact problem of elastic rods

Mourad Chamekh^{a,b}, Mohamed Ali Latrach^b, Yves Renard^c^a Mathematics Department, College of Science and Arts, AlKamel, University of Jeddah, Saudi Arabia^b University of Tunis El Manar, National Engineering School at Tunis, LAMSIN, B.P. 37, 1002 Tunis-Belvédère, Tunisia^c Université de Lyon, CNRS, INSA-Lyon, Institut Camille Jordan UMR5208, 20 Avenue A. Einstein, F-69621 Villeurbanne, France

ARTICLE INFO

Article history:

Received 30 December 2018

Accepted 28 February 2019

Available online 4 March 2019

2010 MSC:

74K10

74M15

74S05

Keywords:

Frictional self-contact

Gap vector

Tangential sliding velocity

Signorini-Coulomb friction

ABSTRACT

In this article, we studied the modeling of a hyperelastic rod during large displacements in the presence of a frictional self-contact described by the Coulomb's law. Compared to other works the self-contact subject with friction is addressed as a problem in itself. This problem generates nonlinear constraints leading to difficulties in mathematical and numerical analysis. To relax the problem of these constraints, an augmented Lagrangian method was used. As a result, we overcame the implicit nature of the Signorini-Coulomb-self-contact formulas. We presented, on the one hand, a result of the existence of the solution. On the other hand, we have proposed an approach that can be useful for a numerical implementation of this problem.

© 2019 The Authors. Production and hosting by Elsevier B.V. on behalf of King Saud University. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

Contents

1. Introduction	829
2. Geometrically exact rod	829
2.1. Preliminaries	829
2.2. Configurations of Cosserat rods	829
2.3. Balance equations	830
3. Frictional self-contact model	830
3.1. Self-contact problem	830
3.2. Characterization of frictional self-contact	831
3.3. Signorini-Coulomb self-contact problem	831
3.4. Non-uniqueness and non-existence of solutions to quasi-static frictional self-contact problem	831
4. Numerical implementation	832
5. Existence and uniqueness solution of penalized frictional self-contact problem	833
6. Linearization of the virtual work of frictional self-contact forces	834

Peer review under responsibility of King Saud University.

E-mail addresses: mohamedali.latrach@enit.utm.tn (M.A. Latrach), yves.renard@insa-lyon.fr (Y. Renard)<https://doi.org/10.1016/j.jksus.2019.02.017>

1018-3647/© 2019 The Authors. Production and hosting by Elsevier B.V. on behalf of King Saud University.

This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

Nomenclature

l, ε	Initial length and the initial radius	$f_N^{\mu_N}, f_T^{\mu_T}$	Normal and tangential penalized stresses
$\mathbf{r}_0, \mathbf{r}_r, \mathbf{r}$	Displacements of each configuration	\mathbf{D}_r	Gap vector
\mathbf{n}	Internal force	\mathbf{d}_r	Gap function
\mathbf{m}	Internal moment	\mathbf{g}_r	Sliding velocity vector
\mathbf{N}, \mathbf{T}	normal and tangential vectors	γ_T	Tangential friction coefficient
f_N, f_T	Normal and tangential stresses	\mathbf{p}, \mathbf{q}	Admissible displacement and rotation variations
μ_N, μ_T	Normal and tangential penalties parameters		

7. Conclusions. 834
 Acknowledgment. 834
 References 834

1. Introduction

Several systems of mechanical structures reveal situations in which an elastic body undergoes self-contact or come in contact with other bodies. Among the recently studied systems, is the DNA supercoiling in chromatin where the change of configuration can influence their local and global structure (Gilbert and Allan, 2014). Another possible example could be the increased use of the stent in the biomedical field. For example, the use of stent-grafts for endovascular repair can lead to serious complications (Demanget et al., 2012; Demanget et al., 2013), despite the large experimental and computational studies. Another example of studying the physical structure of the collision and contact phenomena of different layers of clothing in a virtual garment of the physically based animation. This type of modelization is important to enrich realism provided by physics based simulation in animation (Jiang et al., 2017). Moreover, the nature of the contact plays a determining role in their behavior. The general theory could be attributed to (Duvaut and Lions, 1972) who gave the variational formulations of contact problems and proved the existence and uniqueness of the solution for bilateral contact problems using Tresca’s friction law. In Nečas et al. (1980), they have proved the existence of unilateral contact problem in the static state using a local Coulomb friction law. This result was extended by Jarušek (1983). For a static problem, where a law of normal conformity describes the contact, we can refer to Oden and Martins (1985) and Klarbring et al. (1989). The unilateral contact problems with non-local Coulomb friction have been studied by Duvaut (1980), Demkowicz and Oden (1982), Oden and Pires (1983), Cocu (1984) and Renard (2006). Besides, we cite the work in Béal and Touzani (2003) about the contact in large displacements. The self-contact problem had not been considered analytically as an autonomous topic in the classic scientific literature, but it is considered as a special case, which must, therefore, be dealt with accordingly. Motivated by the self-contact due to the supercoiling phenomena of DNA, the self-contact caused by the writhing of the clamped rods has been studied by Van der Heijden et al. (2003). In Chamekh et al. (2009), a result of the existence for the frictionless self-contact problem and a numerical approach have been given.

This work presents the great difficulties we face in proving even the existence of solutions to quasi-static frictional self-contact problems of elastic rods. One of our basic goals is the aim is to understand, if possible in physical terms, the meaning of non-uniqueness and non-existence in the quasi-static case. The frictional Coulomb’s law does not admit of potential, and even for Tresca’s law does not admit of potential in the case of large displacements. For this reason, we cannot prove the uniqueness and

the existence of solution by the minimization of the total energy method or even if we rely on the balance equations, because we fall into a coupling between displacement and rotation, and which leads to a strong non-linearity.

We propose to extend these results to a particular result of existence whose the frictional self-contact force is known, fixed and proposed as a penalized force to ensure the uniqueness result. That is, solving the non-linear problem in the Uzawa algorithm before updating the Lagrange multipliers. This force is proposed by regularized normal and tangential vectors, this regularization is applied to the contact pressure to extend the scope of the results as well.

2. Geometrically exact rod

2.1. Preliminaries

We consider a direct orthonormal fixed frame $\mathcal{R}_0 = \{\mathbf{O}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the Euclidean space \mathbb{R}^3 . We denote by $\|\cdot\|$ the Euclidean norm on \mathbb{R}^3 . For a vector $\mathbf{a} \in \mathbb{R}^3$, we associate the antisymmetric tensor $\mathbf{a}^\times = skew(\mathbf{a})$, i.e. $\mathbf{a}^\times \mathbf{b} = \mathbf{a} \times \mathbf{b}$ for all vector $\mathbf{b} \in \mathbb{R}^3$. We denote by $axial(\mathbf{A})$ the axial vector associated with the antisymmetric tensor, that is to say, such that $(axial(\mathbf{A}))^\times = \mathbf{A}$. The projection of a vector $\mathbf{a} \in \mathbb{R}^3$ onto a ball centered at the origin and with a radius v is defined as:

$$P_{B(v)}[\mathbf{a}] = \begin{cases} \mathbf{a} & \text{if } \|\mathbf{a}\| < v, \\ v \frac{\mathbf{a}}{\|\mathbf{a}\|} & \text{otherwise.} \end{cases} \quad (1)$$

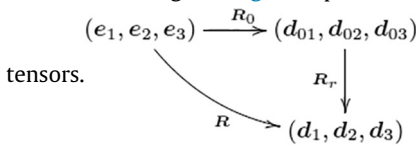
Throughout this report, for any function G differentiation with respect to the time and to the curvilinear abscissa are denoted with a dot ($\dot{G} = \frac{\partial G}{\partial t}$) and prime ($G' = \frac{\partial G}{\partial s}$), respectively.

2.2. Configurations of Cosserat rods

Compared to the work of Mlika et al. (2017) that has treated the frictional self-contact problem by a theory based on Nitsche’s method, we have used the geometrically exact theory (GET) which plays an important role in much recent mechanical research. This theory is based on the Cosserat’s rod (Cosserat and Cosserat, 1909), governed by the equations developed in Reissner (1981) and Simo and Vu-Quoc (1988). The GET has many benefits, has of the advantage to parametrize the 3D configuration of the rod by a 1D parametrization and to take into account the torsion, bending, elongation and shear. The GET could be based on Simo-Reissner theory that we have mentioned above, or also on the Kirchhoff-Love theory (Meier et al., 2016). The studied rod is

initially defined by a right centerline (untwisted) fixed on the spatial basis $(\mathbf{e}_i)_i = \mathbf{R}_0$, the reference configuration that is also called unstressed or stress-free configuration. However, it can be initially twisted and/or curved on the basis $(\mathbf{d}_{0i})_i = \mathbf{R}_r$. Under the action of external efforts, the rod is deformed and moves from the reference configuration to the deformed configuration defined by the deformed centerline on the material basis $(\mathbf{d}_i)_i = \mathbf{R}$. In this model, the sections are initially plane and remain plane with an invariant radius ε . The deformed configuration is then described by the position of each $s \in [0, l]$ (with l is the initial length of the rod) which moves by a partial translational displacement $\mathbf{r}_r \in \mathbb{R}^3$, i.e. $\mathbf{r}(s) = \mathbf{r}_0(s) + \mathbf{r}_r(s)$, and by translational displacement \mathbf{r}_r , and turns with the cross-section and therefore with the local frame $(\mathbf{r}_{0i}; \mathbf{d}_{0i})_i$ by the orthogonal tensor $\mathbf{R}_r = \mathbf{d}_i \otimes \mathbf{d}_{0i}$.

Consequently, the transformation $\mathbf{R} = \mathbf{R}_r \mathbf{R}_0$ is the orthogonal tensor which determines the overall orientation of the cross-section. The diagram Fig. 1 explains the relation between these



Thus, the space of all possible configurations of the rod satisfying a given boundary conditions \mathbf{BC} is

$$\mathcal{C} = \{(\mathbf{r}, \mathbf{R}) \in H^1([0, l]; \mathbb{R}^3 \times SO(3)), \mathbf{BC}\}. \tag{2}$$

2.3. Balance equations

In what follows, we assume that the transformation is very slow so that we can neglect the forces of inertia. The time parameter must not appear even in the equilibrium equations. Nevertheless, we must use an Euler scheme to discretize the tangential velocity field in time (Mlika et al., 2017). The balance equations are the equations, which describe the variation of the internal force $\mathbf{n}(s)$ and the internal moment $\mathbf{m}(s)$ on the centerline of the rod. We obtain the local form of the balance equations under the action of external distributed force \mathbf{f} and torque \mathbf{c}_i for the rod writes

$$\mathbf{n}'(s) + \mathbf{f}(s) + \mathbf{f}_c(s) = \mathbf{0}, \tag{3a}$$

$$\mathbf{m}'(s) + \mathbf{r}'(s) \times \mathbf{n}(s) + \mathbf{d}_i(s) \times \mathbf{c}_i(s) = \mathbf{0}, \tag{3b}$$

with \mathbf{f}_c is contribution force of frictional self-contact.

Concerning constitutive laws, we suppose that exists an elastic energy density $\mathcal{W}(s, \mathbf{u}, \mathbf{v})$ obeys the convexity and coercivity hypotheses with respect to its last two arguments (Antman, 2004) and which verify the following relationships

$$\mathbf{n}(s) = \mathbf{R} \frac{\partial \mathcal{W}}{\partial \mathbf{v}}(s, \mathbf{u}, \mathbf{v}), \quad \mathbf{m}(s) = \mathbf{R} \frac{\partial \mathcal{W}}{\partial \mathbf{u}}(s, \mathbf{u}, \mathbf{v}), \tag{4}$$

where $\mathbf{u} = \mathbf{R}^T \mathbf{u}$ and $\mathbf{v} = \mathbf{R}^T \mathbf{v}$.

3. Frictional self-contact model

3.1. Self-contact problem

Generally, the self-contact do not be frictionless realized unless we neglect the tangential and rotational stress on the interface. We consider for all $s \in [0, l]$ of the centerline of the rod the subset \mathcal{I}_s of $[0, l]$ of the points likely to come into contact with s . Physically, the subset \mathcal{I}_s defined by the points where the self-contact can take place and avoid very close points that cannot come into contact with s (Chamekh et al., 2009).

We define the normal and the tangent vectors to the curve of the centerline of the rod on the point $\mathbf{r}(s)$ which describe the deformation of the rod by (see Fig. 3)

$$\mathbf{N}(s, \bar{s}) = \frac{\mathbf{r}(s) - \mathbf{r}(\bar{s})}{\|\mathbf{r}(s) - \mathbf{r}(\bar{s})\|}, \tag{5a}$$

$$\mathbf{T}(s, \bar{s}) = \frac{\mathbf{r}'(\bar{s})}{\|\mathbf{r}'(\bar{s})\|}, \tag{5b}$$

where \bar{s} is the closest point to s , i.e. $\bar{s} = \operatorname{argmin}_{\sigma \in \mathcal{I}_s} \|\mathbf{r}(s) - \mathbf{r}(\sigma)\|$. The gap vector, associated with the slave point s is given by

$$\mathbf{D}_r(s, \bar{s}) = \mathbf{r}(s) - \mathbf{r}(\bar{s}) - 2\varepsilon \mathbf{N}(s, \bar{s}). \tag{6}$$

Whereas, the self-penetration function is given by

$$d_r(s, \bar{s}) = -(\mathbf{D}_r(s, \bar{s}) \cdot \mathbf{N}(s, \bar{s})). \tag{7}$$

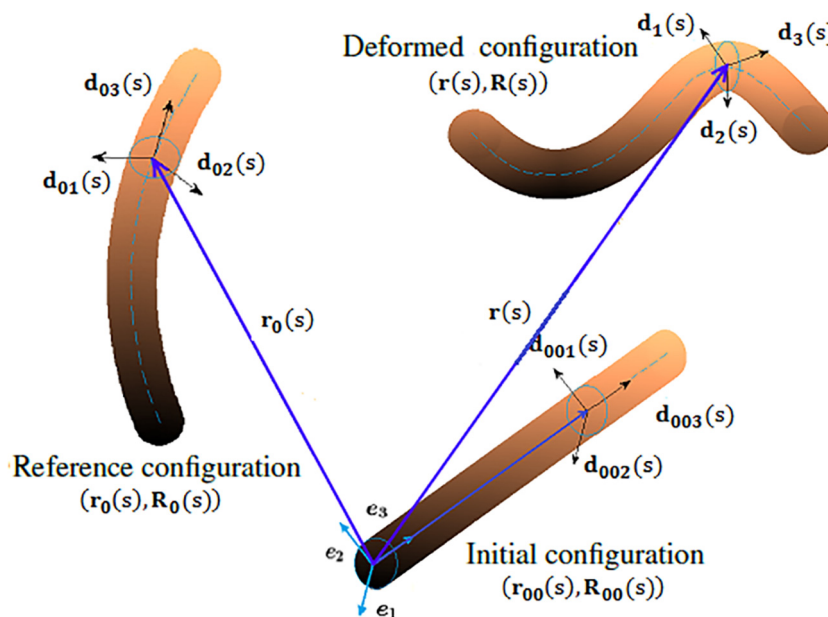


Fig. 1. The different configurations of a Cosserat elastic rod.

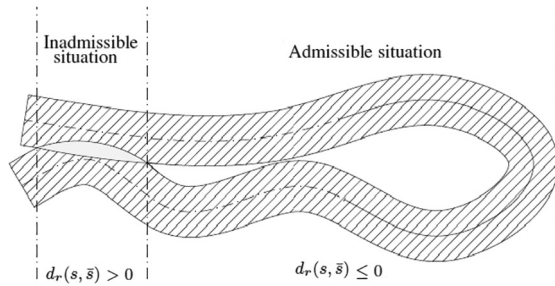


Fig. 2. Admissible and inadmissible situations of elastic rods.

Consequently, this constraint of the non-self-penetration of material leads to the following space of kinematically and physically admissible configurations (see Fig. 2)

$$C_c = \{(\mathbf{r}, \mathbf{R}) \in C, \text{ such that } d_r(s, \bar{s}) \leq 0, \quad \forall s \in [0, l]\}. \quad (8)$$

The minimization problem for get \bar{s} the closest point to s can lead to find the zeros of its gradient, which gives the following orthogonal-ity condition

$$\mathbf{r}'(\bar{s}) \cdot (\mathbf{r}(s) - \mathbf{r}(\bar{s})) = 0. \quad (9)$$

Then, the search for a point of contact consists of resolving, for all $s \in [0, l]$, a scalar orthogonal projection problem.

3.2. Characterization of frictional self-contact

The contact force applied between two points, which are found, respectively, on the cross-sections boundaries $A(\mathbf{r}(s))$ and $A(\mathbf{r}(\bar{s}))$ so that we can continue the contact pressure (normal force) and the frictional efforts (tangential and rotational frictional forces) without falling in the self-penetration problem. If we neglect the rotational friction, the contact force must be defined by (see Fig. 3)

$$\mathbf{f}_c(s) = f_N(s)\mathbf{N}(s, \bar{s}) + \mathbf{f}_T(s).$$

The tangential sliding measurement must be carried out on the plane which is tangent to the cross-section and orthogonal to the plane (\mathbf{N}, \mathbf{T}) . If the self-contact during the sliding is assumed per-sistent, i.e. $d_r(s, \bar{s}) = 0$ and $\dot{d}_r(s, \bar{s}) = 0$. Then, we easy obtain $\dot{\mathbf{D}}_r(s, \bar{s}) = 0$. And we have

$$\mathbf{g}_r(s, \bar{s}) = \dot{\mathbf{r}}(s) - \dot{\mathbf{r}}(\bar{s}) + 2\epsilon\dot{\mathbf{N}}(s, \bar{s}). \quad (10)$$

If $\mathbf{g}_r(s, \bar{s}) = \mathbf{0}$ then the curve is a tangentially self-sticking; else then the curve is a tangentially self-sliding.

3.3. Signorini-Coulomb self-contact problem

Tresca's law has been added to contact problems when normal compressions are important. This leads to a quadratic functional minimization problem. Following Kikuchi and Oden (1988), this is based on a fixed slip potential, assumed to be known, and which does not depend on the normal stress. However, it is very sensitive and can lead to complex instability during large movements. Coulomb's law generalizes the Tresca friction phenomenon and is able to analyze this friction of elastic rods even in case of significant displacement. He describes this threshold dependence with the intensity of normal forces. Unfortunately, also the Coulomb's law does not admit of potential since its friction potential depends on normal compression.

The Kuhn-Tucker and the Signorini-Coulomb frictional conditions as following:

$$d_r(s, \bar{s}) \leq 0, \quad (10a)$$

$$f_N(s) \geq 0, \quad (10b)$$

$$d_r(s, \bar{s})f_N(s) = 0, \quad (10c)$$

$$\|\mathbf{f}_T(s)\| < \gamma_T f_N(s) \quad \text{if } \mathbf{g}_r(s, \bar{s}) = \mathbf{0}, \quad (10d)$$

$$\mathbf{f}_T(s) = \gamma_T f_N(s) \frac{\mathbf{g}_r(s, \bar{s})}{\|\mathbf{g}_r(s, \bar{s})\|} \quad \text{otherwise,} \quad (10e)$$

with γ_T is a tangential friction coefficient. The condition of the per-sistence of self-contact is expressed by the equality, $\dot{d}_r(s, \bar{s})f_N(s) = 0$. This condition is particularly used to develop kinematics of friction.

3.4. Non-uniqueness and non-existence of solutions to quasi-static frictional self-contact problem

The frictional self-contact problems as a particular case of frictional contact problems have a practical as well as a theoretical importance, a large number of algorithms for the numerical solution of the related finite element equations and inequalities as a minimization problem of potential energy have been presented in the literature. But theoretically, the quasi-static frictional self-contact problems are known to exhibit 'unexpected' features such as non-existence and non-uniqueness of solutions.

The typical example for the elastic energy density $\mathcal{W}(\cdot, \mathbf{u}, \mathbf{v}) = \frac{1}{2} \{ \mathbf{v} \cdot \mathbf{B}\mathbf{v} + \mathbf{u} \cdot \mathbf{A}\mathbf{u} \}$, where $\mathbf{B}(s)$ and $\mathbf{A}(s)$ are two coercive, symmetric and positive-definite matrices in $\mathbf{M}_3(\mathbb{R})$. Such an energy functional involves shear, tension, flexure, and torsion. The

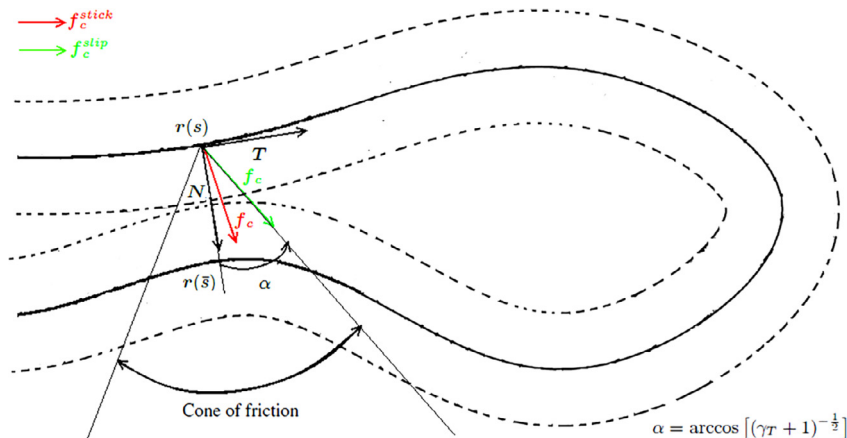


Fig. 3. Domains of the Coulomb's law.

constraints imposed in \mathcal{C} are contained so that if \mathbf{f} and \mathbf{c}_i are, respectively, independent of \mathbf{r} and \mathbf{R} . By the kinematic equations and the Darboux vectors derivative (Chamekh et al., 2009), we are substituting the constitutive laws (4) in the balance equations (3) for obtaining the following weak form of the problem

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{r}, \mathbf{R}) \in \mathcal{C}_c \text{ such that} \\ \int_0^l \mathbf{RBR}^T \mathbf{r}' \cdot \mathbf{p}' ds + \int_0^l \mathbf{f} \cdot \mathbf{p} ds + \int_0^l \mathbf{f}_c \cdot \mathbf{p} ds = \mathbf{0} \\ \int_0^l \mathbf{RAR}^T \mathbf{d}_i' \cdot (\mathbf{q}^* \mathbf{d}_i)' ds + \int_0^l (\mathbf{r}' \times \mathbf{RBR}^T \mathbf{r}') \cdot (\mathbf{q}^* \mathbf{d}_i) ds \\ + \int_0^l \mathbf{c}_i \cdot (\mathbf{q}^* \mathbf{d}_i) ds = \mathbf{0}, \quad \forall (\mathbf{p}, \mathbf{q}) \text{ admissible variation,} \end{array} \right. \quad (11)$$

where \mathbf{p} and \mathbf{q} , respectively, are an infinitesimal displacement and an infinitesimal rotation on the cross-sections $A(\cdot)$, i.e. $\delta \mathbf{R} = \mathbf{q} \times \mathbf{R}$.

To study the existence and uniqueness of the solution for the Signorini-Coulomb contact problems, it would be pointless to think of using classical variational methods, which may also partly explain why fixed-point theorems are widely used by many authors as in Nečas et al. (1980) and Jarušek (1983). The inapplicable fixed-point theorems for problem (11) because of the strong nonlinearity as a coupling between \mathbf{r} and \mathbf{R} at the level of each equation, so that we could not prove that this problem even admits a solution. Note that if we delete the coupling between \mathbf{r} and \mathbf{R} for example by delete the rotation \mathbf{R} in internal force $\mathbf{n}(s)$ and moment $\mathbf{m}(s)$ given by (4) (which is physically not allowed actually), we obtain a purely mathematical linear problem who admits a solution demonstrable by the fixed-point theorem of Schauder for a penalized frictional self-contact force proposed and defined by

$$\mathbf{f}_c^\mu(s) = \frac{1}{\mu} \int_{\mathcal{I}_s} [d_r(s, \bar{s})]_+ (\mathbf{N}^e(s, \bar{s}) + \gamma_T \mathbf{T}^\mu(s, \bar{s})) d\bar{s},$$

$$[x]_+ = \frac{1}{2} (x + |x|), \quad (12)$$

with the following regularizations

$$\mathbf{N}^e(s, \bar{s}) = \begin{cases} \frac{\mathbf{r}(s) - \mathbf{r}(\bar{s})}{2\varepsilon} & \text{if } \|\mathbf{r}(s) - \mathbf{r}(\bar{s})\| \leq 2\varepsilon, \\ \mathbf{N}(s, \bar{s}) & \text{otherwise,} \end{cases} \quad (13a)$$

$$\mathbf{T}^\mu(s, \bar{s}) = \begin{cases} \frac{\mathbf{g}_r(s, \bar{s})}{\mu} & \text{if } \|\mathbf{g}_r(s, \bar{s})\| \leq \mu, \\ \frac{\mathbf{g}_r(s, \bar{s})}{\|\mathbf{g}_r(s, \bar{s})\|} & \text{otherwise.} \end{cases} \quad (13b)$$

After that, we can prove the convergence of the penalized problem where their solution is in the space of all possible configurations \mathcal{C} because the penalty technique authorizes a small penetration. But, the solution of the non-penalized problem must be in the space of kinematically and physically admissible configurations \mathcal{C}_c . This penetration cannot be entirely so that cannot exceed or reach the center line of the rod, this technique is localized by the penalization parameter μ .

4. Numerical implementation

In the literature, the penalty and the augmented Lagrangian methods are the methods most used numerically, and which are able to respect the contact and friction conditions even on a weak form. In this paper, we are interested in these methods which are theoretically detailed in Kikuchi and Oden (1988). The penalty method authorize, firstly, a negligible self-penetration and secondly a lightweight sliding without respect the friction conditions, these permissions depend on the normal and tangential penalty parameters (Fig. 4). The penalty and the augmented Lagrangian methods were used in Wriggers and Zavarise (1997); Zavarise and Wriggers, 2000 for frictionless contact between two beams with circular sections, and recently used in Chamekh et al. (2014) and Chamekh (2015) for the numerical treatment of

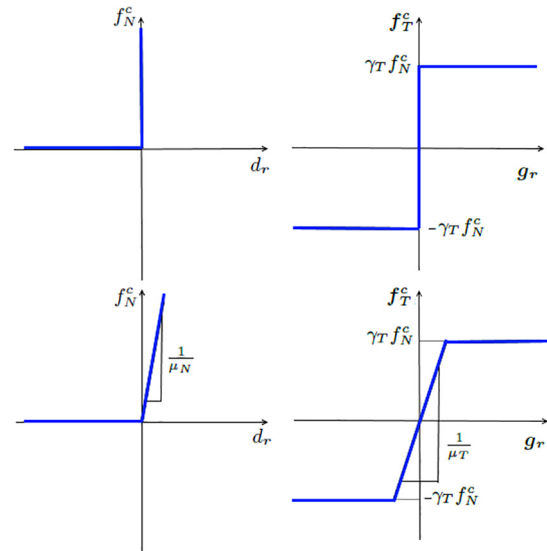


Fig. 4. Schematic of the frictional self-contact regularizations: as the penalty parameters go to zero, the graphs (at the bottom) represent by the method of penalization coincide with the graphs (at the top) represent by the Kuhn-Tucker and friction conditions.

frictionless self-contact of a closed rod with circular sections. We will extend this study to the frictional self-contact case.

The principle of virtual work for the augmented Lagrangian method, it is a necessary condition for minimizing the total energy in the space of kinematically and physically admissible configurations \mathcal{C} given by:

$$\mathcal{G}(\mathbf{r}, \mathbf{R}; \mathbf{p}, \mathbf{q}) = \delta \mathcal{J}(\mathbf{r}, \mathbf{R}; \mathbf{p}, \mathbf{q}) + \int_0^l (f_N^{\mu_N}(s) \mathbf{N}(s, \bar{s}) + \mathbf{f}_T^{\mu_T}(s)) \cdot (\mathbf{p}(\bar{s}) - \mathbf{p}(s)) ds = \mathbf{0}, \quad (14)$$

where $\delta \mathcal{J}(\mathbf{r}, \mathbf{R}; \mathbf{p}, \mathbf{q})$ is the virtual work of the internal and external forces and moments. The integral part in (14) is the virtual work of frictional self-contact forces given by

$$f_N^{\mu_N}(s) = \frac{1}{\mu_N} [d_r(s, \bar{s}) + \mu_N \lambda_N^n(s)]_+, \quad (15)$$

$$\mathbf{f}_T^{\mu_T}(s) = \frac{1}{\mu_T} P_{B_T} [\mathbf{g}_r(s, \bar{s}) + \mu_T \lambda_T^n(s)], \quad (16)$$

where μ_N and μ_T are the normal and tangential penalties parameters and B_T is a ball centered at the origin and with a radius $\mu_T \gamma_T f_N^{\mu_N}$. We define that $B_T^n(s) = B(\mu_T \gamma_T \lambda_T^n(s))$, the iterative approaches $\lambda_N^n(s)$ and $\lambda_T^n(s)$ are updated according to the following iterations:

$$\lambda_N^{n+1}(s) = \frac{1}{\mu_N} [d_r(s, \bar{s}) + \mu_N \lambda_N^n(s)]_+$$

$$\lambda_T^{n+1}(s) = \frac{1}{\mu_T} P_{B_T^{n+1}} [\mathbf{g}_r(s, \bar{s}) + \mu_T \lambda_T^n(s)].$$

Then, the penalized problem that we propose to solve writes

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{r}, \mathbf{R}) \in \mathcal{C} \text{ such that} \\ \mathcal{G}(\mathbf{r}, \mathbf{R}; \mathbf{p}, \mathbf{q}) = 0, \quad \forall (\mathbf{p}, \mathbf{q}) \text{ admissible variation.} \end{array} \right. \quad (17)$$

Note that the penalty method corresponds to the choice $\lambda_N^n \mathbf{N} = \lambda_T^n = \mathbf{0}$ in (15). In the next proposition, the augmented Lagrangian method reassures us about the Kuhn-Tucker and friction conditions:

Proposition 4.1. The self-contact and friction conditions (10) are equivalent to:

$$f_N(s) = \frac{1}{\mu_N} [d_r(s, \bar{s}) + \mu_N f_N(s)]_+ \tag{18a}$$

$$f_T(s) = \frac{1}{\mu_T} P_{B_T} [\mathbf{g}_r(s, \bar{s}) + \mu_T \mathbf{f}_T(s)] \tag{18b}$$

where $B_T(s) := B(\mu_T \gamma_T f_N(s))$.

Proof. Let (\mathbf{r}, \mathbf{R}) be a regular configuration on \mathcal{C} such that $\mathbf{f}_c(s)$, $d_r(s, \bar{s})$ and $\mathbf{g}_r(s, \bar{s})$ verify conditions (10). First, of the condition (10b) we have $f_N(s) > 0$ or $f_N(s) = 0$, suppose that $f_N(s) = 0$, the condition (10a) gives that

$$\frac{1}{\mu_N} [d_r(s, \bar{s}) + \mu_N f_N(s)]_+ = \frac{1}{\mu_N} [d_r(s, \bar{s})]_+ = 0 = f_N(s).$$

Now, suppose that $f_N(s) > 0$, the condition (10c) implies that $d_r(s, \bar{s}) = 0$. then

$$\frac{1}{\mu_N} [d_r(s, \bar{s}) + \mu_N f_N(s)]_+ = \frac{1}{\mu_N} [\mu_N f_N(s)]_+ = f_N(s),$$

so that (18a) holds. Secondly, from the friction conditions in (10) consider the case $\mathbf{g}_r(s, \bar{s}) = \mathbf{0}$, according to the condition (10d) we have $\|\mu_T \mathbf{f}_T(s)\| < \mu_T \gamma_T f_N(s)$, then

$$\frac{1}{\mu_T} P_{B_T} [\mathbf{g}_r(s, \bar{s}) + \mu_T \mathbf{f}_T(s)] = \frac{1}{\mu_T} P_{B_T} [\mu_T \mathbf{f}_T(s)] = \mathbf{f}_T(s).$$

In the case $\mathbf{g}_r(s, \bar{s}) \neq \mathbf{0}$ we have $\mathbf{f}_T(s) = \gamma_T f_N(s) \frac{\mathbf{g}_r(s, \bar{s})}{\|\mathbf{g}_r(s, \bar{s})\|}$, then

$$\begin{aligned} \|\mathbf{g}_r(s, \bar{s}) + \mu_T \mathbf{f}_T(s)\| &= \|\mathbf{g}_r(s, \bar{s})\| + \mu_T \gamma_T f_N(s) \\ &\geq \mu_T \gamma_T f_N(s). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{\mu_T} P_{B_T} [\mathbf{g}_r(s, \bar{s}) + \mu_T \mathbf{f}_T(s)] &= \gamma_T f_N(s) \frac{\mathbf{g}_r(s, \bar{s}) + \mu_T \mathbf{f}_T(s)}{\|\mathbf{g}_r(s, \bar{s}) + \mu_T \mathbf{f}_T(s)\|} \\ &= \gamma_T f_N(s) \frac{\mathbf{g}_r(s, \bar{s})}{\|\mathbf{g}_r(s, \bar{s})\|} = \mathbf{f}_T(s), \end{aligned}$$

so that (18b) holds. Conversely, let $(\mathbf{r}, \mathbf{R}) \in \mathcal{C}$ such that (18) holds. From (18a) we obtain $f_N(s) \geq 0$. Consider the first case $f_N(s) = 0$, then $f_N(s) d_r(s, \bar{s}) = 0$ and $\frac{1}{\mu_N} [d_r(s, \bar{s})]_+ = 0$ which implies $d_r(s, \bar{s}) \leq 0$. In the case $f_N(s) > 0$, we have

$$\begin{aligned} \mu_N f_N(s) &= [d_r(s, \bar{s}) + \mu_N f_N(s)]_+ \\ &= d_r(s, \bar{s}) + \mu_N f_N(s), \end{aligned}$$

which implies that $d_r(s, \bar{s}) = 0$ and $f_N(s) d_r(s, \bar{s}) = 0$. So that the Kuhn-Tucker conditions in (10) holds. Now, from (18b) suppose that $\mathbf{g}_r(s, \bar{s}) = \mathbf{0}$, then $\mathbf{f}_T(s) = \frac{1}{\mu_T} P_{B_T} [\mu_T \mathbf{f}_T(s)]$ which implies that $\|\mathbf{f}_T(s)\| < \gamma_T f_N(s)$ thereby (10d) holds. In the case $\mathbf{g}_r(s, \bar{s}) \neq \mathbf{0}$ we obtain $\|\mathbf{g}_r(s, \bar{s}) + \mu_T \mathbf{f}_T(s)\| \geq \mu_T \gamma_T f_N(s)$ because if $\|\mathbf{g}_r(s, \bar{s}) + \mu_T \mathbf{f}_T(s)\| < \mu_T \gamma_T f_N(s)$ then $\mathbf{g}_r(s, \bar{s}) = \mathbf{0}$ which contradicts the hypothesis. Then, the relationship (18b) can now be written:

$$\mathbf{f}_T(s) = \gamma_T f_N(s) \frac{\mathbf{g}_r(s, \bar{s}) + \mu_T \mathbf{f}_T(s)}{\|\mathbf{g}_r(s, \bar{s}) + \mu_T \mathbf{f}_T(s)\|}$$

Therefore

$$\mathbf{f}_T(s) = \frac{\gamma_T f_N(s)}{\|\mathbf{g}_r(s, \bar{s}) + \mu_T \mathbf{f}_T(s)\|} \mathbf{g}_r(s, \bar{s}). \tag{19}$$

Since $P_{B_T} [\mathbf{g}_r(s, \bar{s}) + \mu_T \mathbf{f}_T(s)] \in \partial B_T$ and from the relationship (18b) we obtain $\|\mathbf{f}_T(s)\| = \gamma_T f_N(s)$ thereby from (19)

$$\|\mathbf{g}_r(s, \bar{s})\| = \|\mathbf{g}_r(s, \bar{s}) + \mu_T \mathbf{f}_T(s)\| - \mu_T \gamma_T f_N(s).$$

Then

$$\mathbf{f}_T(s) = \gamma_T f_N(s) \frac{\mathbf{g}_r(s, \bar{s})}{\|\mathbf{g}_r(s, \bar{s})\|}$$

so that (10e) holds. \square

5. Existence and uniqueness solution of penalized frictional self-contact problem

Here we adopt the augmented Lagrangian formulation described above and the classical variational formulation of the self-contact well detailed in Chamekh et al. (2009). More precisely, the problem of finding the approximated equilibrium configurations leads to the following nonlinear approximated minimization problem

$$\begin{cases} \text{Find } (\mathbf{r}_\mu^n, \mathbf{R}_\mu^n) \in \mathcal{C} \text{ such that} \\ \mathcal{J}_\mu^n(\mathbf{r}_\mu^n, \mathbf{R}_\mu^n) \leq \mathcal{J}_\mu^n(\mathbf{q}, \mathbf{Q}) \quad \forall (\mathbf{q}, \mathbf{Q}) \in \mathcal{C}. \end{cases} \tag{20}$$

where

$$\mathcal{J}_\mu^n(\mathbf{r}, \mathbf{R}) = \int_0^l [\mathcal{W}(\cdot, \mathbf{v}, \mathbf{u}) - \mathbf{f} \cdot \mathbf{r} - \mathbf{f}_c^{\mu, n} \cdot \mathbf{r}] ds,$$

is the energy expression for an elastic rod in balance, and the penalized frictional self-contact force $\mathbf{f}_c^{\mu, n}$ given by

$$\mathbf{f}_c^{\mu, n}(s) = \lambda_N^n(s) \mathbf{N}(s, \bar{s}) + \lambda_T^n(s), \quad \mu_N = \mu_T = \mu. \tag{21}$$

Here, the Lagrange multipliers λ_N^n and λ_T^n given by (15) have been used with regularization similar at (13) to ensure that $\mathbf{f}_c^{\mu, n}$ is in $L^2([0, l]; \mathbb{R}^3)$.

For the sake of simplicity we restrict ourselves to the cases where $\int_0^l \mathbf{c}_i \cdot \mathbf{d}_i ds = 0$, and we recall that $\mathcal{W}(\cdot, \mathbf{u}, \mathbf{v}) = \frac{1}{2} \{\mathbf{v} \cdot \mathbf{B}\mathbf{v} + \mathbf{u} \cdot \mathbf{A}\mathbf{u}\}$.

Theorem 5.1. If the external force $\mathbf{f} \in L^2([0, l]; \mathbb{R}^3)$, then for each $n \in \mathbb{N}$ and $\mu \in]0, 1[$, Problem (20) admits at least one solution.

Proof. A big party of this theorem is proved in Chamekh et al. (2009) for the frictionless self-contact problem based on a generalized Weierstrass minimization theorem (Kikuchi and Oden, 1988 p. 382). The first part of the energy functional is weakly lower semicontinuous and that it is coercive and the weak closure of the set \mathcal{C} is proved in Bourgat et al. (1988). It remains to prove that the penalized second part of the energy functional is weakly lower semicontinuous and that the total energy functional remains coercive after adding the nonlinear terms associated the self-contact. The frictional self-contact force $\mathbf{f}_c^{\mu, n}$ is in $L^2([0, l]; \mathbb{R}^3)$. Then, the mapping

$$\mathbf{r} \in H^1((0, l), \mathbb{R}^3) \mapsto \int_0^l \mathbf{f}_c^{\mu, n}(s) \cdot \mathbf{r}(s) ds \in \mathbb{R}_+,$$

is convex and weakly lower semicontinuous. Furthermore in Chamekh et al. (2009), it is proved that, under certain conditions we assume that they are supported here, there exist constant c_1 and c_2 such that

$$\mathcal{J}_0(\mathbf{r}, \mathbf{R}) \geq c_1 \|\mathbf{r}\|_{1,2}^2 + \frac{c_2}{2} (\|\mathbf{R}\|_{1,2}^2 - l) - \|\mathbf{f}\|_{2,2} \|\mathbf{r}\|_{1,2},$$

where \mathcal{J}_0 is the first part of the energy functional, and $\|\cdot\|_{1,2}$ denotes the usual norm in $H^1([0, l]; \mathbb{R}^3)$ if the argument is in \mathbb{R}^3 , or $H^1([0, l]; M_3(\mathbb{R}))$ if the argument is in $M_3(\mathbb{R})$. Since $\mathbf{f}_c^{\mu, n} \in L^2([0, l]; \mathbb{R}^3)$ we have

$$\mathcal{J}_\mu^n(\mathbf{r}, \mathbf{R}) \geq c_1 \|\mathbf{r}\|_{1,2}^2 + \frac{c_2}{2} (\|\mathbf{R}\|_{1,2}^2 - l) - \|\mathbf{f}\|_{2,2} \|\mathbf{r}\|_{1,2} - \|\mathbf{f}_c^{\mu, n}\|_{2,2} \|\mathbf{r}\|_{1,2}.$$

Therefore the functional \mathcal{J}_μ^n remains coercive which ends the proof. \square

6. Linearization of the virtual work of frictional self-contact forces

The integral part in (14) is the virtual work of the problem (17) that is nonlinear, to solve it, we have used the Newton-Raphson method. This method consists in solving the next linearized problem for (η, θ) at the iteration k with an estimate $(\mathbf{r}^k, \mathbf{R}^k)$.

The frictional self-contact forces rewrites

$$\mathcal{G}_c(\mathbf{r}; \mathbf{p}; \eta) = \int_0^l (f_N^{\mu_N}(s) \mathbf{N}(s, \bar{s}) + \mathbf{f}_T^{\mu_T}(s)) \cdot (\mathbf{p}(\bar{s}) - \mathbf{p}(s)) ds. \tag{22}$$

It follows from that

$$\Delta \mathcal{G}_c(\mathbf{r}; \mathbf{p}; \eta) = \int_0^l [-(\Delta f_N^{\mu_N}(s) \mathbf{N}(s, \bar{s}) + f_N^{\mu_N}(s) \Delta \mathbf{N}(s, \bar{s}) + \Delta \mathbf{f}_T^{\mu_T}(s)) \cdot \bar{\mathbf{p}}(s) + \Delta \bar{s} (f_N^{\mu_N}(s) \mathbf{N}(s, \bar{s}) + \mathbf{f}_T^{\mu_T}(s)) \cdot \mathbf{p}'(\bar{s})] ds \tag{23}$$

where $\bar{h}(s) = h(s) - h(\bar{s})$, for any vector-valued function $h(\cdot)$.

In the following, it is necessary to calculate the directional derivatives $\Delta f_N^{\mu_N}$, $\Delta \mathbf{f}_T^{\mu_T}$, $\Delta \bar{s}$ and $\Delta \mathbf{N}$. For the directional derivative of $\mathbf{f}_T^{\mu_T}$ is given by (see Chamekh et al., 2009):

$$\Delta \mathbf{f}_T^{\mu_T}(s) = \frac{1}{\mu_T} F(\mathbf{g}_r(s, \bar{s}) + \mu_T \lambda_T^n(s)) \Delta \mathbf{g}_r(s, \bar{s}).$$

where the function F is defined by:

$$F(\mathbf{x}) = \begin{cases} 1 & \text{if } \|\mathbf{x}\| \leq \mu_T \lambda_T^n(s), \\ 0 & \text{otherwise.} \end{cases}$$

Using the Euler approximation of the tangential sliding velocity, we obtain:

$$\Delta \mathbf{g}_r(s, \bar{s}) = \frac{1}{\Delta t} (\bar{\eta}(s) - \Delta \bar{s} \mathbf{r}'(\bar{s}) + 2\varepsilon \Delta \mathbf{N}(s, \bar{s})). \tag{24}$$

Therefore

$$\Delta \mathbf{f}_T^{\mu_T}(s) = \frac{F(\mathbf{g}_r(s, \bar{s}) + \mu_T \lambda_T^n(s))}{\mu_T \Delta t} (\bar{\eta}(s) - \Delta \bar{s} \mathbf{r}'(\bar{s}) + 2\varepsilon \Delta \mathbf{N}(s, \bar{s})). \tag{25}$$

We get

$$\Delta \mathcal{G}_c(\mathbf{r}; \mathbf{p}; \eta) = \int_0^l \left[\frac{H(d_r(s, \bar{s}) + \mu_N \lambda_N^n(s))}{\mu_N (2\varepsilon - d_r(s, \bar{s}))} \mathbf{Q}_N(\mathbf{r}; \mathbf{p}; \eta) + \mathbf{Q}_T(\mathbf{r}; \mathbf{p}; \eta) + \frac{F(\mathbf{g}_r(s, \bar{s}) + \mu_T \lambda_T^n(s))}{\mu_T \Delta t (2\varepsilon - d_r(s, \bar{s}))} \bar{\eta}(s) \cdot (2\varepsilon \mathbf{N}(s, \bar{s}) \otimes \mathbf{N}(s, \bar{s}) - d_r \mathbf{I}) \bar{\mathbf{p}}(s) \right] ds, \tag{26}$$

with \mathbf{Q}_N is described in Chamekh et al. (2009) and

$$\mathbf{Q}_T(\mathbf{r}; \mathbf{p}; \eta) = \Omega(s) \begin{pmatrix} \bar{\eta}(s) \\ \eta'(s) \end{pmatrix} \cdot \begin{pmatrix} \Xi(s) \mathbf{r}'(\bar{s}) \otimes \mathbf{r}'(\bar{s}) & \mathbf{r}'(\bar{s}) \otimes \mathbf{f}_T^{\mu_T}(s) \\ \Xi(s) \bar{\mathbf{r}}(s) \otimes \mathbf{r}'(\bar{s}) & \bar{\mathbf{r}}(s) \otimes \mathbf{f}_T^{\mu_T}(s) \end{pmatrix} \begin{pmatrix} \bar{\mathbf{p}}(s) \\ \mathbf{p}'(\bar{s}) \end{pmatrix}$$

$$\Xi(s) = \frac{F(\mathbf{g}_r(s, \bar{s}) + \mu_T \lambda_T^n(s)) (4\varepsilon - d_r(s, \bar{s}))}{\mu_T \Delta t (2\varepsilon - d_r(s, \bar{s}))}, \text{ and}$$

$$\Omega(s) = \frac{1}{\|\mathbf{r}'(\bar{s})\|^2 - \mathbf{r}''(\bar{s}) \cdot \bar{\mathbf{r}}(s)}.$$

7. Conclusions

The fusion of self-contact and friction induces a strong nonlinear problem and rich modeling, along with the nonlinearity of the modeling of a hyperelastic rod in large displacements in three-dimensional space. Theoretically, the existence and uniqueness of the solution to the problem is a major challenge. We have used the augmented Lagrangian method to address the mathematical study of the Signorini-Coulomb-self-contact problem. We have overcome the significant obstacles encountered in the implicit nature of formulations linked the use of Coulomb's law. In addition, we have developed some formulations for the frictional self-contact problem, which can be, implemented numerically in future research.

Acknowledgment

The authors would like to thank Prof. Maher Moakher for his fruitful discussions during this work.

References

Antman, S.S., 2004. *Nonlinear Problems of Elasticity*. Applied Mathematical Sciences, Springer Verlag, New York.

Bourgat, J.F., Le Tallec, P., Mani, S., 1988. Modélisation et calcul des grands déplacements de tuyaux élastiques en flexion-torsion. *J. Mécanique Théor. Appl.* 7 (4), 379–408.

Béal, P., Touzani, R., 2003. Analysis of contact of elastic rods subject to large displacements. *Appl. Math. Lett.* 16, 619–625.

Chamekh, M., Mani-Aouadi, S., Moakher, M., 2009. Modeling and numerical treatment of elastic rods with frictionless self-contact. *Comput. Methods Appl. Mech. Engrgy* 198 (47/48), 3751–3764.

Chamekh, M., Mani-Aouadi, S., Moakher, M., 2014. Stability of elastic rods with self-contact. *Comput. Methods Appl. Mech. Engrgy* 279, 227–246.

Chamekh, M., 2015. Non-linear stability criterion in numerical approximation of hyperelastic rods. *Universal J. Eng. Mech.* 3, 45–52.

Cocu, M., 1984. Existence of solutions of Signorini problems with friction. *Int. J. Eng. Sci.* 22 (5), 567–575.

Cosserat, E., Cosserat, F., 1909. *Théorie des Corps Déformables*. Hermann, Paris.

Duvaut, G., Lions, J.L., 1972. *Les inéquations en mécanique et physique*. Dunod, Paris.

Duvaut, G., 1980. Equilibre d'un solide élastique avec contact unilatéral et frottement de Coulomb. *C.R. Acad. Sci. Paris Sér. A* 290, 263.

Demkowicz, L., Oden, J.T., 1982. On some existence and uniqueness results in contact problems with nonlocal friction. *Nonlinear Anal. Theory Meth. Appl.* 6, 1075.

Demanget, N., Avril, S., Badel, P., et al., 2012. Computational comparison of the bending behavior of aortic stentgrafts. *J. Mech. Behav. Biomed. Mater.* 5, 272–282.

Demanget, N., Duprey, A., Badel, P., et al., 2013. Finite element analysis of the mechanical performances of marketed aortic stent-grafts. *J. Endovasc. Ther.* 20, 523–535.

Gilbert, N., Allan, J., 2014. Supercoiling in DNA and chromatin. *Curr. Opin. Genet. Develop.* 25, 15–21.

Jarušek, J., 1983. Contact problems with bounded friction coercive case. *Czechoslovak Math. J.* 33 (108), 237–261.

Jiang, C., Gast, T., Teran, J., 2017. Anisotropic elastoplasticity for cloth, knit and hair frictional contact. *ACM Trans. Graph.* 36 (4), 14. Article 152.

Kikuchi, N., Oden, J.T., 1988. *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*. SIAM, Philadelphia.

Klarbring, A., Mikić, A., Shillor, M., 1989. On friction problems with normal compliance. *Nonlinear Anal. Theory Meth. Appl.* 13, 935–955.

Meier, C., Wall, W.A., Popp, A., 2016. Geometrically exact finite element formulations for curved slender beams: Kirchhoff-Love theory vs. Simo-Reissner theory. *Arch. Comput. Methods Eng.*

Mlika, R., Renard, Y., Chouly, F., 2017. An unbiased Nitsche's formulation of large deformation frictional contact and self-contact. *Comput. Methods Appl. Mech. Eng.* 325, Elsevier, pp. 265–288.

Nečas, J., Jarušek, J., Haslinger, J., 1980. On the solution of the variational inequality to Signorini problem with small friction. *Bolletino U.M.I.*, 5 17-B. pp. 796–811.

Oden, J.T., Pires, E., 1983. Contact problems in elastostatics with non-local friction law. *J. Appl. Mech.* 50, 67–76.

Oden, J.T., Martins, A.C., 1985. Models and computational methods for dynamic friction phenomena. *Comput. Methods Appl. Mech. Eng.* 50 (1–3), 527–634.

Reissner, E., 1981. On finite deformations of space curved beams. *J. Appl. Math. Phys.* 32, 734–744.

Renard, Y., 2006. A uniqueness criterion for the Signorini problem with Coulomb friction. *SIAM J. Math. Anal.* 38 (2), 452–467.

- Simo, J.C., Vu-Quoc, L., 1988. On the dynamics in space of rods undergoing large motions – a geometrically exact approach. *Comput. Methods Appl. Mech. Eng.* 66 (2), 125–161.
- Van der Heijden, G.H.M., Neukirch, S., Goss, V.G.A., Thompson, J.M.T., 2003. Instability and self-contact phenomena in the writhing of clamped rods. *Int. J. Mech. Sci.* 45, 161.
- Wriggers, P., Zavarise, G., 1997. On contact between three-dimensional beams undergoing large deflection. *Commun. Numer. Methods Eng.* 13, 429–438.
- Zavarise, G., Wriggers, P., 2000. Contact with friction between beams in 3-d space. *Int. J. Numer. Meth. Eng.* 49, 977–1006.