



ORIGINAL ARTICLE

Steady-state response of constant coefficient discrete-time differential systems



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Abstract The problem of steady state output of the discrete-time fractional differential systems is studied in this paper. Based on the fact that the exponentials are the eigenfunctions of such systems, a general algorithm for the output computation when the input is the product “rising factorial, exponential” is presented. The singular case is studied and solved.

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1. Introduction

The discrete-time differential systems were studied in Ortigueira et al. (2015) where we developed a framework parallel to the classic used in continuous-time systems. Those systems are based on the nabla and delta derivatives (Bohner and Peterson, 2001; Hilger, 1990; Ortigueira et al., 2015). Here we resume the study of those systems by considering the steady state responses to exponentials and products of exponentials by rising factorial functions. We will study both

the regular and singular cases in a way similar to the one followed in Ortigueira (2014b). The algorithm is based on the concept of eigenfunction. As shown in Ortigueira et al. (2015) the eigenfunctions of discrete-time differential systems linear systems are exponentials suitably defined and the corresponding eigenvalues are the transfer functions. Such exponentials are defined with the help of the nabla and delta derivatives and lead to nabla and delta Laplace transforms. We will consider the regular and singular cases; these correspond to the situation of infinite eigenvalue.

The paper outline is as follows. In Section 2 we present the nabla and delta derivatives and the corresponding exponentials. Their properties are listed. In Section 3 we show how to compute the output when the input is an exponential or the product of an exponential by a rising factorial function.

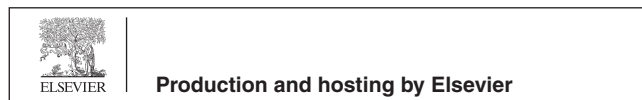
Important remark – The formulation we will present although in a discrete-time setup it mimics the continuous-time counterpart. This leads us to use interchangeably $t = nh$ where h is the underlying time interval.

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2. Fractional nabla and delta derivatives and exponentials

Let $t = nh$ be any generic point in $\mathbf{T} = h\mathbf{Z} = \{kh : k \in \mathbf{Z}\}$. We define the **nabla derivative** (Bohner and Peterson, 2001; Hilger, 1990) by:

$$D_{\nabla}f(t) = f'_{\nabla}(t) := \frac{f(t) - f(t-h)}{h} \quad (1)$$

and the **delta derivative** (Neuman, 1993) by

$$D_{\Delta}f(t) = f'_{\Delta}(t) := \frac{f(t+h) - f(t)}{h} \quad (2)$$

As it can be seen the first one is *causal*, while the second is *anti-causal*. Their generalizations for any real (or complex) order are obtained from the continuous-time Grünwald–Letnikov derivative (Diaz and Osler, 1974; Magin et al., 2011; Ortigueira, 2011; Ortigueira et al., 2015):

$$D_{\nabla}^{(\alpha)}f(t) = f_{\nabla}^{(\alpha)}(t) := \frac{\sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} f(t-nh)}{h^{\alpha}} \quad (3)$$

and

$$D_{\Delta}^{(\alpha)}f(t) = f_{\Delta}^{(\alpha)}(t) := e^{-i\alpha\pi} \frac{\sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} f(t+nh)}{h^{\alpha}} \quad (4)$$

As before (Ortigueira, 2011) we will call these derivatives respectively *forward* and *backward* due to the “time flow”, from past to future or the reverse. This terminology is the reverse of the one used in some mathematical literature. The first is causal while the second is anti-causal.

Attending to the fact that $(-1)^n \binom{\alpha}{n} = \frac{(-\alpha)_n}{n!}$, where $(-\alpha)_n$ is the Pochhammer symbol for the rising factorial $-(a)_k = a(a+1)(a+2)\cdots(a+k-1)^2$; we conclude immediately that these derivatives include as special cases the integer order derivatives and anti-derivatives.

These derivatives enjoy several properties as described in Ortigueira et al. (2015). The eigenfunctions of these derivatives are the **nabla and delta generalized exponentials** defined by Ortigueira et al. (2015):

$$e_{\nabla}(t, s) = [1 - sh]^{-t/h} \quad (5)$$

and

$$e_{\Delta}(t, s) = [1 + sh]^{t/h} \quad (6)$$

The properties of these exponentials are described in Ortigueira et al. (2015).

3. Outputs of differential discrete-time linear systems

3.1. Regular cases

3.1.1. Exponential input

We are going to consider systems with the general format (Magin et al., 2011)

$$\sum_{k=0}^N a_k D^{\alpha_k} y(t) = \sum_{k=0}^M b_k D^{\beta_k} x(t) \quad (7)$$

with $a_N = 1$. The operator D is the nabla derivative defined above. The orders N and M are any positive integers. The α_k and β_k sequences are strictly increasing and positive real numbers.

The discrete-time convolution between two discrete-time functions $f(t)$ and $g(t)$ is given by:

$$f(t) * g(t) = h \sum_{k=-\infty}^{+\infty} f(kh)g(nh - kh) \quad (8)$$

Introduce the discrete delta (impulse) function by:

$$\delta(nh) = D\varepsilon(nh) \quad (9)$$

where $\varepsilon(nh)$ is the discrete-time Heaviside unit step

$$\varepsilon(nh) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \quad (10)$$

Let $g(t)$ be the impulse response of the system defined by (7): $x(t) = \delta(nh)$. The output is the convolution of the input and the impulse response (Ortigueira et al., 2015).

$$y(t) = g(t) * x(t) \quad (11)$$

If $x(t) = e_{\nabla}(nh, s)$ the output is given by:

$$y(t) = e_{\nabla}(nh, s) \left[h \sum_{n=-\infty}^{\infty} g(nh) e_{\Delta}(nh, -s) \right]$$

The summation expression will be called **transfer function** as usually. We write then

$$G(s) = h \sum_{n=-\infty}^{\infty} g(nh) e_{\Delta}(nh, -s) \quad (12)$$

say, the transfer function is the nabla Laplace transform (Ortigueira et al., 2015) of the impulse response. It is important to remark that the nabla Laplace transform uses the delta exponential. There is also the delta Laplace transform (see Ortigueira et al., 2015). With these results we can easily express the transfer function as

$$G(s) = \frac{\sum_{k=0}^M b_k s^{\beta_k}}{\sum_{k=0}^N a_k s^{\alpha_k}} \quad (13)$$

We conclude that:

- The exponentials are the eigenfunctions of the linear systems (7)
- The eigenvalues are the transfer function values.

Putting $s = \frac{1-e^{i\theta}}{h}$ we obtain the usual sinusoidal case. These results exhibit a high degree of coherence with classic results (Ortigueira, 2014a).

Example 1. Let $h = 1$ and consider the differential equation (Ortigueira, 2014a)

$$y'''(t) + y''(t) - 4y'(t) + 2y(t) = x(t)$$

Let $x(n) = 2^{-n}$. This corresponds to $s = -1$. The solution is given by:

$$y(n) = \frac{1}{(-1)^3 + (-1)^2 - 8 + 2} 2^{-n} = \frac{1}{6} 2^{-n}$$

The above result can be generalized.

² We make the convention $(0)_0 = 1$ and $(0)_n = 0$ for any integer n

3.1.2. "Rising factorial.exponential" input

Consider now that the input is the product "rising factorial.exponential" defined by

$$x(nh) = (nh)_{Kh} e_{\nabla}(nh, \beta) = e_{\Delta}(Kh, -\beta) \lim_{s \rightarrow \beta} \frac{d^K}{ds^K} e_{\nabla}(nh, s)$$

where $(nh)_{Kh} = (nh)(nh+h)(nh+2h) \dots (nh+Kh-h)$. We can write

$$y(t) = h \sum_{m=-\infty}^{\infty} g(mh) (nh-mh)_{Kh} e_{\nabla}(nh-mh, \beta)$$

According to [Ortigueira et al. \(2015\)](#) and the theory underlying the Z transform, the summation is uniformly convergent. This means that we can move the derivative operation out from the summation to get:

$$y(t) = e_{\Delta}(Kh, -\beta) \lim_{s \rightarrow \beta} \frac{d^K}{ds^K} G(s) e_{\nabla}(nh, s) \quad (14)$$

Using the usual Leibniz rule for the derivative of the product the particular solution of the differential Eq. (7) when $x(nh) = (nh)_{Kh} e_{\nabla}(nh, \beta)$ is given by:

$$y(nh) = y_0(nh) e_{\nabla}(nh, \beta) \quad (15)$$

with

$$y_0(n) = \sum_{j=0}^K \binom{K}{j} G^{(j)}(\beta) (nh)_{(K-j)h} e_{\Delta}((K-j)h, -\beta) \quad (16)$$

provided that β is not a pole of the transfer function. This result is formally equivalent to the one presented in [Ortigueira \(2014a\)](#).

The general case stated in (7) is very difficult to study due to the problems in computing the poles and zeros. So, we will consider that all the orders are multiple of a given α that we will assume to be real.

$$G(s) = \frac{\sum_{k=0}^M b_k s^{\alpha k}}{\sum_{k=0}^N a_k s^{\alpha k}} \quad (17)$$

The polynomial in the denominator is called *characteristic pseudo-polynomial* ([Ortigueira, 2011](#)).

Example 2. Consider Example 1, but with $x(n) = n2^{-n}$. Using (15) and (16) we obtain $y(n) = 2^{-n} \sum_{j=0}^1 \binom{1}{j} G^{(j)}(\beta) (n)_{K-j} 2^{1-j}$. As $G(-1) = \frac{1}{6}$ and $G'(-1) = \frac{1}{12}$, $y(n) = 2^{-n} \left[\frac{n}{3} - \frac{1}{12} \right]$.

In this kind of systems the above procedure is simple, except in the singular case that we will treat next.

3.2. Singular cases

Let us consider now the problem we have when the *characteristic pseudo-polynomial* in the denominator has an m th order root for $s = \beta$. This means that we can factorize $A(s)$ by putting in evidence the presence of the pole

$$A(s) = (s^{\alpha} - \beta^{\alpha})^m \bar{A}(s)$$

This implies that we can decouple the original equation into two sub-equations

$$\sum_{k=0}^{N-m} \bar{a}_k D^{\alpha k} u(t) = \sum_{k=0}^M b_k D^{\beta k} x(t) \quad (18)$$

and

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \beta^{(m-k)\alpha} D^{k\alpha} y(t) = u(t) \quad (19)$$

The solution, $u(t)$, of (18) is according to (15) and (16) given by

$$u(nh) = u_0(nh) e_{\nabla}(nh, \beta) \quad (20)$$

where

$$u_0(nh) = \sum_{j=0}^K \binom{K}{j} \bar{G}^{(j)}(\beta) e_{\Delta}((K-j)h, -\beta) (nh)_{(K-j)h} \quad (21)$$

and

$$\bar{G}(s) = \frac{B(s)}{A(s)} = \frac{(s^{\alpha} - \beta^{\alpha})^m B(s)}{A(s)} \quad (22)$$

Attending to (20) where $u_0(t)$ can be considered as a "polynomial" with a given degree K , put

$$y(t) = y_0(t) e_{\nabla}(nh, \beta),$$

but now the degree of $y_0(t)$ is $J = K + m$. Using the generalized Leibniz rule ([Ortigueira et al., 2015](#)) for the product

$$D^{\gamma} [f(t)g(t)] = \sum_{i=0}^{\infty} \binom{\gamma}{i} f_{\nabla}^{(i)}(t) g_{\nabla}^{(\gamma-i)}(t - ih)$$

we can write

$$D^{\gamma} [(nh)_{Jh} e_{\nabla}(nh, \beta)] = e_{\nabla}(nh, \beta) \sum_{i=0}^J \binom{\gamma}{i} D^i [(nh)_{Jh}] \beta^{\gamma-i} e_{\Delta}(ih, -\beta)$$

Then the γ order fractional derivative of $y(t)$ is given by

$$D^{\gamma} [y_0(t) e_{\nabla}(nh, \beta)] = e_{\nabla}(nh, \beta) \sum_{i=0}^J \binom{\gamma}{i} D^i y_0(nh) \beta^{\gamma-i} e_{\Delta}(ih, -\beta)$$

where $D[(nh)_{Jh}] = (Jh)(nh)_{(J-1)h}$.

We could express the successive derivatives of the "polynomial", but it is preferable to insert this result into (19) to obtain

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \beta^{(m-k)\alpha} \sum_{i=0}^J \binom{k\alpha}{i} D^i y_0(t) \beta^{k\alpha-i} e_{\Delta}(ih, -\beta) = u_0(t)$$

After some manipulation we get an ordinary integer order differential equation

$$\sum_{i=0}^J \beta^{m\alpha-i} e_{\Delta}(ih, -\beta) \left[\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \binom{k\alpha}{i} \right] y_0^{(i)}(t) = u_0(t)$$

On the other hand,

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \binom{k\alpha}{i} = \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \frac{(-k\alpha)_i}{i!}$$

The Pochhammer symbol $(-k\alpha)_i$ is a polynomial in k with degree equal to i . So, attending to $\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} k^i = 0$ for $i < m$, the above equation can be rewritten as

$$\sum_{i=0}^K \beta^{m\alpha-i-m} e_{\Delta}(ih, -\beta) \gamma_m y_0^{(i+m)}(t) = u_0(t)$$

with $\gamma_m = \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \binom{k\alpha}{i+m}$. Introduce a new function $v(t) = y_0^{(m)}(t)$ verifying the equation

$$\sum_{i=0}^K \beta^{m\alpha-i-m} e_{\Delta}(ih, -\beta) \left[\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \binom{k\alpha}{i+m} \right] v^{(i)}(t) = u_0(t) \quad (23)$$

This new differential equation has the transfer function:

$$G(s) = \frac{1}{\sum_{i=0}^K A_i s^i} \quad (24)$$

with

$$A_i = \beta^{m\alpha-i-m} e_{\Delta}(ih, -\beta) \left[\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \binom{k\alpha}{i+m} \right] \quad (25)$$

for $i = 0, 1, \dots, K$. To obtain the solution of Eq. (7), we use the result stated in (21) showing that $u_0(t)$ is a ‘‘polynomial’’ with order equal to K : $u_0(t) = \sum_{i=0}^K U_i(-nh)_{ih}$.

Then from (21)

$$v(nh) = \sum_{i=0}^K U_i \sum_{j=0}^i \binom{K}{j} \bar{G}^{(j)}(0)(nh)_{(i-j)h}$$

To get the solution, $y(t)$, we are looking for we must realize that $y_0^{(m)}(t) = v(t)$. So, from (3)

$$y_0(t) = h^m \sum_{i=0}^K U_i \sum_{j=0}^i G^{(j)}(0) \frac{(i-j)!}{(i-j+m)!} (nh)_{(i-j+m)h} \quad (26)$$

The particular solution of the differential Eq. (1) when it is singular and $x(t) = (nh)_{Kh} e_{\nabla}(nh, \beta)$ is given by

$$y(t) = e_{\nabla}(nh, \beta) h^m \sum_{i=0}^K U_i \sum_{l=0}^i \frac{l! G^{(i-l)}(0)}{(l+m)!} (nh)_{(l+m)h} \quad (27)$$

with U_i the coefficients of the polynomial in (21) and $G(s)$ given by (24) and (25).

A simple summation change can give another format to the above relation:

$$y(t) = e_{\nabla}(nh, \beta) h^m \sum_{l=0}^K \frac{l!}{(l+m)!} \bar{U}_l (nh)_{(l+m)h} \quad (28)$$

with $\bar{U}_l = \sum_{i=l}^K U_i G^{(i-l)}(0)$.

Example 3. Consider the semi-differential equation (Ortigueira, 2014b):

$$y^{(2)}(t) - 4y^{(3/2)}(t) + 3y + 4y^{(1/2)}(t) - 4y(t) = (n)_2 e_{\nabla}(n, 4)$$

where we assumed $h = 1$ by simplicity.

The characteristic pseudo-polynomial has a second order zero at $s = 4$. Thus $m = 2$ and $K = 2$. Proceeding as described above we have

$$\bar{G}(s) = \frac{(s^{1/2} - 2)^2}{s^2 - 4s^{3/2} + 3s + 4s^{1/2} - 4} = \frac{1}{s + 4s^{1/2} + 3}$$

from where we deduce that $U_0 = \bar{G}''(4) = \frac{79}{15^3}$, $U_1 = G'(4) = -\frac{2}{225}$, and $U_2 = \bar{G}(4) = \frac{1}{15}$.

For $G(s)$, we have

$$\begin{aligned} A_i &= 4^{-1-i} \left[\sum_{k=0}^2 (-1)^{m-k} \binom{m}{k} \binom{k/2}{i+m} \right] \\ &= 4^{-1-i} [1 - 2 \cdot (1/2)_{i+m} + (i+m)!] \end{aligned}$$

that gives $A_0 = \frac{3}{8}$, $A_1 = \frac{13}{64}$, $A_2 = \frac{95}{512}$ and

$$G(s) = \frac{1}{\frac{3}{8} + \frac{13}{64}s + \frac{95}{512}s^2}$$

giving $G(0) = \frac{1}{A_0} = \frac{8}{3}$, $G'(0) = -\frac{A_1}{A_0^2} = -\frac{13}{9}$, and $G''(0) = -\frac{2A_2}{A_0^3} + \frac{2A_1^2}{A_0^3} = \frac{29}{27}$.

With these constants we compute the \bar{U}_l that inserted in (28) gives the searched solution. This algorithm can be formulated in a matricial framework as in Ortigueira (2014b).

4. Conclusions

The steady state output of the discrete-time fractional differential systems was studied in this paper. Based on the fact that the exponentials are the eigenfunctions of such systems, we devised a general output computation when the input is the ‘‘rising factorial.exponential’’ product. The singular case that puts some difficulties was studied and solved.

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