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Original article

Discrete Hardy's inequalities with $0 < p \leqslant 1$

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A R T I C L E I N F O

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A B S T R A C T

We generalize the famous discrete Hardy inequality to $0 < p \leqslant 1$. We obtain this generalization by using the atomic decompositions of the discrete Hardy spaces.

1. Introduction

The main theme of this paper is the following discrete Hardy inequalities.

Let $0 < p \leqslant 1$. There exists a constant $C > 0$ such that for any sequence $a = \{a(n)\}_{n\in\mathbb{Z}}$ with $a(-n) = 0, n \in \mathbb{N} \cup \{0\}$, we have

$$\left(\sum_{m\in\mathbb{N}}\left(\frac{1}{m}\sum_{j=1}^{m}a(j)\right)^{p}\right)^{\frac{1}{p}} \leqslant C\left(\sum_{m\in\mathbb{N}}|a(m)|^{p} + \sum_{min\mathbb{Z}}\left|\sum_{j\neq m}\frac{a(j)}{m-j}\right|^{p}\right)^{\frac{1}{p}}. \quad (1)$$

The family of Hardy inequalities, which consists of the discrete form and the integral form, is one of the most important inequalities in analysis. For the history of the Hardy inequalities, the reader is referred to Kufner et al. (2006, 2007). For the applications and further developments of this famous inequality, the reader may consult (Kufner et al., 2007; Kufner and Persson, 2003; Opic and Kufner, 1990).

Recently, the integral form Hardy inequality had been extended to Hardy spaces on $\mathbb{R}$. In Ho (2016), the Hardy inequalities in Hardy spaces are established by using the atomic decompositions of Hardy spaces. This method is also used in Ho (2016, 2017a,b) to study the Hardy inequalities on Hardy-Morrey spaces with variable exponents and weak Hardy-Morrey spaces.

In this paper, we use the idea from Ho (2016) to obtain the above generalization (1) of Hardy's inequality to $0 < p \leqslant 1$.

To apply the method in Ho (2016), we need to consider the discrete analogue of the classical Hardy spaces on $\mathbb{R}$. The discrete Hardy spaces had been introduced in Boza and Carro (1998, 2002). Moreover, the atomic decompositions of the discrete Hardy spaces were also obtained in Boza and Carro (1998, 2002). These atomic decomposition are precisely what we need to establish the discrete Hardy inequality with $0 < p \leqslant 1$.

Thus, on one hand, we extend the classical discrete Hardy inequalities to $0 < p \leqslant 1$, on the other hand, the main result of this paper gives an application of the atomic decompositions established in Boza and Carro (1998, 2002).

This paper is organized as follows. In Section 2, we recall the definition of the discrete Hardy spaces. The atomic decompositions for the discrete Hardy spaces are also presented in this section. The discrete Hardy inequalities with $0 < p \leqslant 1$ are established in Section 3.

2. Discrete Hardy spaces

In this section, we first recall the definition of the discrete Hardy spaces by discrete Hilbert transform on $\mathbb{Z}$.

For any sequence $a = \{a(n)\}_{n\in\mathbb{Z}}$, the discrete Hilbert transform of a is defined by

$$(H^d a)(m) = \sum_{n\neq m}\frac{a(n)}{m-n}.$$

For any $B \subset \mathbb{Z}$, let $|B|$ denote the cardinality of B.

We use the definition of discrete Hardy spaces from (Boza and Carro, 1998, Definition 3.1).

Definition 2.1. Let $0 < p \leqslant 1$. The discrete Hardy spaces $H^p(\mathbb{Z})$ consists of those sequence $a = \{a(n)\}_{n\in\mathbb{Z}}$ satisfying

$$\|a\|_{H^p(\mathbb{Z})} = \|a\|_{l^p(\mathbb{Z})} + \|H^d a\|_{l^p(\mathbb{Z})} < \infty.$$

In view of the above definition, we see the reason why the second summation on the left hand side of (1) is taking over $\mathbb{Z}$.

We now present the atomic characterization of $H^p(\mathbb{Z})$, we begin with the definition of $H^p(\mathbb{Z})$-atom (Boza and Carro, 1998, Definition 3.9).

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Definition 2.2. Let $0 < p \leqslant 1$. A sequence $a = \{a(k)\}_{k\in\mathbb{Z}}$ is an $H^p(\mathbb{Z})$-atom if it satisfies

(1) supp a is contained in a ball in $\mathbb{Z}$ of cardinality $2n+1, n \geqslant 1$.
(2) $\|a\|_{l^\infty(\mathbb{Z})} \leqslant (2n+1)^{-1/p}$.
(3) $\sum_{n\in\mathbb{Z}} n^\alpha a(n) = 0$ for every $\alpha \in \mathbb{N}$ with $\alpha \leqslant \frac{1}{p} - 1$.

To present the atomic decomposition of $H^p(\mathbb{Z})$, we recall the atomic version of $H^p(\mathbb{Z})$ from (Boza and Carro, 1998, p.43).

Definition 2.3. Let $0 < p \leqslant 1$. The atomic discrete Hardy space $H^p_{\text{at}}(\mathbb{Z})$ consists of those sequence $a = \{a(n)\}_{n\in\mathbb{Z}}$ such that

$$a = \sum_{j\in\mathbb{N}} \lambda_j a_j$$

where a_j are $H^p(\mathbb{Z})$-atoms and

$$\|a\|_{H^p_{\text{at}}(\mathbb{Z})} = \inf \left\{ \left(\sum_{j\in\mathbb{N}} |\lambda_j|^p \right)^{\frac{1}{p}} \right\}$$

where the infimum is taken over all possible representations of a in terms of $H^p(\mathbb{Z})$-atoms.

The following result gives the atomic decomposition of $H^p(\mathbb{Z})$.

Theorem 1. *Let $0 < p \leqslant 1$. Then, there exist constants $B, C > 0$ such that for any sequence $a = \{a(n)\}_{n\in\mathbb{Z}}$, we have*

$$B\|a\|_{H^p(\mathbb{Z})} \leqslant \|a\|_{H^p_{\text{at}}(\mathbb{Z})} \leqslant C\|a\|_{H^p(\mathbb{Z})}.$$

The reader is referred to (Boza and Carro, 1998, Theorems 3.10 and 3.14) for the proof of the above theorem.

We can also characterize discrete Hardy spaces by using Poisson integral and area functions, see (Boza and Carro, 1998, Theorems 3.4 and 3.8). The reader is also referred to Boza (2012) and Kanjin and Satake (2000), Komori (2002) for the factorization theorem and the molecular characterizations of discrete Hardy spaces, respectively.

The reader is also referred to Herz (1973), Ho (2009, 2012), Jiao et al. (2017), Weisz (1994) for some other applications of the atomic decompositions such as the characterizations of *BMO* and martingale *BMO*.

3. Hardy's inequalities

We establish the main result of this paper in this section. We first introduce the Hardy operators in order to simplify our presentation. For any $\alpha \in \mathbb{N} \cup \{0\}$ and $\mu > 0$, define

$$(T_{\alpha,\mu}a)(m) = \frac{1}{m^{\alpha-\mu+1}} \sum_{j=1}^{m} j^\alpha a(j), \quad m \in \mathbb{N}.$$

Notice that when $\alpha = \mu = 0$, we have

$$(T_{0,0}a)(m) = \frac{1}{m} \sum_{j=1}^{m} a(j). \tag{2}$$

It is precisely the Hardy-Littlewood average for the sequence $a = \{a(k)\}_{k=1}^\infty$.

We are now ready to present the main result of this paper, the mapping properties of $T_{\alpha,\mu}$ on discrete Hardy spaces $H^p(\mathbb{Z})$.

Theorem 2. *Let $0 < p \leqslant 1$ and $0 \leqslant \mu < 1$. Suppose that $\alpha \in \mathbb{N} \cup \{0\}$ satisfies $\alpha \leqslant \frac{1}{p} - 1$. If*

$$\frac{1}{p} = \frac{1}{r} + \mu, \tag{3}$$

then there exists a constant $C > 0$ such that for any sequence $a \in H^p(\mathbb{Z})$ with support contained in $\mathbb{N}$, we have

$$\|T_{\alpha,\mu}a\|_{l^r(\mathbb{N})} \leqslant C\|a\|_{H^p(\mathbb{Z})}.$$

When $0 < p \leqslant 1$ and $\alpha = \mu = 0$, we have $p = r$ and the above theorem yields

$$\|T_{0,0}a\|_{l^p(\mathbb{N})} \leqslant C\|a\|_{H^p(\mathbb{Z})}.$$

In view of (2), it establishes the discrete Hardy inequality (1).

We need several supporting results to obtain the proof of Theorem 2. We start with the mapping property of $T_{\alpha,\mu}$ on $H^p(\mathbb{Z})$-atoms.

Lemma 3. *Let $0 < p \leqslant 1$, $0 \leqslant \mu < 1$ and $\alpha \in \mathbb{N} \cup \{0\}$ with $\alpha \leqslant \frac{1}{p} - 1$. Suppose that*

$$\frac{1}{q} - \frac{1}{r} < \mu \leqslant \frac{1}{q}$$

for some $q > 1$. If $a = \{a(n)\}_{n\in\mathbb{Z}}$ satisfies

(1) supp a *is contained in a ball B in $\mathbb{N} \setminus \{0\}$ of cardinality* $2n+1, n \geqslant 1$,
(2) $\|a\|_{l^\infty(\mathbb{N})} \leqslant (2n+1)^{-1/p}$,
(3) $\sum_{n=0}^\infty n^\alpha a(n) = 0$,

then, we have

$$\|T_{\alpha,\mu}a\|_{l^r(\mathbb{N})} \leqslant C|B|^{\mu+\frac{1}{r}-\frac{1}{p}}.$$

Proof. Let $B = \{i \in \mathbb{N} : M \leqslant i \leqslant N\}, M, N \in \mathbb{N}$. We have $|B| = N - M + 1$. For any $i < M$, we have

$$(T_{\alpha,\mu}a)(i) = \frac{1}{i^{\alpha-\mu+1}} \sum_{j=1}^{i} j^\alpha a(j) = 0.$$

Similarly, for any $i > N$, Items (1) and (3) assure that

$$(T_{\alpha,\mu}a)(i) = \frac{1}{i^{\alpha-\mu+1}} \sum_{j=1}^{i} j^\alpha a(j) = \frac{1}{i^{\alpha-\mu+1}} \sum_{j\in\mathbb{N}} j^\alpha a(j) = 0.$$

Therefore, we find that $\text{supp}(T_{\alpha,\mu}a) \subseteq B$.

The Hölder inequality yields

$$|\sum_{j=1}^{m} j^\alpha a(j)| \leqslant \left(\sum_{j=1}^{m} |a(j)|^q \right)^{1/q} \left(\sum_{j=1}^{m} j^{\alpha q'} \right)^{1/q'} \leqslant C\|a\|_{l^q} m^{\alpha+\frac{1}{q'}}$$

for some $C > 0$. Therefore,

$$|(T_{\alpha,\mu}a)(m)| \leqslant C\frac{1}{m^{\alpha-\mu+1}} \|a\|_{l^q} m^{\alpha+\frac{1}{q'}} = Cm^{\mu-\frac{1}{q}} \|a\|_{l^q}.$$

Furthermore, as $\text{supp}(T_{\alpha,\mu}a) \subseteq B$ and $\mu \leqslant \frac{1}{q}$, we obtain

$$\begin{aligned}\|T_{\alpha,\mu}a\|_{l^r}^r &\leqslant C\|a\|_{l^q}^r \sum_{m=M}^{N} m^{r\mu-\frac{r}{q}} \leqslant C \int_M^{N+1} x^{r\mu-\frac{r}{q}} dx \\ &\leqslant C\|a\|_{l^q}^r (N^{r\mu-\frac{r}{q}+1} - M^{r\mu-\frac{r}{q}+1}).\end{aligned}$$

Since $\frac{1}{q} - \frac{1}{r} < \mu \leqslant \frac{1}{q}$, we find that $0 < r\mu - \frac{r}{q} + 1 \leqslant 1$. Consequently,

$$(N^{r\mu-\frac{r}{q}+1} - M^{r\mu-\frac{r}{q}+1}) \leqslant (N - M)^{r\mu-\frac{r}{q}+1} \leqslant |B|^{r\mu-\frac{r}{q}+1}.$$

Hence,

$$\|T_{\alpha,\mu}a\|_{l^r} \leqslant C\|a\|_{l^q}|B|^{\mu-\frac{r}{q}+\frac{1}{r}} \leqslant C\|a\|_{l^\infty}|B|^{\frac{1}{q}}|B|^{\mu-\frac{1}{q}+\frac{1}{r}} \leqslant C|B|^{\mu+\frac{1}{r}-\frac{1}{p}}$$

for some $C > 0$. □

Lemma 3 applies to those $H^p(\mathbb{Z})$-atom $a = \{a(n)\}_{n\in\mathbb{Z}}$ with support in $\mathbb{N}\setminus\{0\}$. On the other hand, for any $a \in H^p(\mathbb{Z})$ with support in $\mathbb{N}\setminus\{0\}$, a does not necessarily possess an atomic decomposition with all the $H^p(\mathbb{Z})$-atoms supported in $\mathbb{N}\setminus\{0\}$. To tackle this difficulty, we consider the odd and the even extensions of $a \in H^p(\mathbb{Z})$.

For any sequence $a = \{a(n)\}_{n\in\mathbb{Z}}$, the even part of a and the odd part of a, denoted by a_e and a_o, are defined by

$$a_e = \left\{\frac{a(n)+a(-n)}{2}\right\}_{n\in\mathbb{Z}} \quad \text{and} \quad a_o = \left\{\frac{a(n)-a(-n)}{2}\right\}_{n\in\mathbb{Z}},$$

respectively.

As the even part is a "reflection" about $k = 0$, the term $a(0)$ will be counted twice in the even part. Thus, for the case when α equals to zero, we need some further modifications. The details of these modifications and the uses of the even and odd parts are given in the subsequent results.

Lemma 4. *Let $0 < p \leqslant 1$ and $\alpha \in \mathbb{N}\cup\{0\}$ with $\alpha \leqslant \frac{1}{p}-1$. If $a \in H^p(\mathbb{Z})$ with* supp$a \subset \mathbb{N}\setminus\{0\}$, *then there exist a sequence of scalars $\{\lambda_j\}_{j\in\mathbb{N}}$ and sequences $\{a_j\}_{j\in\mathbb{N}}$ satisfying*

(1) *supp a_j is contained in a ball in $\mathbb{N}\setminus\{0\}$ of cardinality $2n+1, n \geqslant 1$,*
(2) $\|a_j\|_{l^\infty(\mathbb{Z})} \leqslant 2(2n+1)^{-1/p}$,
(3) $\sum_{n=0}^{\infty} n^\alpha a_j(n) = 0$,

such that $a = \sum_{j\in\mathbb{N}}\lambda_j a_j$ and

$$\left(\sum_{j\in\mathbb{N}}|\lambda_j|^p\right)^{1/p} \leqslant C\|a\|_{H^p(\mathbb{Z})}.$$

Proof. In view of Theorem 1, we have a sequence of scalars $\{\lambda_j\}_{j\in\mathbb{N}}$ and $H^p(\mathbb{Z})$-atoms $\{a_j\}_{j\in\mathbb{N}}$ such that $a = \sum_{j\in\mathbb{N}}\lambda_j a_j$ and $\left(\sum_{j\in\mathbb{N}}|\lambda_j|^p\right)^{1/p} \leqslant C\|a\|_{H^p(\mathbb{Z})}$.

We split the proof into three cases, α is a positive even integer, α is a positive odd integer and α equals zero.

(i) α is a positive even integer. We consider the even part of a,

$$a_e(k) = \sum_{j\in\mathbb{N}}\lambda_j\left(\frac{a_j(k)+a_j(-k)}{2}\right).$$

Since supp$a \subset \mathbb{N}\setminus\{0\}$, we have $a = 2a_e\chi_{\mathbb{N}\setminus\{0\}}$. That is,

$$a(k) = \sum_{j\in\mathbb{N}}\lambda_j(a_j(k)+a_j(-k))\chi_{\mathbb{N}\setminus\{0\}}(k).$$

Write $A_j = \{A_j(k)\}_{k\in\mathbb{Z}}$ where $A_j(k) = (a_j(k)+a_j(-k))\chi_{\mathbb{N}\setminus\{0\}}(k)$. We are going to show that A_j fulfills Item (1)–(3).
As α is even, we find that

$$\sum_{k\in\mathbb{Z}}k^\alpha a_j(k) = \sum_{k\in\mathbb{Z}}k^\alpha a_j(-k) = 0.$$

If supp$a_j \subset \mathbb{N}\setminus\{0\}, A_j = a_j$ satisfies Item (1)–(3). If supp$a_j \subset \mathbb{Z}\setminus(\mathbb{N}\cup\{0\})$, then $A_j(\cdot) = a_j(-\cdot)$ also satisfies Item (1)–(3).
If $0 \in$ suppa_j, we find that

$$\sum_{n=0}^{\infty}n^\alpha A_j(n) = \sum_{n=0}^{\infty}n^\alpha(a_j(n)+a_j(-n)) = \sum_{n\in\mathbb{Z}}n^\alpha a_j(n) = 0$$

because α is a positive even integer.

We have $\|A_j\|_{l^\infty(\mathbb{N})} \leqslant 2\|a_j\|_{l^\infty(\mathbb{N})}$. Moreover, supp$A_j \subseteq$ (supp$a_j\cup$ supp$a_j(-\cdot)) \cap (\mathbb{N}\setminus\{0\})$. Hence, $|\text{supp}A_j| \leqslant |\text{supp}a_j|$.
Consequently, A_j fulfills Item (1)–(3).

(ii) α is a positive odd integer. We consider the odd part of a,

$$a_o(k) = \sum_{j\in\mathbb{N}}\lambda_j\left(\frac{a_j(k)-a_j(-k)}{2}\right).$$

Since supp$a \subset \mathbb{N}\setminus\{0\}$, we have $a = 2a_o\chi_{\mathbb{N}\setminus\{0\}}$. That is,

$$a(k) = \sum_{j\in\mathbb{N}}\lambda_j(a_j(k)-a_j(-k))\chi_{\mathbb{N}\setminus\{0\}}(k).$$

Write $A_j = \{A_j(k)\}_{k\in\mathbb{Z}}$ where $A_j(k) = (a_j(k)-a_j(-k))\chi_{\mathbb{N}\setminus\{0\}}(k)$.
As α is odd, we find that

$$\sum_{k\in\mathbb{Z}}k^\alpha a_j(k) = -\sum_{k\in\mathbb{Z}}k^\alpha a_j(-k) = 0.$$

If supp$a_j \subset \mathbb{N}\setminus\{0\}, A_j = a_j$ satisfies Item (1)–(3). If supp$a_j \subset \mathbb{Z}\setminus(\mathbb{N}\cup\{0\})$, then $A_j(\cdot) = a_j(-\cdot)$ also satisfies Item (1)–(3).
If $0 \in$ suppa_j, we find that

$$\sum_{n=0}^{\infty}n^\alpha A_j(n) = \sum_{n=0}^{\infty}n^\alpha(a_j(n)-a_j(-n)) = \sum_{n\in\mathbb{Z}}n^\alpha a_j(n) = 0$$

because α is a positive odd integer.
Therefore, A_j fulfills Item (1)–(3).

(ii) α equals to zero. In this case, we also consider the even part of a,

$$a_e(k) = \sum_{j\in\mathbb{N}}\lambda_j\left(\frac{a_j(k)+a_j(-k)}{2}\right).$$

Write

$$b_j(k) = \begin{cases} a_j(k), & k \neq 0, \\ \frac{a_j(0)}{2}, & k = 0. \end{cases}$$

When $k \neq 0$, we obviously have

$$a_e(k) = \sum_{j\in\mathbb{N}}\lambda_j\left(\frac{a_j(k)+a_j(-k)}{2}\right) = \sum_{j\in\mathbb{N}}\lambda_j\left(\frac{b_j(k)+b_j(-k)}{2}\right).$$

Since supp$a \subset \mathbb{N}\setminus\{0\}$, we have

$$a(0) = \sum_{j\in\mathbb{N}}\lambda_j a_j(0) = 0.$$

Therefore, for $k = 0$, we have

$$a_e(0) = a(0) = 0 = \sum_{j\in\mathbb{N}}\lambda_j a_j(0) = \sum_{j\in\mathbb{N}}\lambda_j\left(\frac{b_j(0)+b_j(0)}{2}\right).$$

As supp$a \subset \mathbb{N}\setminus\{0\}$, we have $a = 2a_e\chi_{\mathbb{N}\setminus\{0\}}$. That is,

$$a(k) = \sum_{j\in\mathbb{N}}\lambda_j(b_j(k)+b_j(-k))\chi_{\mathbb{N}\setminus\{0\}}(k).$$

Write $B_j = \{B_j(k)\}_{k\in\mathbb{Z}}$ where $B_j(k) = (b_j(k)+b_j(-k))\chi_{\mathbb{N}\setminus\{0\}}(k)$.
It remains to show that B_j satisfies Item (1)–(3).
When supp$a_j \subset \mathbb{N}\setminus\{0\}, B_j = a_j$ satisfies Item (1)–(3). When supp$a_j \subset \mathbb{Z}\setminus(\mathbb{N}\cup\{0\})$, then $B_j(\cdot) = a_j(-\cdot)$ also satisfies Item (1)–(3).
If $0 \in$ suppa_j, we find that

$$\sum_{n=0}^{\infty} B_j(n) = \sum_{n=0}^{\infty}(b_j(n)+b_j(-n)) = \left(\sum_{n=1}^{\infty} a_j(n) + \sum_{n=-\infty}^{-1} a_j(n)\right) + 2b_j(0) = \sum_{n\in\mathbb{Z}} a_j(n) = 0.$$

Thus, B_j fulfills Item (1)–(3). □

Proof of Theorem 2.

In view of Lemma 4, for any $a \in H^p(\mathbb{Z})$ with supp$a \subset \mathbb{N}\setminus\{0\}$, there exist a sequence of scalars $\{\lambda_j\}_{j\in\mathbb{N}}$ and $\{a_j\}_{j\in\mathbb{N}}$ satisfying.

(1) supp a_j is contained in a ball in $\mathbb{N}\setminus\{0\}$ of cardinality $2n+1, n \geqslant 1$,
(2) $\|a_j\|_{l^\infty(\mathbb{Z})} \leqslant 2(2n+1)^{-1/p}$,
(3) $\sum_{n=0}^{\infty} n^\alpha a_j(n) = 0$,

such that $a = \sum_{j\in\mathbb{N}} \lambda_j a_j$ and

$$\left(\sum_{j\in\mathbb{N}} |\lambda_j|^p\right)^{1/p} \leqslant C\|a\|_{H^p(\mathbb{Z})}. \tag{4}$$

When $0 < r \leqslant 1$, Lemma 3 and (3) assure that

$$\|T_{\alpha,\mu}a\|^r_{l^r(\mathbb{N})} \leqslant \sum_{j\in\mathbb{N}} |\lambda_j|^r \|T_{\alpha,\mu}a_j\|^r_{l^r(\mathbb{N})} \leqslant C\sum_{j\in\mathbb{N}} |\lambda_j|^r$$

for some $C > 0$. Furthermore, (3) also guarantees that $r > p$. Thus, (4) yields

$$\|T_{\alpha,\mu}a\|_{l^r(\mathbb{N})} \leqslant C\left(\sum_{j\in\mathbb{N}} |\lambda_j|^r\right)^{1/r} \leqslant C\left(\sum_{j\in\mathbb{N}} |\lambda_j|^p\right)^{1/p} = C\|a\|_{H^p(\mathbb{Z})}$$

for some $C > 0$.

When $r > 1$, we have

$$\|T_{\alpha,\mu}a\|_{l^r(\mathbb{N})} \leqslant \sum_{j\in\mathbb{N}} |\lambda_j| \|T_{\alpha,\mu}a_j\|_{l^r(\mathbb{N})} \leqslant C\sum_{j\in\mathbb{N}} |\lambda_j| \leqslant C\left(\sum_{j\in\mathbb{N}} |\lambda_j|^p\right)^{1/p} = C\|a\|_{H^p(\mathbb{Z})}$$

because $0 < p \leqslant 1$. □

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