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A Baues fibration category of $\mathfrak{P}$-spaces

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Abstract The article is concerned with homotopy in the category $\mathfrak{P}$ whose objects are the pairs $(X, *)$ consisting of a Polish space X and a closed binary operation $*$. Homomorphisms in $\mathfrak{P}$ are continuous maps compatible with the operations. The result showed that the category $\mathfrak{P}$ admits the structure of a fibration category in the sense of H. Baues. The notions of fibration and weak equivalence are defined in the category $\mathfrak{P}$ and showed to satisfy fundamental properties that the corresponding notions satisfy in the category *Top* of topological spaces.

1. Introduction

Several ubiquitous spaces such as $Homeo(M)$ and $Diff(M)$ (where M is a compact manifold), have been studied intensively from many perspectives. However, in the last few decades, they come together as a unified class of metric spaces called Polish spaces. A Polish space is a separable completely metrizable topological space. These spaces are the natural setting for descriptive set theory and its applications. See for instant Kechris (2012), Moschovakis (2009) and Srivastava (2013). In this paper, we focus our attention on the homotopy theory of Polish spaces. In particular, we take the viewpoint of axiomatic homotopy theory.

The definition of Polish spaces depends essentially on topological notions of metrizability and separability which are not preserved by homotopy (Gottlieb, 1964, Theorem 3.1 and Hu, 1961, Section 2). Another observation is that taking subsets of Polish spaces need not yield a Polish space.

Therefore, we impose a mild condition on the category of Polish spaces and continuous maps, *Pol*, which is the existence of a closed operation. The objects in the new category $\mathfrak{P}$ are the pairs $(X, *)$ consisting of a Polish space X and an operation $* : X \times X \to X$ defined for each $(x, y) \in X \times X$. The homomorphisms in $\mathfrak{P}$ are continuous maps compatible with the operations. Sometimes such an algebra is referred to as a binary groupoid or a magma. However, we will call it a $\mathfrak{P}$-space here, so we do not confuse it with a completely different approach proposed in Ramsay (1990) named Polish groupoids.

The idea here is that we choose a nice algebraic category $\mathfrak{P}$ endowed with a reasonable notion of homotopy, together with a functor $F : Pol \to \mathfrak{P}$ such that if X is homotopic to Y in *Pol*, then $F(X)$ is homotopic to $F(Y)$ and vice versa, where X and Y could be either objects or morphisms.

In Quillen (1967), Quillen introduced the notion of model category as a framework for axiomatic homotopy theory. A modern theory of Quillen's homotopical algebra can be found

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in Dwyer and Spalinski (1995) or Hirschhorn (2003). Baues (Baues, 1989) developed the concept of fibration category and cofibration category. His approach is regarded as doing one half of Quillen's axioms. These categories satisfy certain axioms that rely on the notions of fibration, cofibration, and weak equivalence.

Baues showed that *Top* has the structure of a fibration category where fibration means Hurewicz fibration, cofibration means the usual cofibration, and weak equivalence refers to homotopy equivalence. This paper attempts to examine whether the category $\mathfrak{P}$ satisfies the axioms of Baues. Focusing on fibration and regarding the $\mathfrak{P}$ category as a subcategory of *Top*, it is natural to ask whether the fibration structure of *Top* restricted to $\mathfrak{P}$ is a fibration category. This question was addressed for different spaces (cf. Andersen and Groda (1997) and Kahl (2009)).

The result showed that the category $\mathfrak{P}$ has the structure of a fibration category (Theorem 6) provided that the fibration means a $\mathfrak{P}$-fibration, and weak equivalence means a $\mathfrak{P}$-homotopy equivalence.

We should point out that the techniques used in this paper do not make serious use of deep results from descriptive set theory. The results here depend only on the topological properties of the spaces.

Our main reference for the classical homotopy theory is the book (Hu, 1959). All the spaces in this work are assumed to be separable and completely metrizable.

2. The category *P*

Definition 1. A pair $(X, *)$ is called a $\mathfrak{P}$-space where X is a separable, completely metrizable space and $*$ is a closed operation on X.

$* : X \times X \to X$, such that $(x_1, x_2) \mapsto x_1 * x_2 \in X$,

$\forall (x_1, x_2) \in X \times X$. A $\mathfrak{P}$-map $f : (X, *_X) \to (Y, *_Y)$ between two $\mathfrak{P}$-spaces is a continuous map $f : X \to Y$ that preserves the operations, i.e., $f(x_1 *_X x_2) = f(x_1) *_Y f(x_2)$ for all $x_1, x_2 \in X$.

Remark 1.

1. Clearly, the composition of two $\mathfrak{P}$-maps is a $\mathfrak{P}$-map. The identity map $id_X : X \to X$ is a $\mathfrak{P}$-map $(X, *) \to (X, *)$ for any closed operation on X.
2. Since the spaces under study are metrizable, (and hence they are Hausdorff and paracompact), the diagonal set $\Gamma = \{(x_1, x_2) \in X \times X | x_1 = x_2\}$ is closed in $X \times X$. Therefore the pair (X, Γ) is a $\mathfrak{P}$-space and the set of $\mathfrak{P}$-maps $(X, \Gamma) \to (Y, *)$ is the same as the set of continuous maps $X \to Y$. Moreover, X is said to have a G_δ-diagonal since Γ is G_δ-set ((Gruenhage, 2014)).
3. The two natural projections given by $\pi_1(x_1, x_2) = x_1$ and $\pi_2(x_1, x_2) = x_2$ for all $x_1, x_2 \in X$ define closed operations that make X into the $\mathfrak{P}$-space (X, π_1) and (X, π_2).

Example 1. The space $C(X, Y)$ of continuous maps $X \to Y$ with the compact-open topology is separable, completely metrizable Kechris, 2012, Theorem 4.19. Then $(C(X, Y), \circ)$ is a $\mathfrak{P}$-space, where $\circ$ is the composition of maps.

It is known that a subspace of a Polish space need not be a Polish space, it is Polish if and only if it is a G_δ in the relative topology (Kechris, 2012, Section 3C).

Definition 2. Let $(X, *_X)$ be a $\mathfrak{P}$-space and let A be a subset of X. Then $(A, *_A)$ is a $\mathfrak{P}$-subspace of $(X, *_X)$ provided A is a G_δ (in relative topology), and $a_1 *_A a_2 = a_1 *_X a_2$ for all $a_1, a_2 \in A$.

Henceforth, we shall identify the operation map on the $\mathfrak{P}$-subspace with that on the original $\mathfrak{P}$-space.

The category $\mathfrak{P}$ is formed by the $\mathfrak{P}$-spaces as objects and the $\mathfrak{P}$-maps as morphisms. The $\mathfrak{P}$-maps $f : (X, \pi_i) \to (Y, \pi_i), (i = 1, 2)$, defines a continuous map $f : X \to Y$. There are two functors can be defined between the categories $\mathfrak{P}$ and *Pol*:

1. The functor $\mathcal{G}_i : Pol \to \mathfrak{P}$, such that $\mathcal{G}_i(X) = (X, \pi_i)$, $(i = 1, 2)$.
2. The forgetful functor $\mathcal{F} : \mathfrak{P} \to Pol$, such that $\mathcal{F}(X, *) = X$.

Propostion 1. *The category $\mathfrak{P}$ is closed under pullbacks.*

Proof. Let $g : (X, *_X) \to (Y, *_Y)$ and $f : (Z, *_Z) \to (Y, *_Y)$ be two $\mathfrak{P}$-maps. Consider the fiber product space $P = Z \times_Y X = \{(z, x) \in Z \times X | f(z) = g(x)\}$ with the operation $*$ defined by $(z_1, x_1) * (z_2, x_2) = (z_1 *_Z z_2, x_1 *_X x_2)$ Then $(P, *)$ is a $\mathfrak{P}$ space. Indeed, the product of Polish spaces is Polish (Kechris, 2012), and the induced operation $*$ on P is closed as a subset of $P \times P$. Consider the diagram in $\mathfrak{P}$.

$$\begin{tikzcd}
(Q, *_Q) \arrow[drr, bend left, "\alpha"] \arrow[dr, dotted, "\mu"] \arrow[ddr, bend right, "\beta"'] & & \\
& (P, *) \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"'] & (X, *_X) \arrow[d, "g"] \\
& (Z, *_Z) \arrow[r, "f"'] & (Y, *_Y)
\end{tikzcd}$$

where $\tilde{f}$ and $\tilde{g}$ are projections, and hence, $\mathfrak{P}$-maps. To show that $(P, *)$ has the universal property of the pullback, let $(Q, *_Q)$ be a $\mathfrak{P}$-space and let $\alpha : (Q, *_Q) \to (X, *_X)$ and $\beta : (Q, *_Q) \to (Z, *_Z)$ be two $\mathfrak{P}$-maps such that $g \circ \alpha = f \circ \beta$. Assume that there exists a unique continuous map $\mu : (Q, *_Q) \to (P, *)$ such that $\alpha = \tilde{f} \circ \mu$ and $\beta = \tilde{g} \circ \mu$. To show that μ is a $\mathfrak{P}$-map; let $q \in Q$. Since the spaces are Polish, and both α and β are $\mathfrak{P}$-maps, then there exist neighborhoods U and V of q such that α is compatible with the operations on U and β is compatible with the operations on V. Since $U \cap V$ is a neighborhood of q, and since α and β are compatible with the operations on $U \cap V$, then μ is compatible with the operations on $U \cap V$. Therefore $(P, *)$ is the pullback of g and f in $\mathfrak{P}$. $\square$

3. The *P*-Homotopy

Using the fact that the path space X^I of all continuous maps $I = [0, 1] \to X$ with the compact-open topology is a Polish space (Example 1). Given a continuous map $*_{X^I} : X^I \times X^I \to X^I$, such that $\epsilon_1 *_{X^I} \epsilon_2 = \epsilon_1(t) *_X \epsilon_2(t)$, for all paths $\epsilon_1, \epsilon_2 \in X^I$ and for all $t \in I$. Obviously, $*_{X^I}$ defines a closed operation on X^I and, hence, the pair $(X^I, *_{X^I})$ is a $\mathfrak{P}$-space.

Now we have the $\mathfrak{P}$-maps $j_s : (X^I, *_{X^I}) \to (X, *_X), (s = 0, 1)$ such that $j_i(\epsilon) = \epsilon(s)$. Let a $\mathfrak{P}$-map $(X, *_X) \to (Y, *_Y)$ be given. The continuous map $\sigma : (X, *_X) \to (Y^I, *_{Y^I})$ such that $\sigma(x) = C_{f(x)}$, where $C_a(t) = a$ is the constant path.

$$\begin{aligned}\sigma(x *_X x') &= C_{f(x *_X x')} \\ &= C_{[f(x) *_Y f(x')]} \\ &= C_{f(x)} *_{Y^I} C_{f(x')} \\ &= \sigma(x) *_{Y^I} \sigma(x').\end{aligned}$$

Thus σ is a $\mathfrak{P}$-map.

Definition 3. Two $\mathfrak{P}$-maps $f, g : (X, *_X) \to (Y, *_Y)$ are said to be $\mathfrak{P}$-homotopic, (notation: $f \simeq_{\mathfrak{P}} g$), if there exists a $\mathfrak{P}$-map $H : (X, *_X) \to (Y^I, *_{Y^I})$ called a $\mathfrak{P}$-homotopy joining f and g such that for all $x \in X$; $H(x)(0) = f(x)$ and $H(x)(1) = g(x)$.

Definition 4. A $\mathfrak{P}$-map $f : (X, *_X) \to (Y, *_Y)$ is a $\mathfrak{P}$-homotopy equivalence if there exists a $\mathfrak{P}$-map $g : (Y, *_Y) \to (X, *_X)$ such that $g \circ f : (X, *_X) \to (X, *_X)$ and $f \circ g : (Y, *_Y) \to (Y, *_Y)$ are homotopic to id_X and id_Y respectively.

The $\mathfrak{P}$-homotopy has the standard properties required of a homotopy notion, namely:

Theorem 1.

1. *The relation $\simeq_{\mathfrak{P}}$ is an equivalence relation.*
2. *Let $f_1, f_2 : (X, *_X) \to (Y, *_Y)$ and $g_1, g_2 : (Y, *_Y) \to (Z, *_Z)$ be $\mathfrak{P}$-maps. If $f_1 \simeq_{\mathfrak{P}} f_2$ and $g_1 \simeq_{\mathfrak{P}} g_2$, then $g_1 f_1 \simeq_{\mathfrak{P}} g_2 f_2$.*
3. *Every invertible $\mathfrak{P}$-map in $\mathfrak{P}$ is a $\mathfrak{P}$-homotopy equivalence. Moreover, if two of the $\mathfrak{P}$-maps f, h, and $f \circ h$ are $\mathfrak{P}$-homotopy equivalence, then so is the third.*

Proof. (1) symmetry and reflexivity are straightforward. To check transitivity: Let $f \simeq_{\mathfrak{P}} g$ and $g \simeq_{\mathfrak{P}} h$ by $\mathfrak{P}$-homotopies H and F respectively, and let K be given by

$$K(x)(t) = \begin{cases} H(x)(2t), & 0 \leqslant t \leqslant 1/2 \\ F(x)(2t-1), & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Then K is continuous and also a $\mathfrak{P}$-homotopy between f and h.

(2) Let $H : (X, *_X) \to (Y^I, *_{Y^I})$ be a $\mathfrak{P}$-homotopy joining f_1 and f_2. Similarly, let $F : (Y, B) \to (Z^I, *_{Z^I})$ be a $\mathfrak{P}$-homotopy joining g_1 and g_2. Define

$$K(x)(t) = \begin{cases} g_1 H(x)(2t), & 0 \leqslant t \leqslant 1/2 \\ F(f_2(x))(2t-1), & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Then $K(x)(t)$ is a $\mathfrak{P}$-homotopy $(X, A) \to (Z^I, *_{Z^I})$ joining $g_1 f_1$ and $g_2 f_2$.

(3) Follows from the corresponding fact for isomorphisms and from the case of *Top* category. □

If $f : (X, *_X) \to (Y, *_Y)$ is a $\mathfrak{P}$-map, then the $\mathfrak{P}$-homotopy class consisting of all $\mathfrak{P}$-maps homotopic to f is denoted by $[f]_{\mathfrak{P}}$. The set of $\mathfrak{P}$-homotopy classes of $\mathfrak{P}$-maps from $(X, *_X)$ to $(Y, *_Y)$ is denoted by $[X, Y]_{\mathfrak{P}}$.

The $\mathfrak{P}$-homotopy relation defines the homotopy category $\mathfrak{P}/\simeq_{\mathfrak{P}}$ where the objects are the $\mathfrak{P}$-spaces and the morphisms are the $\mathfrak{P}$-homotopy classes of $\mathfrak{P}$-maps with composition rule $[g]_{\mathfrak{P}} \circ [f]_{\mathfrak{P}} = [g \circ f]_{\mathfrak{P}}$. A $\mathfrak{P}$-map is a $\mathfrak{P}$-homotopy equivalence if and only if its $\mathfrak{P}$-homotopy class is an isomorphism in $\mathfrak{P}/\simeq_{\mathfrak{P}}$.

Theorem 2. *Given two $\mathfrak{P}$-maps $f, g : (X, *_X) \to (Y, *_Y)$. If f and g are $\mathfrak{P}$-homotopic, then $f, g : X \to Y$ are homotopic in Top.*

Proof. Since f and g are $\mathfrak{P}$-homotopic through a $\mathfrak{P}$-homotopy $F : (X, *_X) \to (Y^I, *_{Y^I})$, then the function $H : X \times I \to Y$ defined by $H(x, t) = F(x)(t)$ for every $(x, t) \in X \times I$, is a homotopy between the maps $f, g : X \to Y$. This is implied by the properties of the compact-open topology on the path space Y^I and the Hausdorffness of the Polish spaces. □

The converse of Theorem 2 need not be true, consider the following counter example:

Example 2. Given a $\mathfrak{P}$-map $f : (\mathbb{T}, \pi_1) \to (\mathbb{I}, \cdot)$ where $\mathbb{T}$ is the unit circle with the first projection operation, and $\mathbb{I}$ is the unit interval with the usual multiplication. Both $\mathbb{I}$ and $\mathbb{T}$ are connected Polish spaces. Then, for every $y \in \mathbb{T}$ $f(\pi_1(y, y)) = f(y) \cdot f(y) = f(y)$. Because f is by definition continuous and the spaces are connected, this implies that the value of $f(y)$ is either 0 or 1 and hence, f must be constant. Consequently, there are only two $\mathfrak{P}$-maps between $(\mathbb{T}, \pi_1)$ and $(\mathbb{I}, \cdot)$. Let f_0 and f_1 be the constant $\mathfrak{P}$-maps that send $\mathbb{T}$ into 0 and 1 respectively. In a similar manner, we can show that there are two $\mathfrak{P}$-maps between $(\mathbb{T}, \pi_1)$ and $(\mathbb{I}^I, *_{\mathbb{I}^I})$. The space $(\mathbb{I}^I, *_{\mathbb{I}^I})$ is Hausdorff with distinct points, whereas $\mathbb{T}$ is connected. Thus, the $\mathfrak{P}$-maps $f_0$and f_1 are homotopic but not $\mathfrak{P}$-homotopic.

The following theorem provides the condition under which the converse of Theorem 2 is true.

Theorem 3. *The $\mathfrak{P}$-maps $f, g : (X, \pi_i) \to (Y, \pi_i), (i = 1, 2)$, are $\mathfrak{P}$-homotopic if and only if the maps $f, g : X \to Y$ are homotopic.*

Proof. We will prove the case when the projection is π_1. The case of π_2 will follow similarly.

For one assertion, assume that $F : X \times I \to Y$ is a homotopy joining f and g. To show the existence of a $\mathfrak{P}$-homotopy between (X, π_1) and (Y^I, π_1); recall that by the definition of π_1 we have: $(\pi_1 Y)^I = \pi_1(Y^I) = Y^I$. Hence, define a map $H : (X, \pi_1) \to (Y^I, \pi_1)$ by $H(x)(t) = F(x, t)$ for every $x \in X$ and every $t \in I$. On the other hand, the assertion that $\mathfrak{P}$-homotopy implies homotopy is given by Theorem 2.

In the proofs of the last two theorems we applied the fact that for any topological spaces A and B, the map $f : A \times I \to B$ implies the existence of a map $g : A \to B^I$ defined by $g(a) = f(a, t)$ for all $a \in A$ and $t \in I$ ([Strom, 2011]). Since I is compact and regular, then the statement is reversible. $\square$

Remark 2. As a consequence of Theorem 3, it is clear that the functor $\mathcal{G}_i$ reflects the homotopy while the functor $\mathcal{F}$ does not in general.

4. The *P*-fibration

Definition 5. A $\mathfrak{P}$-map $f : (X, *_X) \to (Y, *_Y)$ is a $\mathfrak{P}$-fibration with respect to every $\mathfrak{P}$-space $(Z, *_Z)$ if for a $\mathfrak{P}$-map $g : (Z, *_Z) \to (X, *_X)$ and a $\mathfrak{P}$-homotopy $H : (Z, *_Z) \to (Y^I, *_{Y^I})$ with $H(z)(0) = f \circ g$, there exists a $\mathfrak{P}$-map $F : (Z, *_Z) \to (X^I, *_{X^I})$ such that $F(z)(0) = g$ and $f(F(z)(t)) = H(z)(t)$ for every $z \in Z$ and every $t \in I$.

Theorem 4. *The $\mathfrak{P}$-map $f : (X, \pi_1) \to (Y, \pi_1)$ is a $\mathfrak{P}$-fibration if and only if $f : X \to Y$ is a Hurewicz fibration.*

Proof. Since the definition of fibration and $\mathfrak{P}$-fibration depend on homotopy and $\mathfrak{P}$-homotopy respectively, then the result follows easily by applying Theorem 3. $\square$

Lemma 1. *Let $f : (X, *_X) \to (Y, *_Y)$ be a $\mathfrak{P}$-fibration. For any $\mathfrak{P}$-subspace $(Z, *_Y)$ of $(Y, *)$. The $\mathfrak{P}$-map $\bar{f} : (f^{-1}(Z), *_X) \to (Z, *_Y)$ is a $\mathfrak{P}$-fibration.*

Proof. Since the spaces are paracompact, the result follows from the general case of fibration (Hu, 1959, Proposition 8.1 p. 73) and the property of Polish subspaces, see e.g., Berberian, 1988, Proposition A.1.13). $\square$

The concept of $\mathfrak{P}$-lifting function can be defined in the presence of operations as follows. Given a $\mathfrak{P}$-fibration $f : (X, *_X) \to (Y, *_Y)$. Define the space $(\lambda, *_\lambda)$ as: $\lambda = \{(x, \epsilon) \in X \times Y^I | f(x) = \epsilon(0)\}$, where $*_\lambda$ is the induced operation $(*_X \times *_{Y^I})$. Then λ is Polish because it is a product of Polish spaces, and hence, $(\lambda, *_\lambda)$ defines a $\mathfrak{P}$-space.

Definition 6. A $\mathfrak{P}$-map $\mathfrak{F} : (\lambda, *_\lambda) \to (X^I, *_{X^I})$ is called a $\mathfrak{P}$-lifting function if:

1. $\mathfrak{F}(x, \epsilon)(0) = x, \forall (x, \epsilon) \in \lambda$,
2. $f \circ \mathfrak{F}(x, \epsilon) = \epsilon, \forall (x, \epsilon) \in \lambda$.

Remark 3. The $\mathfrak{P}$-extension property of $\mathfrak{P}$-cofibrations is dual to the $\mathfrak{P}$-lifting property that is used to define $\mathfrak{P}$-fibrations. However, as the focus of this work is $\mathfrak{P}$-fibrations and not $\mathfrak{P}$-cofibrations; $\mathfrak{P}$-extension property was not found to be of critical importance to the contribution of the paper. This will serve as a topic for further studies in future.

As it was shown by Hurewicz (Hurewicz, 1955), a fibration is regular if it admits a regular lifting function, that is, if $\mathfrak{F}$ has the property that $\lambda(x, \epsilon)$ is a constant path whenever ϵ is a constant path, then $\mathfrak{F}$ is said to be regular.

Lemma 2. *Every $\mathfrak{P}$-fibration is regular.*

Proof. Let $f : (X, *_X) \to (Y, *_Y)$ be a $\mathfrak{P}$-fibration. Using the fact that $Y \times Y$ is a normal space and that Γ is a G_δ-set (Remark 1), then $(Y, *_Y)$ admits a ϕ-function (Tulley, 1965, Theorem 3.1, p.134). Therefore, f is a regular $\mathfrak{P}$-fibration. $\square$

Next, we verify that $\mathfrak{P}$-fibrations have the properties usually required of a fibration.

Theorem 5.

1. *Every isomorphism of $\mathfrak{P}$-maps is a $\mathfrak{P}$-fibration.*
2. *The composition of two $\mathfrak{P}$-fibrations is a $\mathfrak{P}$-fibration.*
3. *If $(g : X, *_X) \to (Y, *_Y)$ is a $\mathfrak{P}$-fibration and $f : (Z, *_Z) \to (Y, *_Y)$ is any $\mathfrak{P}$-map, then the $\mathfrak{P}$-map $\tilde{g} : (P, *) \to (Z, *_Z)$ is a $\mathfrak{P}$-fibration. Moreover, if g (resp., f) is a $\mathfrak{P}$-homotopy equivalence, so is $\tilde{g}$ (resp., $\tilde{f}$).*

Proof. (1) is obvious.

(2) Given $\mathfrak{P}$-fibrations $u : (X, *_X) \to (Y, *_Y)$ and $v : (Y, *_Y) \to (Z, *_Z)$. Let $v \circ u : (X, *_X) \to (Z, *_Z)$ be a $\mathfrak{P}$-map and let $(S, *_S)$ be any $\mathfrak{P}$-space. Assume that we are given a $\mathfrak{P}$-map $h : (S, *_S) \to (X, *_X)$ and that $H : (S, *_S) \to (Z^I, *_{Z^I})$ is a $\mathfrak{P}$-homotopy with $H_0 = g \circ u \circ h$. Since v is a $\mathfrak{P}$-fibration, this implies the existence of a $\mathfrak{P}$-homotopy $F : (S, *_S) \to (Y^I, *_{Y^I})$ such that $F_0 = \widehat{h} : (S, *_S) \to (Y, *_Y)$ and $v(F(s)(t)) = H(x)(t) for all s \in S,\ t \in I$ Since u is a $\mathfrak{P}$-fibration, then there is a $\mathfrak{P}$-homotopy $G : (S, *_S) \to (X^I, *_{X^I})$ such that $G_0 = h$ and $u(G(x)(t)) = F(x)(t)$ for all $s \in S,\ t \in I$. Hence, we have the composition

$$\begin{aligned} v[u(G(x)(t))] &= vu[G(x)(t)] \\ &= H(x)(t) \end{aligned}$$

for all $s \in S,\ t \in I$. This implies that $v \circ u : (X, *_X) \to (Z, *_Z)$ is a $\mathfrak{P}$-fibration.

(3) The second assertion follows easily from the *Top* category case, see e.g. [Baues, 1989]. The proof of the first assertion follows by analyzing the diagram

$$\begin{tikzcd}
(X, *_X) \arrow[dddd, "g"'] & & & & (X^I, *_{X^I}) \arrow[llll] \arrow[dddd] \\
& (P, *) \arrow[ul, "\tilde{f}"'] \arrow[dd, "\tilde{g}"'] & & (P^I, *_{P^I}) \arrow[ll] \arrow[ur] \arrow[dd] & \\
& & (M, *_M) \arrow[ul, "\alpha"] \arrow[ur, dashed, "F"] \arrow[dr, "H"'] & & \\
& (Z, *_Z) \arrow[dl, "f"] & & (Z^I, *_{Z^I}) \arrow[ll] \arrow[dr, "\beta"] & \\
(Y, *_Y) & & & & (Y^I, *_{Y^I}) \arrow[llll]
\end{tikzcd}$$

where $P = Z \times_g X$ is as defined in Proposition 1.

Given a $\mathfrak{P}$-homotopy $H : (M, *_M) \to (Z^I, *_{Z^I})$ and a $\mathfrak{P}$-lifting $F_0 : (M, *_M) \to (P^I, *_{P^I})$ of H_0, i.e., $j_0 \circ F_0 = H_0$. Then we can find a $\mathfrak{P}$-homotopy $F : (M, *_M) \to (P^I, *_{P^I})$ such that the first coordinate is given by H, and the second coordinate is given by the lift of $\beta \circ H$. Since g is a $\mathfrak{P}$-fibration, this implies that any $\mathfrak{P}$-homotopy $(M, *_M) \to (Y^I, *_{Y^I})$ gets lifted to a $\mathfrak{P}$-homotopy $(M, *_M) \to (X^I, *_{X^I})$.

Propostion 2. *Let $g : (X, *_X) \to (Y, *_Y)$ be an arbitrary $\mathfrak{P}$-map. Then g can be factorized into a $\mathfrak{P}$-homotopy equivalence $f : (X, *_X) \to (Z, *_Z)$ followed by a $\mathfrak{P}$-fibration $h : (Z, *_Z) \to (Y, *_Y)$.*

Proof. Let $(Z, *_Z)$ be the $\mathfrak{P}$-space $(\lambda, *_\lambda)$ where $\lambda = X \times_Y Y^I$. To show that f is a $\mathfrak{P}$-homotopy equivalence; for any point $y \in Y$, there is a constant path given by $\delta : I \to Y$ such that $\delta_y(t) = y$. Define f by $f(x) = (x, \delta_{g(x)})$ for each $x \in X$ Now, let $\pi : (\lambda, *_\lambda) \to (X, *_X)$ be the projection. Then $\pi \circ f = id_X$. Let $H : (\lambda, *_\lambda) \to (\lambda^I, *_{\lambda^I})$ be a $\mathfrak{P}$-homotopy given by

$$H(x, \epsilon)(t) = \begin{cases} (x, \delta_{\epsilon(0)}), & 0 \leqslant t \leqslant \frac{1}{2} \\ (x, \epsilon_{2t-1}), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

where $\epsilon_t : I \to Y$ is given by $\epsilon_t(s) = \epsilon(st), s \in I$. This implies that $f \circ \pi \simeq_{\mathfrak{P}} id_\lambda$. Thus, f is a $\mathfrak{P}$-homotopy equivalence.

To show that h is a $\mathfrak{P}$-fibration; consider the following diagram.

$$\begin{tikzcd}
(\lambda, *_\lambda) \arrow[d, "h"'] & & (\lambda^I, *_{\lambda^I}) \arrow[ll, "j_\circ"'] \arrow[d, "h'"] \\
& (M, *_M) \arrow[ul, "\alpha"] \arrow[ur, dashed, "F"] \arrow[dr, "H"'] & \\
(Y, *_Y) & & (Y^I, *_{Y^I}) \arrow[ll, "j_\circ"]
\end{tikzcd}$$

where the $\mathfrak{P}$-maps α and H are given such that $h \circ \alpha = j_\circ H$. We need to construct a lift F of H. Let $\alpha(m) = (\alpha_1(m), \alpha_2(m)) \in \lambda$ and let

$$F(m)(t) = (\alpha_1(m), \mathfrak{f}(m, t))$$

where

$$\mathfrak{f}(m, t)(s) = \begin{cases} \alpha_2(m)(s + st), & 0 \leqslant s \leqslant \frac{1}{(1+t)} \\ H(m, s + ts - 1), & \frac{1}{(1+t)} \leqslant s \leqslant 1. \end{cases}$$

Then $F(m)(t)$ is a $\mathfrak{P}$-map because its components are all $\mathfrak{P}$-maps. It follows that h is a $\mathfrak{P}$-fibration.

By a trivial $\mathfrak{P}$-fibration we mean a $\mathfrak{P}$-fibration which is also a $\mathfrak{P}$-homotopy equivalence.

Propostion 3. *If the $\mathfrak{P}$-map $f : (X, *_X) \to (Y, *_Y)$ is a trivial $\mathfrak{P}$-fibration, then f has a section, i.e., a $\mathfrak{P}$-map $s : (Y, *_Y) \to (X, *_X)$ such that $s \circ f \simeq_{\mathfrak{P}} id_X$.*

Proof. Let $g : (Y, *_Y) \to (X, *_X)$ be the inverse of f such that $f \circ g \simeq_{\mathfrak{P}} id_Y$ and $g \circ f \simeq_{\mathfrak{P}} id_X$. Let H be the homotopy between $f \circ g$ and id_Y. Since f is a $\mathfrak{P}$-fibration, then there exists a $\mathfrak{P}$-homotopy $G : (Y, *_Y) \to (X^I, *_{X^I})$ such that $G(y)(0) = g(y)$ and $f \circ G = H$. Let $s : (Y, *_Y) \to (X, *_X)$ such that $s(y) = G(y)(1)$. Then $s \simeq_{\mathfrak{P}} g$ and hence $s \circ f \simeq_{\mathfrak{P}} g \circ f \simeq_{\mathfrak{P}} id_X$. □

5. Baues fibration category

Using the acquired information from previous sections we can now state and prove our main theorem. First, we recall the definition of fibration category in sense of Baues.

Definition 7 Baues, 1989. Let *fib* and *w.e.* denote classes of morphisms in a category $\mathcal{C}$, called fibrations and weak equivalences respectively. A fibration category is a category $\mathcal{C}$ with the structure $(\mathcal{C}, fib, w.e.)$ that satisfies the following axioms:

(F1) Composition axiom. Isomorphisms are both fibrations and weak equivalences. For two maps in $\mathcal{C}$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

if any two of f, g, and gf are weak equivalences, then so is the third. The composition of fibrations is a fibration.

(F2) Pullback axiom. For a fibration $g : X \to Y$ and any map $f : Z \to Y$, there exists a pullback diagram in $\mathcal{C}$

$$\begin{tikzcd}
Z \times_Y X \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"'] & X \arrow[d, "g"] \\
Z \arrow[r, "f"'] & Y
\end{tikzcd}$$

such that

1. $\tilde{g}$ is a fibration,
2. if f is a weak equivalence, so is $\tilde{f}$,
3. if g is a weak equivalence, so is $\tilde{g}$.

(F3) Factorization axiom. Every map $g : X \to Y$ in $\mathcal{C}$ can be factorized into a weak equivalence f followed by a fibration h such that next diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & Y \\
Z \arrow[ur, "h"'] &
\end{tikzcd}$$

(F4) Axioms on cofibrant models. For any object Y in $\mathcal{C}$ there exists a trivial fibration $X \to Y$ where X is cofibrant in $\mathcal{C}$. An object X in $\mathcal{C}$ is a cofibrant model if every trivial fibration $f : E \to X$ admits a section.

Theorem 6. *The category $\mathfrak{P}$ with the structure*

$$\begin{aligned} fib &= \mathfrak{P} - \textit{fibrations with the unique path lifting in } \mathfrak{P}, \text{and} \\ w.e. &= \mathfrak{P} - \textit{homotopy equivalences in } \mathfrak{P}, \end{aligned}$$

is a fibration category in which all objects are $\mathfrak{P}$-fibrant and $\mathfrak{P}$-cofibrant.

Proof. Axiom $F1$ is proved in Theorem 1(3), Theorem 5(1), and Theorem 5(2). Combining Proposition 1 and Theorem 5 (3) yields the proof of axiom $F2$. $F3$ is Proposition 2. $F4$ follows from Proposition 3 together with the general case of *Top*. □

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