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Stochastic orders and aging classes based on quantile residual life in a proportional context

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A B S T R A C T

A new proportional stochastic order is defined based on the concept of quantile residual life. In addition, a new aging class of life distributions, namely, decreasing proportional failure rate distributions, was defined. Moreover, two other aging classes based on the proposed stochastic ordering are defined and investigated: decreasing proportional α-quantile residual life and its dual increasing proportional α-quantile residual life. The relationship between the proportional failure rate and proportional α-quantile residual life has been discussed. Some well-known and useful models are classified in terms of the proportional α-quantile residual life function. Finally, a new modified Pareto distribution with a decreasing proportional failure rate and an increasing proportional α-quantile residual life function was presented and studied.

1. Introduction

The concept of stochastic orders has aroused great interest in the fields of reliability engineering and survival analysis. Based on various related concepts, different orders are defined and extended, for example, the distribution function, failure rate (FR) function, mean residual life (MRL) function, and α-quantile residual life (α-QRL) function. Meaningful relationships exist between the different orders defined by these concepts, which have been adequately addressed in the literature (Lai and Xie, 2006; Shaked and Shantikumar, 2007). Let T be a lifetime random variable with density function f_T and distribution function F_T. The simplest reliability measure, reliability function (RF), is defined by $R_T(t) = 1 - F_T(t)$. The conditional residual life, $T_t = T - t|T > t$, is the cornerstone of many concepts in reliability theory, e.g., the FR function which indicates the instantaneous risk of failure at time t, is the density function of T_t at zero,

$$\lambda(t) = f_{T_t}(0) = \frac{f_T(t)}{R_T(t)}, \quad t \geq 0. \tag{1}$$

In reliability engineering, survival analysis, and other related fields, examining the shape of the FR function of a data set can be very helpful for researchers to find an appropriate model. Some common shapes of the FR function include increasing, decreasing, bathtub, and unimodal (Lai and Xie, 2006). Another well-known measure of conditional residual life is the MRL

$$m(t) = E(T_t) = \frac{\int_t^\infty R_T(x)dx}{R_T(t)}, \quad t \geq 0. \tag{2}$$

The MRL function and its shape are closely linked to the FR function. For example, when FR increases (decreases), MRL decreases (increases). MRL is very useful in reliability theory and survival analysis, but for some models, it may not exist or researchers may face some shortcomings. For example, if the data are censored or the underlying distribution is skewed or heavily tail-heavy, the empirical MRL function cannot be calculated, or a single long-term survivor has an unusual influence (Schmittlein and Morrison, 1981; Lai and Xie, 2006; Lillo, 2005). In such situations, it is better to apply the α-QRL function or median residual life function, which represents the special case $\alpha = 0.5$. The α-QRL is defined as the α-quantile of T_t and can be expressed as follows:

$$q_\alpha(t) = R_T^{-1}(\overline{\alpha} R_T(t)) - t, \quad 0 < \alpha < 1, \, t \geq 0, \, \overline{\alpha} = 1 - \alpha. \tag{3}$$

In the context of reliability and survival analysis, $q_\alpha(t)$ indicates, among the devices that have survived to time t, the time at which we expect $(1 - \alpha)\%$ of the remaining devices to survive. Similar to other

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well-known reliability measures, the α-QRL function has been extensively studied in the literature. A class of lifetime distributions characterized by the decreasing (increasing) α-QRL function over time has been introduced and studied (Haines and Singpurwalla, 1984). The problem of determining the distribution by its α-QRL or median residual life function, the α-QRL function and the concept of α-QRL order have been investigated and studied (Arnold and Brockett, 1983; Gupta and Langford, 1984; Joe and Proschan, 1984a, 1984b). In addition, the life distribution characterized by its α-QRL functions and a Bayesian regression model for the median residual life span investigated (Joe, 1985; Gelfand and Kottas, 2003). The α-QRL function for a class of life distributions with the property of resetting the clock to zero was investigated (Rao et al., 2006). The Kaplan-Meier survival estimator used to estimate the median residual life in a nonparametric framework (Jeong et al., 2008) and the α-QRL measure considered to introduce and investigate the α-QRL order (Franco-Pereira et al., 2011). They investigated the relationship between FR and other orders of the α-QRL order. The α-QRL concept extended to include a multivariate context (Shafaei et al., 2018). On the other hand, the concept of proportional FR function and proportional orders was proposed and studied (Belzunce et al., 1995; Belzunce et al., 1998; Lariviere and Porteus, 2001; Ramos-Romero and Sordo-Diaz, 2001; Belzunce et al., 2002; Lariviere, 2006). In addition, the concept of proportional mean residual life (PMRL) order defined and discussed (Kayid et al., 2014). They demonstrated that the PMRL order is stronger than the MRL order.

Given the applicability and preference of the α-QRL function over the MRL function, this study aims to define a new family of stochastic orders based on a comparison of their α-QRL functions in a proportional framework. The rest of this study is organized as follows. Section 2 contains preliminary remarks on stochastic orderings and some necessary results. Section 3 proposes and investigates the new stochastic ordering, namely the α-proportional quantile residual lifetimes (α-PQRL). This order is indexed by $\alpha \in (0,1)$, and a well-known special case occurs when $\alpha = 0.5$, which refers to the proportional median residual life order. In addition, a new aging class of distributions with decreasing proportional FR (DPFR) is defined. Based on the new order, two new aging classes, decreasing proportional α-QRL (α-DPQRL) and increasing proportional α-QRL (α-IPQRL) are defined. The relationship between the PFR and the α-PQRL was investigated. Then, the behavior of the PFR and α-PQRL functions was explored for some known lifetime distributions. In Section 4, a new modified Pareto (MP) distribution in the DPFR class was introduced and discussed. Finally, Section 5 concludes the paper.

2. Stochastic and proportional orders

First, we need to check some predefined stochastic orderings and associated outcomes. Let $F_1(R_1)$ and $F_2(R_2)$ be the distribution functions (RFs) of T_1 and T_2, respectively. Then, T_1 is smaller than T_2 in the usual stochastic order, $T_1 \leq_{st} T_2$, if and only if

$$R_1(t) \leq R_2(t), \ \text{for all}\, t \in \mathrm{R}. \tag{4}$$

Equivalently, in the view of the inverse of the distribution function, $T_1 \leq_{st} T_2$, if and only if

$$F_1^{-1}(p) \leq F_2^{-1}(p), \ \text{for all}\, p \in (0,1). \tag{5}$$

As a stronger order, T_1 is said to be smaller than T_2 in the FR order, $T_1 \leq_{FR} T_2$, if and only if the fraction $R_2(t)/R_1(t)$ be an increasing function in t.

Let $T_{i,s} = T_i - s | T_i > s$ be the conditional residual life of T_i, $i = 1,2$. It can be checked that $T_1 \leq_{FR} T_2$ if and only if

$$T_{1,s} \leq_{st} T_{2,s}, \ \text{for all}\, s \in \mathrm{R},$$

which shows that the FR order is stronger than the ordinary stochastic order. Based on the MRL function, T_1 is said to be smaller than T_2 in the MRL order, $T_1 \leq_{MRL} T_2$, if and only if $m_1(t) \leq m_2(t)$, for all $t \geq 0$. It can be verified that $m_1(t) \leq m_2(t)$ if and only if

$\frac{\int_t^\infty R_2(u)du}{\int_t^\infty R_1(u)du}$, is increasing in t.

From this discussion, we see that the usual stochastic and MRL orders are weaker than the FR order, i.e.

$T_1 \leq_{FR} T_2 \rightarrow T_1 \leq_{st} T_2$ and $T_1 \leq_{MRL} T_2$.

The concept of proportional FR ordering was proposed (Belzunce et al., 1995). T_1 is smaller than T_2 in the proportional FR order, $T_1 \leq_{PFR} T_2$, if for each $x \in (0,1]$, $xT_1 \leq_{FR} T_2$. Furthermore, a random variable T is said to be increasing proportional failure rate (IPFR) if for each $x \in (0,1]$, $xT \leq_{FR} T$. This corresponds to the condition that $t\lambda(t)$ increases in t. The function $g(t) = t\lambda(t)$ is defined as a generalized FR function (Lariviere and Porteus, 2001).

The following definition is needed for stating some results related to proportional orders.

Definition 1. A function g is star-shaped if $g(pt) \leq pg(t)$, for all t and $p \in [0,1]$, or equivalently $g(t)/t$ is non-decreasing in t. Similarly, g is anti-starshaped if $g(pt) \geq pg(t)$, for all t and $p \in [0,1]$, or equivalently if $g(t)/t$ is non-increasing in t.

Using the MRL, T_1 is said to be smaller than T_2 in the PMRL order if for every $x \in (0,1]$, $xT_1 \leq_{MRL} T_2$. This ordering and some results in reliability engineering and renewal theory was demonstrated and the anti-starshaped MRL distributions was considered and discussed their relationship with the proposed ordering (Kayid et al., 2014).

From another point of view, T_1 is smaller than T_2 in the α-QRL order, $T_1 \leq_{\alpha-QRL} T_2$, if and only if $q_{1,\alpha}(t) \leq q_{2,\alpha}(t)$, for all $t \geq 0$. Franco-Pereira et al. [5] have discussed the relationship of this order to some other known orders. It is clear that $T_1 \leq_{\alpha-QRL} T_2$ for every $\alpha \in (0,1)$ if and only if there exists a $0 < \beta < 1$ such that $T_1 \leq_{\alpha-QRL} T_2$ for every $\alpha \in (0,\beta)$.

It is a well-known result that $T_1 \leq_{FR} T_2$ implies $T_1 \leq_{\alpha-QRL} T_2$, $\alpha \in (0,1)$. The following result shows that the inverse is also true under a mild condition.

Lemma 1. $T_1 \leq_{FR} T_2$ if and only if there exists a $0 < \beta < 1$ such that $T_1 \leq_{\alpha-QRL} T_2$ for every $\alpha \in (0,\beta)$.

3. The α-PQRL order

Assume that $q_{1,\alpha}(t)$ and $q_{2,\alpha}(t)$ stand for the α-QRL functions of random variables T_1 and T_2, respectively. Here, we propose a new class of stochastic orders.

Definition 2. T_1 is smaller than T_2 in α-PQRL order if $xT_1 \leq_{\alpha-QRL} T_2$, for all $x \in (0,1]$ or equivalently, $T_1 \leq_{\alpha-QRL} xT_2$, for all $x \in [1,\infty)$.

Definition 3. A random variable T is said to be α-DPQRL if $xT \leq_{\alpha-QRL} T$, for all $x \in (0,1]$ or equivalently, $T \leq_{\alpha-QRL} xT$, for all $x \geq 1$.

The IPFR has been defined and studied (Lariviere and Porteus, 2001). Here, we propose a new dual class for IPFRs, namely, the DPFR class of distributions.

Definition 4. A random variable T is said to be DPFR if $T \leq_{FR} xT$, for all $x \in (0,1]$ or equivalently, $xT \leq_{FR} T$, for all $x \in [1,\infty)$.

Definition 5. A random variable T is said to be α-IPQRL if $T \leq_{\alpha-QRL} xT$, for all $x \in (0,1]$ or equivalently, $xT \leq_{\alpha-QRL} T$, for all $x \in [1,\infty)$.

However, the DPFR and α −IPQRL classes seem strange at the first sight, we propose a MP model for these classes that might be useful for real-world data problems (see Section 4). It has been demonstrated that lifetime span T is an IPFR if and only if its PFR function is increasing (Lariviere, 2006). The following result shows that T is DPFR if, and only if its PFR function is decreasing.

Theorem 1. A random variable T is DPFR if and only if the PFR function $t\lambda(t)$ be decreasing in t.

Proof. *Let T be DPFR,* i.e.*, $T\leq_{FR}xT$, for every $x\in(0,1]$. It is equivalent to*.

$$\lambda(t)\geq\frac{1}{x}\lambda(t/x),\ \ t\geq0,x\in(0,1],$$

where $\lambda(t)$ is the FR of T. Multiplying both sides by t, we have

$$t\lambda(t)\geq\frac{t}{x}\lambda(t/x),\ \ t\geq0,x\in(0,1].$$

Now, let $t\leq s$ be two arbitrarry points. Then we take $x=t/s$, thus $t\lambda(t)\geq s\lambda(s)$ which shows that $t\lambda(t)$ is decreasing. The only if part follows in an inverse manner.

Definition 6. A random variable T is α-QRL starshaped (α-QRLS) if its α-QRL function is starshaped. Also, T is α-QRL anti-starshaped (α-QRLAS) if its α-QRL function is anti-starshaped.

The following result proves that for an α-IPQRL (α-DPQRL) random variable T, the function $q^*_\alpha(t)=q_\alpha(t)/t$ is increasing (decreasing) in t and vice versa. Thus we call it the α-proportional quantile residual life (α-PQRL) function. In the light of Definition 6, T is α-IPQRL (α-DPQRL) if and only if it is α-QRLS (α-QRLAS).

Theorem 2. The random variable T is:

(i). α-IPQRL if and only if the α-PQRL function $q^*_\alpha(t)$ be increasing in t.

(ii). α-DPQRL if and only if the α-PQRL function $q^*_\alpha(t)$ be decreasing in t.

Proof. (i). Assume that T is α-IPQRL, i.e., $T\leq_{\alpha-QRL}xT$ for every $x\in(0,1]$. Note that the α-QRL function of xT is $xq_\alpha(t/x)$ where $q_\alpha(t)$ is the α-QRL function of T. Thus,

$$xq_\alpha(t/x)\geq q_\alpha(t),$$

and by dividing both sides to t, we have

$$\frac{q_\alpha(t/x)}{(t/x)}\geq\frac{q_\alpha(t)}{t}.$$

Now, let $t<s$ be two arbitrarry points. Then, by taking $x=t/s$ we have

$$q_\alpha(s)/s\geq q_\alpha(t)/t,$$

which shows that $q^*_\alpha(t)$ is increasing. The proof of (ii) is similar.

Similar to other stochastic orderings, if T_1 and T_2 are lifetime random variables, the components with lifetime T_2 are more reliable than the components with lifetime T_1. If you fix $x=1$, $T_1\leq_{\alpha-PQRL}T_2$ implies $T_1\leq_{\alpha-QRL}T_2$. The following example shows that $T_1\leq_{\alpha-QRL}T_2$ does not imply that $T_1\leq_{\alpha-PQRL}T_2$. This implies that the α-PQRL order is stronger than the α-QRL order.

Example 1. Let T_1 and T_2 be two random variables with the median residual life functions $M_1(t)=I_{(0,2)}(t)(t^2+1)+I_{[2,\infty)}(t)5$ and $M_2(t)=t^2+1$, respectively. Note that $M_1(t)$ characterizes a class of distributions satisfying

$$2R_1(t^2+t+1)I_{(0,2)}(t)+2R_1(t+5)I_{[2,\infty)}(t)=R_1(t),\ \ t\geq0,$$

and $M_2(t)$ characterizes a class of distributions satisfying

$$2R_2(t^2+t+1)=R_2(t),\ \ t\geq0.$$

It is clear that $T_1\leq_{0.5-QRL}T_2$. Now we take $x=0.5$ and show that $xT_1\nleq_{0.5-QRL}T_2$. Let $M^*_1(t)$ be the median residual life function of $0.5T_1$. Then, $M^*_1(t)=I_{(0,1)}(t)\left(2t^2+\frac{1}{2}\right)+I_{[1,\infty)}(t)2.5$. Note that $M^*_1(0.9)=2.12>M_2(0.9)$, i.e., $xT_1\nleq_{0.5-QRL}T_2$. It shows that $T_1\nleq_{0.5-PQRL}T_2$. Similarly, we can show that neither T_1 nor T_2 are 0.5-DPQRL.

The following theorem illustrates the relationship between the IPFR and α-DPQRL classes. More precisely, IPFR is a subclass of α-DPQRL. It was found that the role of IPFR class in maximizing α-DPQRL and supply chain contract problems (Lariviere, 2006). The following conclusions can easily be drawn from Theorem 3 (Joe and Proschan, 1984), so the proof is omitted.

Theorem 3. *(i). If T is IPFR (DPFR) then for every α, it is α-DPQRL (α-IPQRL).*

(ii). IPFR (DPFR) if and only if it is α-DPQRL (α-IPQRL), for all $\alpha\in(0,\beta)$ for some $0<\beta<1$.

These results state that if the PFR function $t\lambda(t)$ be increasing (decreasing) then the α-PQRL function $q_\alpha(t)/t$ is decreasing (increasing).

The following result shows the relationship between PFR and α-PQRL orders.

Lemma 2. *(i). $T_1\leq_{PFR}T_2$ implies $T_1\leq_{\alpha-PQRL}T_2$, $\alpha\in(0,1)$.*

(ii). $T_1\leq_{PFR}T_2$ if and only if there exists a $0<\beta<1$ such that $T_1\leq_{\alpha-PQRL}T_2$ for every $\alpha\in(0,\beta)$.

Proof. In view of Corollary 3.6 (Franco-Pereira et al., 2011), the proof is straightforward.

The following theorem presents equivalent conditions for the proposed order.

Theorem 4. The following assertaions are equivalent:

(i). $T_1\leq_{\alpha-PQRL}T_2$.

(ii). $xq_{1,\alpha}(t)\leq q_{2,\alpha}(xt)$, for all $x\in(0,1]$.

(iii). $R_2(xR_1^{-1}(\overline{\alpha}u))\geq\overline{\alpha}R_2(xR_1^{-1}(u))$, for all $x\in(0,1]$.

Proof. The statement $T_1\leq_{\alpha-PQRL}T_2$ is equivalent to $xT_1\leq_{\alpha-QRL}T_2$, for each $x\in(0,1]$. The α-QRL function of xT_1 is $xq_{1,\alpha}(t/x)$. Thus, the condition (i) is equivalent to the following statement for every $x\in(0,1]$ and $t\geq0$.

$$xq_{1,\alpha}(t/x)\leq q_{2,\alpha}(t),$$

which results in the statement (i) by a simple transformation t/x to t. To show that (ii) and (iii) are the same, note that (ii) can be written as

$$x(R_1^{-1}(\overline{\alpha}R_1(t))-t)\leq R_2^{-1}(\overline{\alpha}R_2(xt))-xt,$$

which by taking R_2 from both sides of this inequality and assuming $R_1(t)=u$, we have

$$R_2(xR_1^{-1}(\overline{\alpha}u))\geq\overline{\alpha}R_2(xR_1^{-1}(u)),$$

which is (iii).

If the α-QRL function $q_\alpha(t)$ is decreasing, then clearly $q_\alpha(t)/t$ is decreasing, i.e., $q_\alpha(t)$ is anti-starshaped. Thus, the class of distributions with decreasing α-QRL functions is a subset of α-DPQRL or equivalently α-QRL anti-starshaped (α-QRLAS) distributions. The following theorem provides a more general equivalent condition for this class of distributions.

Theorem 5. T is α-DPQRL if and only if $x_1T\leq_{\alpha-QRL}x_2T$ for every $x_1\leq x_2$.

Proof. *Let $T_i=x_iT$, $i=1,2$. It is easy to check that the α-QRL of T_i is $q_{i,\alpha}(t)=x_iq_\alpha(t/x_i)$ where $q_\alpha(t)$ is the corresponding α-QRL of T. Thus $x_1T\leq_{\alpha-QRL}x_2T$ is equivalent to.*

$$x_1q_\alpha(t/x_1)\leq x_2q_\alpha(t/x_2)\ \ \text{for every t}\geq0.$$

Taking $x=x_1/x_2$ and $r=t/x_1$, it can be rewritten as

$$xq_\alpha(r)\leq q_\alpha(xr)\ \ \text{for every r}\geq0\,\text{and x}\in(0,1].$$

Appealing to Theorem 2, the result follows. □

The following result provides a sufficient condition under which $T_1\leq_{\alpha-QRL}T_2$ implies $T_1\leq_{\alpha-PQRL}T_2$.

Theorem 6. If $T_1\leq_{\alpha-QRL}T_2$ and T_1 or T_2 be α-QRLAS, then $T_1\leq_{\alpha-PQRL}T_2$.

Proof. *Let T_1 be α-QRLAS, then we have.*

$$xT_1 \leq_{\alpha-QRL} T_1 \leq_{\alpha-QRL} T_2, \quad \text{for every } 0 < x \leq 1,$$

which by transitivity of the α-QRL order, it results that $xT_1 \leq_{\alpha-QRL} T_2$, for every $0 < x \leq 1$.

Let T_2 be α-QRLAS, then

$$T_1 \leq_{\alpha-QRL} T_2 \leq_{\alpha-QRL} xT_2, \quad \text{for every } x \geq 1,$$

which implies that $T_1 \leq_{\alpha-QRL} xT_2$ for every $x \geq 1$, i.e., $T_1 \leq_{\alpha-PQRL} T_2$. □

The following theorem provides another necessary and sufficient condition for α-QRLAS class by applying a random coefficient.

Theorem 7. A random variable T is α-QRLAS if and only if for every random variable X with support $(0,1]$ which is independent of T, $XT \leq_{\alpha-QRL} T$.

Proof. *Let T be α-QRLAS. Then for every $x \in (0,1]$, $xT \leq_{\alpha-QRL} T$, or equivalently.*

$$R_Z^{-1}(\overline{\alpha} R_Z(t)) \leq R_T^{-1}(\overline{\alpha} R_T(t)), \quad \text{for every } x \in (0,1], \tag{6}$$

where $Z = xT$. We show that for every random variable X independent of T, we have $XT \leq_{\alpha-QRL} T$, i.e.,

$$R_Y^{-1}(\overline{\alpha} R_Y(t)) \leq R_T^{-1}(\overline{\alpha} R_T(t)), \tag{7}$$

where $Y = XT$. The inequality (6) can be rewritten as in the following.

$$R_Z^{-1}(\overline{\alpha} R_Z(t)) = \inf\{s : R_T(s/x) \leq \overline{\alpha} R_T(t/x)\} \leq C_t, \quad \text{for every } x \in (0,1], \tag{8}$$

where C_t is constant with respect to x. Note that $R_Y(y) = \int_0^1 R_T(y/x) f_X(x) dx$ where f_X shows the density function of X. On the other hand, the inequality (7) is equivalent to the following:

$$R_Y^{-1}(\overline{\alpha} R_Y(t)) = \inf\{s : R_Y(s) \leq \overline{\alpha} R_Y(t)\} \tag{9}$$

$$= \inf\left\{s : \int_0^1 R_T(s/x) f_X(x) dx \leq \overline{\alpha} \int_0^1 R_T(t/x) f_X(x) dx\right\} \leq C_t. \tag{10}$$

Let $\overline{Q} = \sup_{0<x\leq 1} R_Z^{-1}(\overline{\alpha} R_Z(t))$. Then it is clear that $R_Y^{-1}(\overline{\alpha} R_Y(t)) \leq \overline{Q}$. It shows that $R_Y^{-1}(\overline{\alpha} R_Y(t)) \leq C_t$ as claimed.

Now, suppose that $XT \leq_{\alpha-QRL} T$ for every random variable X independent of T. For every $x \in (0,1]$, take X be degenerate at x, i.e., $P(X = x) = 1$. It implies that $xT \leq_{\alpha-QRL} T$ for every $x \in (0,1]$ and in turn T is α-QRLAS.

4. Reliability models

In this section, some well-known distributions that are used extensively in reliability theory, survival analysis and many other scientific fields are considered and the behavior of the PFR and α-PQRL functions are examined. Assume T from the Weibull distribution with the RF

$$R(t) = \exp\{-(bt)^a\}, \quad a > 0, \ b > 0, \ t \geq 0.$$

Then, the PFR function is

$$t\lambda(t) = ab^a t^a, \quad a > 0, \ b > 0, \ t \geq 0,$$

which clearly is increasing in t. The α-PQRL function is

$$q_\alpha(t) \Big/ t = (-\ln\overline{\alpha} + (bt)^a / bt^a)^{\frac{1}{a}} - 1, \quad a > 0, \ b > 0, \ t \geq 0,$$

and decreases in t, for all parameter values, so T is α-DPQRL and, by Theorem 2, α-QRLAS.

The gamma distribution is determined by the density function

$$f(t) = \frac{b^a}{\Gamma(a)} t^{a-1} e^{-bt}, \quad a > 0, b > 0, t \geq 0.$$

The PFR function of the gamma model could be written as in the following:

$$t\lambda(t) = \left(\int_0^\infty \frac{1}{t}\left(1 + \frac{u}{t}\right)^{a-1} e^{-bu} du\right)^{-1},$$

which is increasing in t, for all parameter values. This shows that, by Theorem 3, the gamma model is α-DPQRL.

Consider the inverse Weibull model with the distribution function

$$F(t) = \exp(-\beta t^{-\alpha}), \quad t \geq 0.$$

Its PFR function is

$$t\lambda(t) = \alpha\beta t^{-\alpha}[\exp(\beta t^{-\alpha}) - 1]^{-1}, \quad t \geq 0,$$

that is increasing in t, for all parameter values. Thus the inverse Weibull model is IPFR and hence α-DPQRL. On the other hand, the truncated normal distribution with the density function

$$f(t) = \frac{1}{a\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(t-\mu)^2\right), \quad \mu \in R, \sigma > 0, t \geq 0,$$

is IFR and IPFR and in turn α-DPQRL.

In addition, the Lognormal distribution is defined by the following density function:

$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(\ln t - \mu)^2\right), \quad \mu \in \mathrm{R}, \sigma > 0, t \geq 0.$$

The PFR is

$$t\lambda(t) = \frac{1}{\sqrt{2\pi}\sigma} \frac{\exp\left(-\frac{1}{2\sigma^2}(\ln at)^2\right)}{1 - \Phi\left(\frac{\ln at}{\sigma}\right)},$$

where $a = \exp(-\mu)$. To show that this function has the form of an upside down bathtub shaped (unimodal), we write the PFR function as follows:

$$t\lambda(t) = \frac{\exp\left(-\frac{1}{2}t^2\right)}{\int_t^\infty \exp\left(-\frac{1}{2}z^2\right) dz},$$

where $y = \ln(at)/\sigma$. By some straightforward algebra, it could be checked that $t\lambda(t)$ is upside down bathtub shaped.

The Gompertz distribution is identified with the RF

$$R(t) = \exp(-\beta(c^t - 1)), \quad \beta > 0, c > 1, t \geq 0.$$

Then, the PFR function of the Gompertz model is

$$t\lambda(t) = \ln(c)\beta t c^t, \quad \beta > 0, c > 1, t \geq 0,$$

which is increasing in t. Thus, the Gompertz model is also has the α-DPQRL property.

The Lomax distribution is defined by the RF

$$R(t) = (1 + \beta t)^{-\alpha}, \quad \alpha > 0, \beta > 0, t \geq 0.$$

The PFR function is

$$t\lambda(t) = \frac{\alpha\beta t}{1 + \beta t}, \quad \alpha > 0, \beta > 0, t \geq 0.$$

which is an increasing function for all parameter values, so the Lomax model is α-DPQRL.

The Log-logistic distribution is recognized by the RF

$$R(t) = \frac{1}{1 + (at)^b}, \quad a > 0, b > 0, t \geq 0.$$

The PFR function is

$$t\lambda(t) = b\frac{a^b t^b}{1 + a^b t^b}, \quad a > 0, b > 0, t \geq 0,$$

which is an increasing function for all parameter values and it follows that the Log-logistic distribution is α-DPQRL.

Let T follows the Pareto model with the reliability

$$R(t) = (c/t)^a, \quad c > 0, a > 0, t \geq c. \tag{11}$$

Then, the PFR function is

$$t\lambda(t) = a, \quad c > 0, a > 0, t \geq c,$$

which is constant in the support. In addition, the α-PQRL function is constant and equals to

$$q_\alpha(t)/t = \overline{\alpha}^{-\frac{1}{a}} - 1, \quad c > 0, a > 0, t \geq c.$$

It could be checked that the constant PFR function characterizes this Pareto model. This point hints that there may be a modified version of Pareto exhibiting decreasing PFR function.

In the rest of this section, we propose a new MP model which is characterized by the RF

$$R(t) = \left(\frac{c}{bt}b^{\frac{c}{t}}\right)^a, \quad t \geq c, \ a > 0, \ b > 1, \ c > 0. \tag{12}$$

Note that, for $b = 1$, the proposed model reduces to the Pareto reliability model (11). The probability density function (PDF) and the PFR function of the proposed MP model are respectively as follows:

$$f(t) = \left(\frac{c}{b}\right)^a a t^{-a} b^{\frac{ac}{t}}\left(\frac{1}{t} + \frac{c\ln b}{t^2}\right), \quad t \geq c, \ a > 0, \ b > 1, \ c > 0,$$

and

$$t\lambda(t) = a\left(1 + \frac{c\ln b}{t}\right), \quad t \geq c,$$

which is a decreasing function, i.e., the model is DPFR.

Fig. 1 shows the density and FR functions of the MP model (12) for some parameter values. The PFR and proportional median residual life functions are shown in Fig. 2, showing a decreasing shape for the PFR and an increasing shape for the proportional median residual life functions.

Let T follows from the proposed MP model (12). Then, the k th moment is

$$E\left(T^k\right) = c^k R(c) + \int_c^\infty kt^{k-1}R(t)dt.$$

It can be shown that when $k \geq a$, $E\left(T^k\right)$ is infinite. But, for $k < a$, it is finite and we have the following upper bound for it.

$$E\left(T^k\right) < c^k R(c) + \frac{kc^k}{a-k}.$$

Specially, $E(T)$ is infinite for $a \leq 1$ and finite for $a > 1$.

The quantile at point p of the MP model can be computed numerically by solving the equation $R(t) = 1 - p$, which simplifies to solving the following equation in terms of $t \geq c$.

$$\frac{c}{t}b^{\frac{c}{t}} - 1 - p = 0,$$

which has not algebraic solution. The α-PQRL function of the MP model is

$$q_\alpha^*(t) = \frac{1}{t}R^{-1}(\overline{\alpha}R(t)) - 1 \quad t \geq c, a > 0, b > 1, c > 0,$$

which could be computed numerically. Since the MP model is DPFR, by Theorem 7, it is α-IPQRL, i.e., $q_\alpha^*(t)$ is an increasing function.

4.1. Simulations study of MP

For generating a sample from the MP distribution, one random sample of the standard uniform distribution is simulated. Then the sample of MP is computed by solving the equation $F(X) = U$. In this simulation study, some parameter valus are selected. Then, in every run $r = 1000$ replicates of samples of sizes $n = 50$ or 100 are generated. The maximum likelihood estimates of the parameters are computed for every sample and finally the bias (B) and mean squared error (MSE) are calculated. Every cell of Table 1 shows one run. The maximum likelihood estimates are computed by optimizing the log-likelihood function and applying the "optim" function with the default method "Nelder-Mead" in R. The initial values applied in the optimization process are taken randomly from the uniform distribution. The simulation results are reported in Table 1 and clearly indicate that the estimator is consistent.

4.2. Applications

Table 2 shows intervals between successive failures of the air conditioning system of Boeing 720 jet airplanes, reported and described (Proschan, 1963). The maximum likelihood estimates of the parameters of MP model and three alternatives Pareto, gamma and Weibull are calculated and abstracted in Table 3. Also, the Akaike information criterion (AIC), the Kolmogorov Smirnov (KS) and Cramer Von Mises (CVM) statistics are computed. The p-value of the KS and CVM for PM

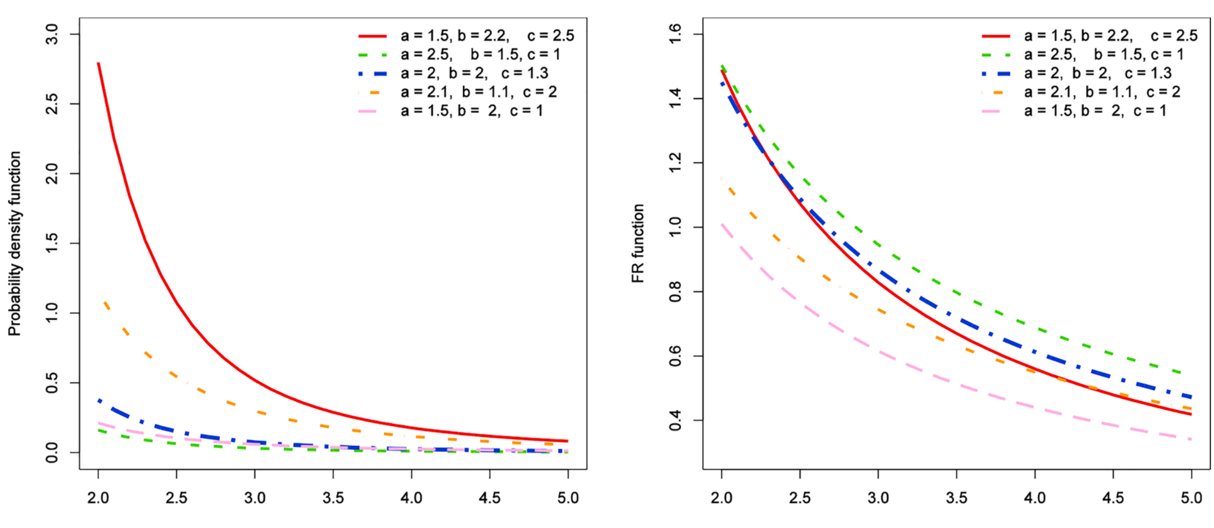

Fig. 1. The density and FR function of the MP model for some values of parameters.

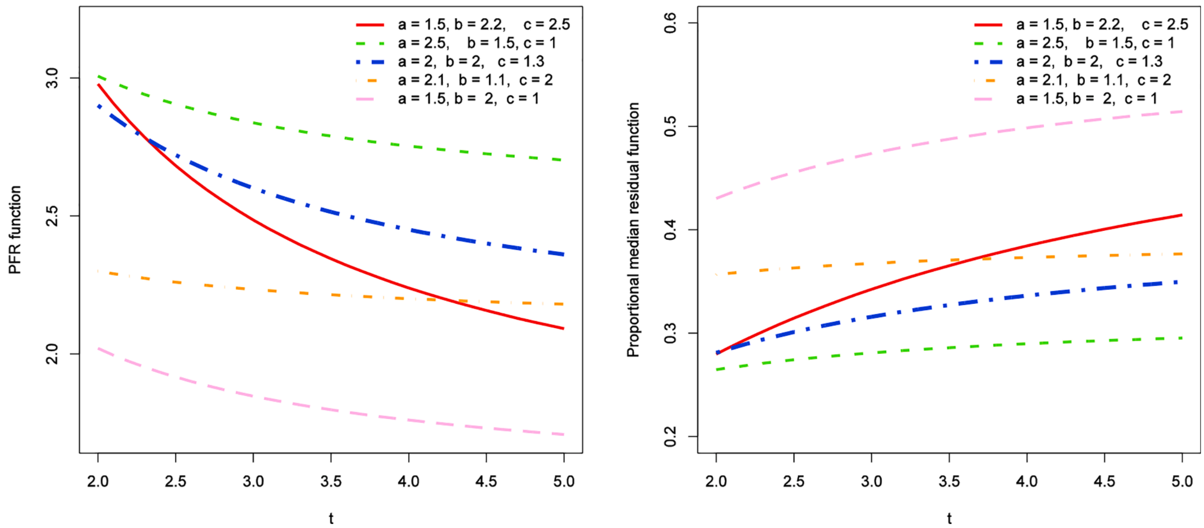

Fig. 2. The PFR and median residual life function of the MP model for some values of parameters.

Table 1
The simulation results for estimation the parameters of the MP model. In every cell, the first, second and third lines are related to parameters a, b and c respectively.

Parameters	n			
	50		100	
	B	MSE	B	MSE
$a = 0.9, b = 1.05, c = 2$	−0.0368	0.0321	−0.0326	0.0168
	0.6685	1.7411	0.2417	0.4767
	0.0442	0.0043	0.0219	0.0009
$a = 1.5, b = 1.02, c = 3$	−0.1086	0.1142	−0.0884	0.0658
	0.6174	3.0881	0.3180	0.6736
	0.0412	0.0034	0.0197	0.0008
$a = 2, b = 1.01, c = 1$	−0.1604	0.2401	−0.1658	0.1392
	0.7042	4.7020	0.3678	0.8959
	0.0102	0.0002	0.0047	0.00004
$a = 1, b = 1.07, c = 1.5$	−0.0470	0.0419	−0.0401	0.0215
	0.7156	3.1275	0.2752	0.5649
	0.0276	0.0016	0.0143	0.0004

Table 2
Interval between failures.

47	57	48	29	502	12	70	21	29	386
59	27	153	26	326					

Table 3
Comparing the proposed MP model with alternatives.

Model	$\hat{a}$	$\hat{b}$	$\hat{c}$	AIC	K-S p-value	CVM p-value
MP	0.5382	1.0699	11.99	176.29	0.2199	0.1315
					0.4627	0.4554
Pareto	0.5979	—	12	176.85	0.2368	0.1581
					0.3692	0.3687
Gamma	0.9109	0.0075	—	177.85	0.2617	0.1866
					0.2555	0.2969
Weibull	0.8884	0.0149	—	177.53	0.2374	0.1497
					0.3662	0.3938

model shows relatively large values and based on these statistics, the PM outperforms the Pareto, gamma and Weibull models. The empirical distribution function and the estimated distribution functions of MP, Pareto, gamma and Weibull are plotted in Fig. 3and graphically indicates a good fit for MP and Pareto.

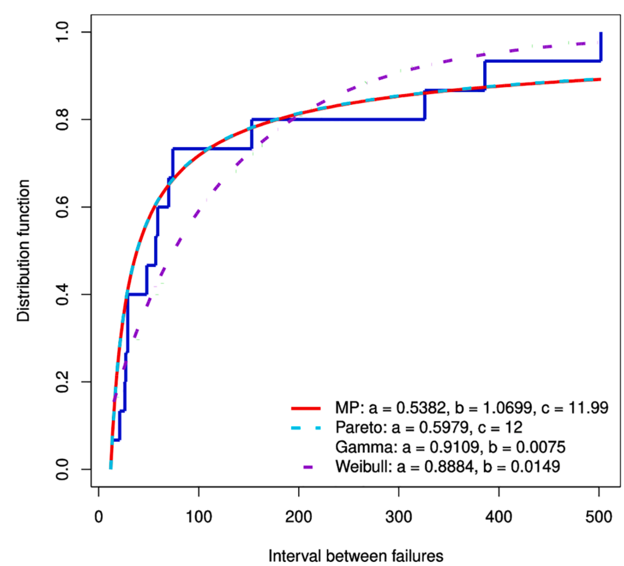

Fig. 3. The empirical distribution function of the interval between failures data and the estimated disribution functions of considered models.

5. Conclusion

The proposed α-PQRL order can be applied to extend the aging classes of the distributions. Based on this ordering, two aging classes α-DPQRL and α-IPQRL are defined and analyzed. In addition, a subset of α-IPQRL, namely the DPFR class of distributions, is defined and discussed. As investigated, many of the known lifetime models belong to the IPFR and α-DPQRL class. We have proposed a new lifetime model in the DPFR and α-IPQRL class to show that this class of distributions could be very useful in reliability theory, survival analysis, and other related fields. The preservation properties of the proposed stochastic ordering and aging classes as well as a family of exponentiality tests for these new classes are interesting topics in reliability engineering that are still open problems.

CRediT authorship contribution statement

Mohamed Kayid: Writing – review & editing, Writing – original draft, Visualization, Supervision, Project administration, Investigation, Formal analysis, Conceptualization. **Nassr S. Al-Maflehi:** Writing –

original draft, Visualization, Software, Methodology, Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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