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Original article

The metric dimension of the circulant graph with 2k generators can be less than k

Tomáš Vetrík[a], Muhammad Imran[b,*], Martin Knor[c], Riste Škrekovski[d,e]

[a] Department of Mathematics and Applied Mathematics, University of the Free State, Bloemfontein, South Africa
[b] Department of Mathematical Sciences, Colleges of Science, United Arab Emirates University, Al Ain, United Arab Emirates
[c] Slovak University of Technology, Bratislava, Slovakia
[d] Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia
[e] Faculty of Information Studies, Novo Mesto, Slovenia

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ABSTRACT

Circulant graphs are useful networks because of their symmetries. For $k \geqslant 2$ and $n \geqslant 2k+1$, the circulant graph $C_n(1,2,\ldots,k)$ consists of the vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ and the edges $v_i v_{i+1}, v_i v_{i+2}, \ldots, v_i v_{i+k}$, where $i = 0,1,2,\ldots,n-1$, and the subscripts are taken modulo n. The metric dimension $\beta(C_n(1,2,\ldots,k))$ of the circulant graphs $C_n(1,2,\ldots,k)$ for general k (and n) has been studied in several papers. In 2017, Chau and Gosselin proved that $\beta(C_n(1,2,\ldots,k)) \geqslant k$ for every k, and they conjectured that if $n = 2k+r$, where k is even and $3 \leqslant r \leqslant k-1$, then $\beta(C_n(1,2,\ldots,k)) = k$. We disprove both by showing that for every $k \geqslant 9$, there exists an $n \in [2k+5, 2k+8] \subset [2k+3, 3k-1]$ such that $\beta(C_n(1,2,\ldots,k)) \leqslant \lceil \frac{2k}{3} \rceil + 2$. We conjecture that for $k \geqslant 6$, $\beta(C_n(1,2,\ldots,k))$ cannot be less than $\lceil \frac{2k}{3} \rceil + 2$.

1. Introduction

The metric dimension is an invariant which has applications in pharmaceutical chemistry (see Chartrand et al., 2000), pattern recognition and image processing (see Melter and Tomescu, 1984), robot navigation (see Khuller et al., 1996), and Sonar and coast guard Loran (see Slater, 1975).

In a graph G having vertex set $V(G)$, the number of edges in a shortest path connecting two vertices $u, v \in V(G)$ is the distance $d(u,v)$ between u and v. If $d(u,w) \neq d(v,w)$, then a vertex w resolves two vertices u and v. For an ordered set $W = \{w_1, w_2, \ldots, w_z\}$, the ordered z-tuple

$$r(v|W) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_z))$$

is the representation of distances of v in terms of W. If all the vertices of G have different representations, then $W \subset V(G)$ is a resolving set of G. The metric dimension $\beta(G)$ is the number of vertices in a smallest resolving set.

Circulant graphs are very useful networks because of their symmetries. For $k \geqslant 2$ and $n \geqslant 2k+1$, the circulant graph $C_n(1,2,\ldots,k)$ has the vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ and the edges $v_i v_{i+1}, v_i v_{i+2}, \ldots, v_i v_{i+k}$, where $i = 0,1,2,\ldots,n-1$, and the subscripts are taken modulo n. We assume that $n \geqslant 2k+1$, because for $n \leqslant 2k$, $C_n(1,2,\ldots,k)$ would contain multiple edges. So, $\{\pm 1, \pm 2, \ldots, \pm k\}$ is the set of generators of $C_n(1,2,\ldots,k)$.

In this paper, we focus on the following question: For a fixed k, find

$$\min\{\beta(C_n(1,2,\ldots,k)) \mid n \geqslant 2k+1\}.$$

By Borchert and Gosselin (2018) and Javaid et al. (2008), we have

$$\beta(C_n(1,2)) = \begin{cases} 4 & \text{if } n \equiv 1 \pmod 4, \\ 3 & \text{if } n \equiv 0,2,3 \pmod 4. \end{cases}$$

By Borchert and Gosselin (2018) and Imran et al. (2012), for $n \geqslant 8$,

$$\beta(C_n(1,2,3)) = \begin{cases} 5 & \text{if } n \equiv 1 \pmod 6, \\ 4 & \text{if } n \equiv 0,2,3,4,5 \pmod 6. \end{cases}$$

* Corresponding author.
E-mail addresses: vetrikt@ufs.ac.za (T. Vetrík), imrandhab@gmail.com (M. Imran), knor@math.sk (M. Knor).

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Since $\beta(C_n(1,2)) \geqslant 3$ and $\beta(C_n(1,2,3)) \geqslant 4$, Grigorious et al. (2014) assumed and proved that $\beta(C_n(1,2,\ldots,k)) \geqslant k+1$. Their result does not hold. In Vetrík (2017) it was proved that $\beta(C_n(1,2,\ldots,k))$ can be k, and that for $n \geqslant k^2+1$ where $k \geqslant 2$,

$$\beta(C_n(1,2,\ldots,k)) \geqslant k,$$

and for $n \equiv r \pmod{2k}$ where $k+2 \leqslant r \leqslant 2k+1$ and $k \geqslant 2$, we have

$$\beta(C_n(1,2,\ldots,k)) \geqslant k+1.$$

Chau and Gosselin (2017) obtained several strong results, but their inequality (1) is incorrect. They proved that for $n \equiv r \pmod{2k}$,

$$\beta(C_n(1,2,\ldots,k)) \geqslant k \tag{1}$$

if $3 \leqslant r \leqslant k$, and

$$\beta(C_n(1,2,\ldots,k)) \geqslant k+1$$

if $k+1 \leqslant r \leqslant 2k+2$. They also conjectured that if $n = 2k+r$ where k is even and $3 \leqslant r \leqslant k-1$, then

$$\beta(C_n(1,2,\ldots,k)) = k. \tag{2}$$

We disprove (1) and (2) by showing that for every $k \geqslant 9$, there exists an $n \in [2k+5, 2k+8] \subset [2k+3, 3k-1]$ such that

$$\beta(C_n(1,2,\ldots,k)) \leqslant \left\lceil \frac{2k}{3} \right\rceil + 2.$$

The importance of this result is that it has not been known before that $\beta(C_n(1,2,\ldots,k))$ can be less than k.

Let us note that the graphs $C_n(1,3)$ were studied in Javaid et al. (2012), $C_n(1,4)$ in Azhar and Javaid (2018), $C_n(2,3)$ in Du Toit and Vetrík (2019), $C_n(1,\frac{n}{2})$ for even n in Salman et al. (2012), $C_n(1,2,4)$ in Imran and Bokhary (2014), $C_n(1,2,5)$ in Imran et al. (2018), $C_n(1,2,3,4)$ in Grigorious et al. (2017) and Vetrík (2017), $C_n(1,2,\ldots,k)$ also in Vetrík (2020), and some interesting networks were investigated also in Arulperumjothi et al. (2023), Azeem et al. (2022), Koam et al. (2022), Prabhu et al. (2022) and Prabhu et al. (2022).

2. New bounds

We show that for every $k \geqslant 7$, there exists an n such that $\beta(C_n(1,2,\ldots,k)) \leqslant \lceil \frac{2k}{3} \rceil + 2$. The case $k \equiv 1 \pmod 3$ is studied in Theorem 1, $k \equiv 2 \pmod 6$ is studied in Theorem 2, $k \equiv 3 \pmod 6$ is investigated in Theorem 3, $k \equiv 5 \pmod 6$ is considered in Theorem 4 and $k \equiv 0 \pmod 6$ is studied in Theorem 5. Note that the subscripts of the vertices v_i are taken modulo n.

Theorem 1. *Let $n = 2k+5$ where $k \equiv 1 \pmod 3$ such that $k \geqslant 7$. Then*

$$\beta(C_n(1,2,\ldots,k)) \leqslant \frac{2k+1}{3} + 2.$$

Proof. Let $W = \{v_0\} \cup \{v_i : i = 2,5,\ldots,2k+3\}$. We have $n = 2k+5$, so for any vertex $v_i \in W$ (where $i = 2,5,\ldots,2k+3$ or 0), there are 4 vertices $v_{i+k+1}, v_{i+k+2}, v_{i+k+3}, v_{i+k+4}$ at distance 2 from v_i (and $2k$ vertices at distance 1 from v_i). Let $V_i = \{v_{i+k+1}, v_{i+k+2}, v_{i+k+3}, v_{i+k+4}\}$. It follows that the representations of a vertex of $V(C_n(1,2,\ldots,k)) \setminus V_i$ and a vertex of V_i are not the same in terms of W.

For the vertices of V_i where $i = 5,8,\ldots,2k$ and the ordered set $\{v_{i-3}, v_{i+3}\} \subset W$, we get

$$r(v_{i+k+1}|\{v_{i-3}, v_{i+3}\}) = (2,1),$$
$$r(v_{i+k+2}|\{v_{i-3}, v_{i+3}\}) = (1,1),$$
$$r(v_{i+k+3}|\{v_{i-3}, v_{i+3}\}) = (1,1),$$
$$r(v_{i+k+4}|\{v_{i-3}, v_{i+3}\}) = (1,2).$$

Since $v_{i+k+2} \in W$ for $i = 5,8,\ldots,k+1$, and $v_{i+k+3} \in W$ for $i = k+4, k+7, \ldots, 2k$, the vertices of V_i are resolved (for $i = 5,8,\ldots,2k$). We have

$$V_5 \cup V_8 \cup \ldots \cup V_{2k} = \{v_{k+6}, v_{k+7}, \ldots, v_{3k+4}\}.$$

Note that $3k+4 = k-1$. Finally, we need to resolve $v_k, v_{k+1}, \ldots, v_{k+5}$. We have $v_{k+1}, v_{k+4} \in W$ and

$$r(v_k|\{v_{2k+3}, v_0, v_2\}) = (2,1,1),$$
$$r(v_{k+2}|\{v_{2k+3}, v_0, v_2\}) = (2,2,1),$$
$$r(v_{k+3}|\{v_{2k+3}, v_0, v_2\}) = (1,2,2),$$
$$r(v_{k+5}|\{v_{2k+3}, v_0, v_2\}) = (1,1,2).$$

Thus all the vertices of $C_n(1,2,\ldots,k)$ are resolved. So, W is a resolving set and we obtain $\beta(C_n(1,2,\ldots,k)) \leqslant |W| = \frac{2k+1}{3} + 2$. $\square$

Theorem 2. *Let $n = 2k+8$ where $k \equiv 2 \pmod 6$ such that $k \geqslant 8$. Then*

$$\beta(C_n(1,2,\ldots,k)) \leqslant \frac{2k+2}{3} + 2.$$

Proof. For $i = 0,1,2,\ldots,\frac{k+1}{3}$, let $W_i = \{v_{6i}, v_{6i+2}\}$. We show that $W = W_0 \cup W_1 \cup \ldots \cup W_{\frac{k+1}{3}}$ resolves $C_n(1,2,\ldots,k)$. We have $n = 2k+8$, so for any vertex v_j where $j = 0,1,\ldots,n-1$, there are 7 vertices $v_{j+k+1}, v_{j+k+2}, v_{j+k+3}, v_{j+k+4}, v_{j+k+5}, v_{j+k+6}, v_{j+k+7}$ at distance 2 from v_j (and $2k$ vertices at distance 1 from v_j). Thus there are exactly 7 vertices at distance 2 from $v_{6i} \in W_i$ where $i = 0,1,2,\ldots,\frac{k+1}{3}$. Those vertices are the vertices of the set

$$V_i = \{v_{6i+k+1}, v_{6i+k+2}, v_{6i+k+3}, v_{6i+k+4}, v_{6i+k+5}, v_{6i+k+6}, v_{6i+k+7}\}.$$

So, the representations of a vertex of $V(C_n(1,2,\ldots,k)) \setminus V_i$ and a vertex of V_i are not the same in terms of W. For the vertices of V_i and $\{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\} \subset W$, we get

$$r(v_{6i+k+1}|\{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) = (2,2,1,1),$$
$$r(v_{6i+k+2}|\{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) = (1,2,1,1),$$
$$r(v_{6i+k+3}|\{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) = (1,2,2,1),$$
$$r(v_{6i+k+4}|\{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) = (1,1,2,1),$$
$$r(v_{6i+k+5}|\{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) = (1,1,2,1),$$
$$r(v_{6i+k+6}|\{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) = (1,1,2,1),$$
$$r(v_{6i+k+7}|\{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) = (1,1,2,2).$$

The vertices $v_{6i+k+4}, v_{6i+k+6} \in W$, so the vertices of V_i are resolved (where $i = 0,1,2,\ldots,\frac{k+1}{3}$). Since $V_0 \cup V_1 \cup \ldots \cup V_{\frac{k+1}{3}} = V(C_n(1,2,\ldots,k))$, the set W resolves the vertices of $C_n(1,2,\ldots,k)$. Therefore $\beta(C_n(1,2,\ldots,k)) \leqslant |W| = 2(\frac{k+1}{3}+1)$. $\square$

Theorem 3. *Let $n = 2k+6$ where $k \equiv 3 \pmod 6$ such that $k \geqslant 9$. Then*

$$\beta(C_n(1,2,\ldots,k)) \leqslant \frac{2k}{3} + 2.$$

Proof. For $i = 0, 1, 2, \ldots, \frac{k}{3}$, let $W_i = \{v_{6i}, v_{6i+2}\}$. We show that $W = W_0 \cup W_1 \cup \ldots \cup W_{\frac{k}{3}}$ resolves $C_n(1, 2, \ldots, k)$. We have $n = 2k + 6$, so for any vertex v_j where $j = 0, 1, \ldots, n-1$, there are 5 vertices $v_{j+k+1}, v_{j+k+2}, v_{j+k+3}, v_{j+k+4}, v_{j+k+5}$ at distance 2 from v_j. For each $W_i = \{v_{6i}, v_{6i+2}\}$ where $i = 0, 1, 2, \ldots, \frac{k}{3}$, there are 7 vertices at distance 2 from at least one of v_{6i}, v_{6i+2}. Those vertices are the vertices of the set

$$V_i = \{v_{6i+k+1}, v_{6i+k+2}, v_{6i+k+3}, v_{6i+k+4}, v_{6i+k+5}, v_{6i+k+6}, v_{6i+k+7}\}.$$

So, the representations of a vertex of $V(C_n(1, 2, \ldots, k)) \setminus V_i$ and a vertex of V_i are not the same in terms of W. For the vertices of V_i and the ordered set $\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\} \subset W$, we get

$$r(v_{6i+k+1}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (2, 2, 1, 1),$$
$$r(v_{6i+k+2}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1, 2, 1, 1),$$
$$r(v_{6i+k+3}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1, 2, 2, 1),$$
$$r(v_{6i+k+4}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1, 2, 2, 1),$$
$$r(v_{6i+k+5}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1, 2, 2, 1),$$
$$r(v_{6i+k+6}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1, 1, 2, 1),$$
$$r(v_{6i+k+7}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1, 1, 2, 2).$$

The vertices $v_{6i+k+3}, v_{6i+k+5} \in W$, so the vertices of V_i are resolved (where $i = 0, 1, 2, \ldots, \frac{k}{3}$). Since $V_0 \cup V_1 \cup \ldots \cup V_{\frac{k}{3}} = V(C_n(1, 2, \ldots, k))$, W resolves the vertices of $C_n(1, 2, \ldots, k)$. Therefore $\beta(C_n(1, 2, \ldots, k)) \leqslant |W| = 2(\frac{k}{3} + 1)$. □

Theorem 4. *Let $n = 2k + 6$ where $k \equiv 5 \pmod 6$ such that $k \geqslant 11$. Then*

$$\beta(C_n(1, 2, \ldots, k)) \leqslant \frac{2k+2}{3} + 2.$$

Proof. For $i = 0, 1, 2, \ldots, \frac{k-5}{6}$, let $W_i = \{v_{6i}, v_{6i+2}\}$ and $W'_i = \{v_{6i+k+3}, v_{6i+k+5}\}$. Let $W' = \{v_{k-1}, v_{2k+2}\}$ and

$$W = (W_0 \cup W_1 \cup \ldots \cup W_{\frac{k-5}{6}}) \cup (W'_0 \cup W'_1 \cup \ldots \cup W'_{\frac{k-5}{6}}) \cup W'.$$

Note that $|W| = 2(\frac{k-5}{6} + 1) + 2(\frac{k-5}{6} + 1) + 2 = \frac{2k+2}{3} + 2$. We prove that W resolves $C_n(1, 2, \ldots, k)$. We have $n = 2k + 6$, so for any vertex v_j where $j = 0, 1, \ldots, n-1$, there are 5 vertices $v_{j+k+1}, v_{j+k+2}, v_{j+k+3}, v_{j+k+4}, v_{j+k+5}$ at distance 2 from v_j.

For each $W_i = \{v_{6i}, v_{6i+2}\}$ where $i = 0, 1, 2, \ldots, \frac{k-5}{6}$, there are exactly 7 vertices at distance 2 from at least one of v_{6i}, v_{6i+2}. Those vertices are the vertices of the set

$$V_i = \{v_{6i+k+1}, v_{6i+k+2}, v_{6i+k+3}, v_{6i+k+4}, v_{6i+k+5}, v_{6i+k+6}, v_{6i+k+7}\}.$$

So, the representations of a vertex in $V(C_n(1, 2, \ldots, k)) \setminus V_i$ and a vertex of V_i are not the same in terms of W. For the vertices of V_i where $i = 0, 1, 2, \ldots, \frac{k-11}{6}$ in terms of the ordered set $\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\} \subset W$, we get

$$r(v_{6i+k+1}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (2, 2, 1, 1),$$
$$r(v_{6i+k+2}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1, 2, 1, 1),$$
$$r(v_{6i+k+3}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1, 2, 2, 1),$$
$$r(v_{6i+k+4}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1, 2, 2, 1),$$
$$r(v_{6i+k+5}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1, 2, 2, 1),$$
$$r(v_{6i+k+6}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1, 1, 2, 1),$$
$$r(v_{6i+k+7}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1, 1, 2, 2).$$

The vertices $v_{6i+k+3}, v_{6i+k+5} \in W$, so the vertices of V_i are resolved by W (where $i = 0, 1, 2, \ldots, \frac{k-11}{6}$).

Similarly, for each $W'_i = \{v_{6i+k+3}, v_{6i+k+5}\}$ where $i = 0, 1, 2, \ldots, \frac{k-5}{6}$, there are exactly 7 vertices at distance 2 from at least one of v_{6i+k+3}, v_{6i+k+5}. Those vertices are the vertices of the set

$$V'_i = \{v_{6i-2}, v_{6i-1}, v_{6i}, v_{6i+1}, v_{6i+2}, v_{6i+3}, v_{6i+4}\}.$$

So, the representations of a vertex of V'_i and a vertex of $V(C_n(1, 2, \ldots, k)) \setminus V'_i$ are not the same in terms of W. For the vertices of V'_i where $i = 0, 1, 2, \ldots, \frac{k-11}{6}$ in terms of the ordered set $\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\} \subset W$, we get

$$r(v_{6i-2}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (2, 2, 1, 1),$$
$$r(v_{6i-1}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (1, 2, 1, 1),$$
$$r(v_{6i}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (1, 2, 2, 1),$$
$$r(v_{6i+1}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (1, 2, 2, 1),$$
$$r(v_{6i+2}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (1, 2, 2, 1),$$
$$r(v_{6i+3}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (1, 1, 2, 1),$$
$$r(v_{6i+4}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (1, 1, 2, 2).$$

The vertices $v_{6i}, v_{6i+2} \in W$, so the vertices of V'_i are resolved (where $i = 0, 1, 2, \ldots, \frac{k-11}{6}$).

We have

$$V_0 \cup V_1 \cup \ldots \cup V_{\frac{k-11}{6}} = \{v_{k+1}, v_{k+2}, \ldots, v_{2k-4}\}$$

and

$$V'_0 \cup V'_1 \cup \ldots \cup V'_{\frac{k-11}{6}} = \{v_{2k+4}, v_{2k+5}, v_0, v_1 \ldots, v_{k-7}\}.$$

Finally, we resolve the vertices

$v_{k-6}, v_{k-5}, \ldots, v_k$ and $v_{2k-3}, v_{2k-2}, \ldots, v_{2k+3}$.

Note that $v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}, v_{2k+2} \in W$ and

$$r(v_{k-6}|\{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) = (1, 1, 1, 2, 1),$$
$$r(v_{k-4}|\{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) = (1, 1, 1, 2, 2),$$
$$r(v_{k-2}|\{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) = (1, 1, 1, 1, 2),$$
$$r(v_{k}|\{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) = (1, 1, 1, 1, 1),$$
$$r(v_{2k-3}|\{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) = (2, 1, 1, 1, 1),$$
$$r(v_{2k-1}|\{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) = (2, 2, 1, 1, 1),$$
$$r(v_{2k+1}|\{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) = (1, 2, 2, 1, 1),$$
$$r(v_{2k+3}|\{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) = (1, 1, 2, 1, 1).$$

Thus all the vertices of $C_n(1, 2, \ldots, k)$ are resolved. So $\beta(C_n(1, 2, \ldots, k)) \leqslant |W| = \frac{2k+2}{3} + 2$. □

The proof of Theorem 5 is longer than the previous proofs, because the resolving sets used in the proof of Theorem 5 are not as simple as resolving sets used in the previous proofs.

Theorem 5. *Let $n = 2k + 6$ where $k \equiv 0 \pmod 6$ such that $k \geqslant 12$. Then*

$$\beta(C_n(1, 2, \ldots, k)) \leqslant \frac{2k}{3} + 2.$$

Proof. For $i = 0, 1, \ldots, \frac{k}{6} - 1$, let $W_i = \{v_{6i}, v_{6i+2}\}$. For $i = 1, 2, \ldots, \frac{k}{6} - 2$, let $W'_i = \{v_{6i+k+3}, v_{6i+k+5}\}$. Let $W' = \{v_{k-2}, v_{k+2}, v_{k+5}, v_{2k-3}, v_{2k}, v_{2k+4}\}$ and

$$W = (W_0 \cup W_1 \cup \ldots \cup W_{\frac{k}{6}-1}) \cup (W'_1 \cup W'_2 \cup \ldots \cup W'_{\frac{k}{6}-2}) \cup W'.$$

Note that $|W| = 2(\frac{k}{6}) + 2(\frac{k}{6} - 2) + 6 = \frac{2k}{3} + 2$. We prove that W resolves $C_n(1, 2, \ldots, k)$. We have $n = 2k + 6$, so for any vertex v_j where $j = 0, 1, \ldots, n-1$, there are 5 vertices $v_{j+k+1}, v_{j+k+2}, v_{j+k+3}, v_{j+k+4}, v_{j+k+5}$ at distance 2 from v_j.

For each $W_i = \{v_{6i}, v_{6i+2}\}$ where $i = 1, 2, \ldots, \frac{k}{6} - 2$, there are 7 vertices at distance 2 from at least one of v_{6i}, v_{6i+2}. Those vertices are the vertices of the set

$$V_i = \{v_{6i+k+1}, v_{6i+k+2}, v_{6i+k+3}, v_{6i+k+4}, v_{6i+k+5}, v_{6i+k+6}, v_{6i+k+7}\}.$$

So, the representations of a vertex of $V(C_n(1,2,\ldots,k)) \setminus V_i$ and a vertex of V_i are not the same in terms of W. For the vertices in V_i and the ordered set $\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\} \subset W$, we get

$$r(v_{6i+k+1}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (2,2,1,1),$$
$$r(v_{6i+k+2}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1,2,1,1),$$
$$r(v_{6i+k+3}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1,2,2,1),$$
$$r(v_{6i+k+4}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1,2,2,1),$$
$$r(v_{6i+k+5}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1,2,2,1),$$
$$r(v_{6i+k+6}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1,1,2,1),$$
$$r(v_{6i+k+7}|\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) = (1,1,2,2).$$

The vertices $v_{6i+k+3}, v_{6i+k+5} \in W$, so the vertices of V_i are resolved by W (where $i = 1, 2, \ldots, \frac{k}{6} - 2$).

Similarly, for each $W'_i = \{v_{6i+k+3}, v_{6i+k+5}\}$ where $i = 1, 2, \ldots, \frac{k}{6} - 2$, there are 7 vertices at distance 2 from at least one of v_{6i+k+3}, v_{6i+k+5}. Those vertices are the vertices of the set

$$V'_i = \{v_{6i-2}, v_{6i-1}, v_{6i}, v_{6i+1}, v_{6i+2}, v_{6i+3}, v_{6i+4}\}.$$

For the vertices of V'_i and the ordered set $\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\} \subset W$, we get

$$r(v_{6i-2}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (2,2,1,1),$$
$$r(v_{6i-1}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (1,2,1,1),$$
$$r(v_{6i}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (1,2,2,1),$$
$$r(v_{6i+1}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (1,2,2,1),$$
$$r(v_{6i+2}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (1,2,2,1),$$
$$r(v_{6i+3}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (1,1,2,1),$$
$$r(v_{6i+4}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) = (1,1,2,2).$$

The vertices $v_{6i}, v_{6i+2} \in W$, so the vertices of V'_i are resolved.

We have

$$V_1 \cup V_2 \cup \ldots \cup V_{\frac{k}{6}-2} = \{v_{k+7}, v_{k+8}, \ldots, v_{2k-5}\}$$

and

$$V'_1 \cup V'_2 \cup \ldots \cup V'_{\frac{k}{6}-2} = \{v_4, v_5, \ldots, v_{k-8}\}.$$

Finally, we resolve the vertices

$$v_{k-7}, v_{k-6}, \ldots, v_{k+6} \text{ and } v_{2k-4}, v_{2k-3}, \ldots, v_{2k+5}.v_0, v_1, v_2, v_3$$

if $k \geqslant 18$, and it remains to resolve all the vertices if $k = 12$. Note that

$$v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}, v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2 \in W,$$

so those vertices are resolved. We obtain

$$r(v_{k-7}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) = (2,1,1,1,1),$$
$$r(v_{k-5}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) = (2,2,1,1,1),$$
$$r(v_{k-3}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) = (1,2,1,1,1),$$
$$r(v_{k-1}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) = (1,2,2,1,1),$$
$$r(v_{k}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) = (1,1,2,1,1),$$
$$r(v_{k+1}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) = (1,1,2,2,1),$$
$$r(v_{k+3}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) = (1,1,2,2,2),$$
$$r(v_{k+4}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) = (1,1,1,2,2),$$
$$r(v_{k+6}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) = (1,1,1,1,2).$$

and

$$r(v_{2k-4}|\{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) = (2,1,1,1,1),$$
$$r(v_{2k-2}|\{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) = (2,2,1,1,1),$$
$$r(v_{2k-1}|\{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) = (2,2,2,1,1),$$
$$r(v_{2k+1}|\{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) = (1,2,2,1,1),$$
$$r(v_{2k+2}|\{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) = (1,1,2,1,1),$$
$$r(v_{2k+3}|\{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) = (1,1,2,2,1),$$
$$r(v_{2k+5}|\{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) = (1,1,1,2,1),$$
$$r(v_{1}|\{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) = (1,1,1,2,2),$$
$$r(v_{3}|\{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) = (1,1,1,1,2).$$

For $i = k-7, k-5, k-3, k-1, k, k+1, k+3, k+4, k+6$, we obtain

$$r(v_i|\{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) = (1,1,1,1,1),$$

and for $i = 2k-4, 2k-2, 2k-1, 2k+1, 2k+2, 2k+3, 2k+5, 1, 3$,

$$r(v_i|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) = (1,1,1,1,1).$$

Thus all the vertices of $C_n(1,2,\ldots,k)$ are resolved for $k \geqslant 18$.

If $k = 12$, it remains to consider v_4 and v_{19}. We have $r(v_4|\{v_{k+5}, v_{2k-3}\}) = r(v_4|\{v_{17}, v_{21}\}) = (2,2)$

and

$$r(v_{19}|\{v_2, v_{k-6}\}) = r(v_{19}|\{v_2, v_6\}) = (2,2).$$

It can be seen above that no other vertex has such representation in terms of the ordered sets $\{v_{k+5}, v_{2k-3}\}$ and $\{v_2, v_{k-6}\}$. So, W resolves all the vertices of $C_n(1,2,\ldots,k)$. Thus $\beta(C_n(1,2,\ldots,k)) \leqslant |W| = \frac{2k}{3} + 2$. □

From Theorems 1–5, we obtain the following corollary.

Corollary 1. *For every $k \geqslant 7$, there exists an $n \in [2k+5, 2k+8]$ such that*

$$\beta(C_n(1,2,\ldots,k)) \leqslant \left\lceil \frac{2k}{3} \right\rceil + 2.$$

Let us note that for $k = 7$ and 8, we have $\lceil \frac{2k}{3} \rceil + 2 = k$, so Corollary 1 is important for $k \geqslant 9$.

3. Concluding remarks and further work

We showed that for every $k \geqslant 7$, there is an $n \in [2k+5, 2k+8]$ such that $\beta(C_n(1,2,\ldots,k)) \leqslant \lceil \frac{2k}{3} \rceil + 2$. The importance of this result is that it has not been known before that $\beta(C_n(1,2,\ldots,k))$ can be less than k. We believe that $\beta(C_n(1,2,\ldots,k))$ cannot be less than $\lceil \frac{2k}{3} \rceil + 2$ for $k \geqslant 6$, thus we provide Conjecture 1 as a possible future work. Note that $n \geqslant 2k+1$, otherwise $C_n(1,2,\ldots,k)$ would contain multiple edges.

Conjecture 1. For every $k \geqslant 6$ (and $n \geqslant 2k+1$),

$$\beta(C_n(1,2,\ldots,k)) \geqslant \left\lceil \frac{2k}{3} \right\rceil + 2.$$

Conjecture 1 would not hold for $k = 4$ and $k = 5$. For example, for $k = 5$ and $n = 13$, the set $\{v_0, v_1, v_2, v_4, v_5\}$ is one possible resolving set of $C_{13}(1,2,\ldots,5)$.

Data availability statement

The data used to find the results are included in this paper.

Author contributions

All the authors contributed equally to this work. All the authors read and approved the final version.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at https://doi.org/10.1016/j.jksus.2023.102834.

References

Arulperumjothi, M., Klavžar, S., Prabhu, S., 2023. Redefining fractal cubic networks and determining their metric dimension and fault-tolerant metric dimension. Appl. Math. Comput. 452, 128037.

Azeem, M., Imran, M., Nadeem, M.F., 2022. Sharp bounds on partition dimension of hexagonal Möbius ladder. J. King Saud Univ. Sci. 34 (2), 101779.

Azhar, M.N., Javaid, I., 2018. Metric basis in circulant networks. Ars Combin. 136, 277–286.

Borchert, A., Gosselin, S., 2018. The metric dimension of circulant graphs and Cayley hypergraphs. Util. Math. 106, 125–147.

Chartrand, G., Eroh, L., Johnson, M.A., Oellermann, O.R., 2000. Resolvability in graphs and the metric dimension of a graph. Discrete Appl. Math. 105 (1–3), 99–113.

Chau, K., Gosselin, S., 2017. The metric dimension of circulant graphs and their Cartesian products. Opuscula Math. 37 (4), 509–534.

Du Toit, L., Vetrík, T., 2019. On the metric dimension of circulant graphs with 2 generators. Kragujevac J. Math. 43 (1), 49–58.

Grigorious, C., Kalinowski, T., Ryan, J., Stephen, S., 2017. The metric dimension of the circulant graph $C(n, \pm\{1,2,3,4\})$. Australas. J. Combin. 69 (3), 417–441.

Grigorious, C., Manuel, P., Miller, M., Rajan, B., Stephen, S., 2014. On the metric dimension of circulant and Harary graphs. Appl. Math. Comput. 248, 47–54.

Imran, M., Baig, A.Q., Bokhary, S.A., Javaid, I., 2012. On the metric dimension of circulant graphs. Appl. Math. Lett. 25, 320–325.

Imran, M., Baig, A.Q., Rashid, S., Semaničová-Feňovčíková, A., 2018. On the metric dimension and diameter of circulant graphs with three jumps. Discrete Math. Algorithms Appl. 10 (1), 1850008.

Imran, M., Bokhary, S.A.U.H., 2014. On resolvability in double-step circulant graphs, U.P.B. Sci. Bull. Series A 76 (2), 31–42.

Javaid, I., Azhar, M.N., Salman, M., 2012. Metric dimension and determining number of Cayley graphs. World Appl. Sci. J. 18, 1800–1812.

Javaid, I., Rahim, M.T., Ali, K., 2008. Families of regular graphs with constant metric dimension. Util. Math. 75, 21–33.

Khuller, S., Raghavachari, B., Rosenfeld, A., 1996. Landmarks in graphs. Discrete Appl. Math. 70 (3), 217–229.

Koam, A.N.A., Ahmad, A., Asim, M.A., Azeem, M., 2022. Computation of vertex and edge resolvability of benzenoid tripod structure. J. King Saud Univ. Sci. 34 (6), 102208.

Melter, R.A., Tomescu, I., 1984. Metric bases indigital geometry. Comput. Vision Graphics Image Process. 25, 113–121.

Prabhu, S., Deepa, S., Arulperumjothi, M., Susilowati, L., Liu, J.-B., 2022. Resolving-power domination number of probabilistic neural networks. J. Intell. Fuzzy Syst. 43 (5), 6253–6263.

Prabhu, S., Manimozhi, V., Arulperumjothi, M., Klavžar, S., 2022. Twin vertices in fault-tolerant metric sets and fault-tolerant metric dimension of multistage interconnection networks. Appl. Math. Comput. 420, 126897.

Salman, M., Javaid, I., Chaudhry, M.A., 2012. Resolvability in circulant graphs. Acta Math. Sin. (Engl. Ser.) 28 (9), 1851–1864.

Slater, P.J., 1975. Leaves of trees. Congr. Numer. 14, 549–559.

Vetrík, T., 2020. On the metric dimension of directed and undirected circulant graphs. Discuss. Math. Graph Theory 40 (1), 67–76.

Vetrík, T., 2017. The metric dimension of circulant graphs. Canad. Math. Bull. 60 (1), 206–216.

Vetrík, T., 2017. On the metric dimension of circulant graphs with 4 generators. Contrib. Discrete Math. 12 (2), 104–114.