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On the global convergence of a fast Halley's family to solve nonlinear equations

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ABSTRACT

The purpose of this paper is to suggest an approach for increasing the convergence speed of Halley's method to solve a non-linear equation. This approach is based on the second order Taylor polynomial and on Halley's formula. By applying it a certain number of times, we obtain a new family of methods. The originality of this family is manifested in the fact that all its sequences are generated from one exceptional formula that depends on a natural integer parameter p. In addition, under certain conditions, the convergence speed of its sequences increases with p. The convergence analysis shows that the order of convergence of all proposed methods is three. A study on their global convergence is carried out. To illustrate the performance of this family, several numerical comparisons are made with other third and higher order methods.

1. Introduction

One of the most encountered problems in science and engineering is solving nonlinear equation

$$f(x) = 0 \tag{1}$$

where f is a real analytic function. One of the best ways to approximate a simple solution α of Eq. (1) is to use a fixed-point method. In this method, we find another function$F$, called an iteration function (I.F) for f, and from an initial value x_0 (Traub, 1964), we define a sequence

$$x_{n+1} = F(x_n) \quad for \;\; n = 0, 1, 2\cdots \tag{2}$$

The second order Newton's method (Traub, 1964) is also well known. In order to increase the convergence speed, new algorithms have been developed: Halley, super-Halley, Chebyshev, Euler, Chun, Sharma (Sharma et al. (2012)) and Dubeau (2013) have proposed some third order methods. Ghanbari (2011), Fang et al. (2008) Solaiman and Hashim (2019), Noor et al. (2007), Chun and Ham (2007), Kou and Li (2007), Wang and Zhang (2014), proposed families of higher-order methods. Zhou and Zhang (2020) have constructed some interesting algorithms with variable convergence rate ((1 + 2p)-order). Zhang (2020) has recently elaborated a fully derivative-free conjugate residual method, using secant condition.

In this paper, based on Halley's method and Taylor polynomial, we construct an interesting family to find simple roots of nonlinear equations with cubical convergence. The originality of this family is manifested in its special formula which depends on a natural integer parameter p, and in the augmentation of the convergence speed of its sequences with the increase in p, if some hypotheses are satisfied.

The rest of this article is organized as follow: Section 2 features the family's derivation of Halley's method, Section 3 provides the convergence study of the new methods, the advantages of the new family of this article are presented in Section 4. The numerical results of this work are provided in Section 5 while the last section gives our conclusion.

2. Family's derivation of Halley's method

Among the famous third order methods, we quote Halley's method$B0$, given by

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$$\begin{cases} x_{n+1}^0 = x_n - \frac{f(x_n)}{f'(x_n)} W_0(L_n) \\ W_0(L_n) = \frac{1}{1-\frac{1}{2}L_n} \\ L_n = L_f(x_n) = \frac{f(x_n)f''(x_n)}{f'(x_n)^2} \end{cases} \quad (3)$$

where L_n is the degree of logarithmic convexity of f at x_n (Hernández, 1991).

The second-order Taylor polynomial of f at x_n is given by:

$$y(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{f''(x_n)}{2}(x - x_n)^2$$

The goal is to find a point $(x_{n+1}, 0)$, where the curve of y passes through the x_axis (Scavo and Thoo, 1995), which is the solution of

$$0 = f(x_n) + (x_{n+1} - x_n)(f'(x_n) + \frac{f''(x_n)}{2}(x_{n+1} - x_n))$$

simplifying the above yields

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + \frac{f''(x_n)}{2}(x_{n+1} - x_n)} \quad \text{for } n = 0, 1, 2 \cdots \quad (4)$$

Eq. (4) is an implicit scheme because it does not allow to directly explain x_{n+1} as a function of x_n. In order to make it explicit, we replace x_{n+1} placed on the right side of (4) by Halley's method $B0$ (3), we get the Super-Halley's method $B1$:

$$x_{n+1}^1 = x_n - \frac{f(x_n)}{f'(x_n) + \frac{f''(x_n)}{2}(x_{n+1}^0 - x_n)} = x_n - \frac{f(x_n)}{f'(x_n)} W_1(L_n), \quad n \in \mathbb{N} \quad (5)$$

where $W_1(L_n) = \frac{1-\frac{1}{2}L_n}{1-L_n}$

By repeating the above procedure p times and each time replace $(x_{n+1} - x_n)$ located on the right side of (4) with the last correction found, we derive a following general family of Halley's method $\{Bp\}$:

$$x_{n+1}^p = x_n - \frac{f(x_n)}{f'(x_n) + \frac{f''(x_n)}{2}\left(x_{n+1}^{p-1} - x_n\right)} \quad (6)$$

where x_{n+1}^0 is given by (3), and p is a non-zero natural integer parameter.

Theorem 1. *Let p be a natural integer parameter and f a real function sufficiently smooth in some neighborhood of zero, α. The family of Halley's method $\{Bp\}$, defined by the sequences (6), can be expressed in the following form:*

$$x_{n+1}^p = x_n - W_p(L_n)\frac{f(x_n)}{f'(x_n)} \quad (7)$$

where $L_n = \frac{f(x_n)f''(x_n)}{f'(x_n)^2}$ and $\begin{cases} W_p(L_n) = \frac{T_p(L_n)}{T_{p+1}(L_n)} \\ T_0(L_n) = 1 \text{ and } T_1(L_n) = 1 - \frac{L_n}{2} n \in \mathbb{N} \\ T_{p+2}(L_n) = T_{p+1}(L_n) - \frac{L_n}{2}T_p(L_n) \end{cases}$

Proof. *Let $n \in \mathbb{N}$, $(v_p)_{p\in\mathbb{N}}$ and $(v'_p)_{p\in\mathbb{N}}$ be defined by the sequences $\left\{x_{n+1}^p\right\}$ given by (6) and (7) respectively. We will prove by induction that, for all $p \in \mathbb{N}$, $v'_p = v_p$.*

If$p = 1$, the formula (6) leads to the (5) one given by:

$v_1 = x_{n+1}^1 = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{1-\frac{1}{2}L_n}{1-L_n}\right)$

Furthermore, according to (7), we have

$T_1(L_n) = 1 - \frac{L_n}{2}$ and $T_2(L_n) = 1 - L_n$ then

$$v'_1 = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{1 - \frac{1}{2}L_n}{1 - L_n}\right)$$

So

$$v'_1 = v_1$$

Now, we assume that, for a given p, we have $v'_p = v_p$, we will show that $v'_{p+1} = v_{p+1}$.

From (7), we have:

$$v'_{p+1} = x_{n+1}^{p+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{T_{p+1}(L_n)}{T_{p+2}(L_n)}\right)$$

where

$$T_{p+2}(L_n) = T_{p+1}(L_n) - \frac{L_n}{2}T_p(L_n)$$

and, from (6), we have:

$$v_{p+1} = x_n - \frac{f(x_n)}{f'(x_n) + \frac{f''(x_n)}{2}(x_{n+1}^p - x_n)} = x_n - \frac{f(x_n)}{f'(x_n) + \frac{f''(x_n)}{2}(v_p - x_n)}$$

As

$$v_p = v'_p = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{T_p(L_n)}{T_{p+1}(L_n)}\right)$$

then

$$\begin{aligned} v_{p+1} &= x_n - \frac{f(x_n)}{f'(x_n) - \frac{f''(x_n)f(x_n)}{2f'(x_n)}\left(\frac{T_p(L_n)}{T_{p+1}(L_n)}\right)} \\ &= x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{T_{p+1}(L_n)}{T_{p+1}(L_n) - \frac{L_n}{2}T_p(L_n)}\right) \end{aligned}$$

So

$$v_{p+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{T_{p+1}(L_n)}{T_{p+2}(L_n)}\right)$$

where

$$T_{p+2}(L_n) = T_{p+1}(L_n) - \frac{L_n}{2}T_p(L_n)$$

Consequently $v_{p+1} = v'_{p+1}$ and the induction is completed.

Now, let's try to find the general expression of the polynomial T_p as a function of L_n.

From (7), we obtain

$$T_p(L_n) = \sum_{k=0}^{\left[\frac{p+1}{2}\right]} R_k^p.(L_n)^k$$

where $[x]$ is integer part of x, and

$$\begin{cases} R_0^p = 1 \quad \text{for } p \geq 0 \\ R_1^p = \frac{-p}{2} \quad \text{for } p \geq 1 \\ R_{\left[\frac{p+1}{2}\right]}^{2\left[\frac{p-1}{2}\right]} = 0 \text{ and } R_{\left[\frac{p+1}{2}\right]}^{2\left[\frac{p+1}{2}\right]-1} = \left(\frac{-1}{2}\right)^{\left[\frac{p+1}{2}\right]} \quad \text{for } p \geq 1 \\ R_k^p - R_k^{p-1} = -\frac{1}{2}R_{k-1}^{p-2} \quad \text{for } p \geq 3 \text{ and } 2 \leq k \leq \left[\frac{p+1}{2}\right] \end{cases} \quad (8)$$

Thus,

$$\text{for } k = 0, R_0^p = 1 = \frac{(-1)^0}{2^0}.\frac{(p+1)!}{(p+1)!0!}$$

$$\text{for } k = 1, \ R_1^p = \frac{-p}{2} = \frac{(-1)^1}{2^1}.\frac{p!}{(p-1)!1!}$$

$$\text{for } k = 2, \begin{cases} R_2^p - R_2^{p-1} = -\frac{1}{2}R_1^{p-2} \\ R_2^{p-1} - R_2^{p-2} = -\frac{1}{2}R_1^{p-3} \\ \vdots \\ R_2^4 - R_2^3 = -\frac{1}{2}R_1^2 \end{cases}$$

We deduce that

$$R_2^p = R_2^3 - \frac{1}{2}\left[R_1^2 + \cdots + R_1^{p-2}\right]$$

knowing that

$$R_{\left[\frac{p+1}{2}\right]}^{2\left[\frac{p+1}{2}\right]-1} = \left(\frac{-1}{2}\right)^{\left[\frac{p+1}{2}\right]}$$

for $p = 3$, we find $R_2^3 = \frac{1}{4}$, so

$$R_2^p = \frac{1}{4}\left[1 + \sum_{i=2}^{p-2}(i)\right] = \frac{(-1)^2}{2^2}\cdot\frac{(p-2)(p-1)}{2!}$$
$$= \frac{(-1)^2}{2^2}\cdot\frac{(p-1)!}{(p-3)!2!} \quad \text{for } p \geq 3 \tag{9}$$

We admit that:

$$\sum_{k=1}^{n} k(k+1)\cdots(k+i) = \frac{n(n+1)\cdots(n+i)(n+i+1)}{i+2} \quad \text{for } i \geq 0 \tag{10}$$

By using (8)–(10) and by applying the same method, we obtain: For $k = 3$ and $p \geq 5$,

$$R_3^p = R_3^5 - \frac{1}{2}\left[R_2^4 + \cdots + R_2^{p-2}\right] = \frac{(-1)^3}{2^3}\cdot\frac{(p-4)(p-3)(p-2)}{3!}$$
$$= \frac{(-1)^3}{2^3}\cdot\frac{(p-2)!}{(p-5)!3!}$$

For $k = 4$ and $p \geq 7$

$$R_4^p = R_4^7 - \frac{1}{2}\left[R_3^6 + \cdots + R_3^{p-2}\right] = \frac{(-1)^4}{2^4}\cdot\frac{(p-6)(p-5)(p-4)(p-3)}{3!}$$
$$= \frac{(-1)^4}{2^4}\cdot\frac{(p-3)!}{(p-7)!4!}$$

and by conjecture, we obtain for $p \geq 2k - 1$:

$$R_k^p = \frac{(-1)^k(p-k+1)!}{2^k(p-2k+1)!k!}$$

Corollary 1. *Let p be a natural integer parameter and f a real function sufficiently smooth in some neighborhood of zero,α. The family of Halley's method {Bp}, defined by the sequences (7), can be expressed in the following explicit form:*

$$x_{n+1}^p = x_n - W_p(L_n)\frac{f(x_n)}{f'(x_n)} \quad \text{for } n \in \mathbb{N} \tag{11}$$

where $\begin{cases} W_p(L_n) = \frac{T_p(L_n)}{T_{p+1}(L_n)} \\ T_p(L_n) = \sum_{k=0}^{\left[\frac{p+1}{2}\right]} R_k^p (L_n)^k \\ R_k^p = \frac{(-1)^k(p-k+1)!}{2^k(p-2k+1)!k!} \end{cases}$ and L_n is defined in (3).

Proof. *We will prove, by induction, that (7) is expressed by (11) for all* $p \in \mathbb{N}$.

From (7), we have

$$T_0(L_n) = 1 \text{ and } T_1(L_n) = 1 - \frac{L_n}{2}.$$

From (11), we obtain the same result:

$$T_0(L_n) = R_0^0 = 1 \text{ and } T_1(L_n) = R_0^1 + R_1^1 L_n = 1 - \frac{L_n}{2}$$

Now, we assume that, for a given p, $T_p(L_n)$ and $T_{p+1}(L_n)$ given by (11) is equal to the one defined by (7). We will show that $T_{p+2}(L_n)$ is also the same.

From (7), we have:

$$T_{p+2}(L_n) = T_{p+1}(L_n) - \frac{L_n}{2}T_p(L_n)$$

Furthermore, according (11), we have

$$T_{p+1}(L_n) - T_{p+2}(L_n) = \sum_{k=0}^{\left[\frac{p+2}{2}\right]} R_k^{p+1}(L_n)^k - \sum_{k=0}^{\left[\frac{p+3}{2}\right]} R_k^{p+2}(L_n)^k$$

If p is even, we have

$$\left[\frac{p+3}{2}\right] = \left[\frac{p+2}{2}\right] \text{and } R_0^{p+1} = R_0^{p+2}$$

So

$$T_{p+1}(L_n) - T_{p+2}(L_n) = \sum_{k=1}^{\left[\frac{p+2}{2}\right]}\left(R_k^{p+1} - R_k^{p+2}\right)(L_n)^k$$
$$= L_n \sum_{k=0}^{\left[\frac{p}{2}\right]}\left(R_{k+1}^{p+1} - R_{k+1}^{p+2}\right)(L_n)^k$$

As $R_{k+1}^{p+1} - R_{k+1}^{p+2} = \frac{1}{2}R_k^p$ and $\left[\frac{p}{2}\right] = \left[\frac{p+1}{2}\right]$, then

$$T_{p+1}(L_n) - T_{p+2}(L_n) = \frac{L_n}{2}\sum_{k=0}^{\left[\frac{p+1}{2}\right]} R_k^p (L_n)^k$$

Consequently

$$T_{p+2}(L_n) = T_{p+1}(L_n) - \frac{L_n}{2}T_p(L_n)$$

The case p odd is analogous. We conclude, by induction, that (7) is expressed by (11) for all $p \in \mathbb{N}$.

The scheme (11) is powerful because it regenerates the Halley's method $B0$, the Super-Halley method $B1$, and several new methods such as $B2$ and $B8$ given by

$$x_{n+1}^2 = x_n + \frac{f(x_n)}{f'(x_n)}\left(\frac{4L_n - 4}{L_n^2 - 6L_n + 4}\right) \tag{12}$$

$$x_{n+1}^8 = x_n - \frac{f(x_n)}{f'(x_n)}\left(\frac{16 - 56L_n + 60L_n^2 - 20L_n^3 + L_n^4}{16 - 64L_n + 84L_n^2 - 40L_n^3 + 5L_n^4}\right) \tag{13}$$

3. Convergence study of new methods

3.1. Order of convergence

Theorem 2. *Let $\boldsymbol{f}$ be a real function. Assuming that $\boldsymbol{f}$ is sufficiently smooth in some neighborhood of a simple zero $\boldsymbol{\alpha}$. Further, assume that the initial value $\boldsymbol{x}_0$ is sufficiently close to $\boldsymbol{\alpha}$. Then the iteration process defined by (11) converges cubically to $\boldsymbol{\alpha}$.*

Proof. *According to Sharma et al. (2012), the iteration process defined by:*

$$x_{n+1} = x_n - W(L_n)\frac{f(x_n)}{f'(x_n)}$$

converges cubically to α, provided that

$$W(0) = 1, W'(0) = \frac{1}{2} \text{ and} |W''(0)| < \infty.$$

From (11), for all $p \in \mathbb{N}$, we have

$$T_p(0) = 1, T_p^{'}(0) = -\frac{p}{2}, T_0^{''}(0) = 0$$

and for all$p \in \mathbb{N}^*$:

$$T_p^{''}(0) = \frac{(p-1)(p-2)}{4}$$

So

$$\begin{cases} For\ all\ p \in \mathbb{N} & W_p(0) = 1 \quad and \quad W_p^{'}(0) = \frac{1}{2} \\ For\ all\ p \in \mathbb{N}^* & W_p^{''}(0) = 1 \quad and \quad W_0^{''}(0) = \frac{1}{2} \end{cases}$$

Consequently, the sequences defined by (11) are **cubically** convergent for all $p \in \mathbb{N}$.

3.2. Global convergence

The following lemmas will be useful for the future.

3.2.1. Important lemmas

Lemma 1. *Let $p \in \mathrm{IN}^*$. We assume that the polynomial T_p, defined in (11), admits real roots of which b_p is the smallest. Then, the root b_p is strictly positive and the polynomial T_p is strictly positive over the interval $(-\infty, b_p)$.*

Proof. *Let $p \in \mathbb{N}^*$, we have*

$$T_p(x) = \sum_{k=0}^{\left[\frac{p+1}{2}\right]} R_k^p x^k = \sum_{k=0}^{\left[\frac{p+1}{4}\right]} R_{2k}^p x^{2k} + x.\sum_{k=0}^{\left[\frac{p-1}{4}\right]} R_{2k+1}^p x^{2k}$$

and $T_p(b_p) = 0$.So

$$b_p = -\frac{\sum_{k=0}^{\left[\frac{p+1}{4}\right]} R_{2k}^p.(b_p)^{2k}}{\sum_{k=0}^{\left[\frac{p-1}{4}\right]} R_{2k+1}^p (b_p)^{2k}}.$$

As for all $(0 \le \mathrm{k} \le \left[\frac{p+1}{2}\right])$, we have

$R_{2k}^p > 0$ and $R_{2k+1}^p < 0$

Then

$b_p > 0$

We also have

$\lim_{x \to -\infty} T_p(x) = +\infty$ and $T_p(b_p) = 0$ where b_p is the smallest root of T_p. As, in addition, the function T_p is continuous over $(-\infty, b_p)$, then for all $x \in (-\infty, b_p)$,we have

$T_p(x) > 0$

This end the proof of Lemma 1.

In lemma 1, we have assumed that, for a given p, the function T_p admits at least one real root. Now we will show that this is always true.

Lemma 2. *The polynomials T_p, defined by (11) for different values of non-zero natural integer p, each admit at least one real root, and the sequence $\{b_p\}$, constituted by their smallest positive real roots, is strictly decreasing.*

Proof. By induction, *we have*

$T_1(L_n) = 1 - L_n/2$ and $T_2(L_n) = 1 - L_n$

Then

$b_1 = 2$ and $b_2 = 1$

So

$(b_1$ and $b_2)$ exist and $b_2 < b_1$

Let $p \in \mathbb{N}^*$, we suppose that for every $k \le p+1$, we have

b_k exist and $b_k < b_{k-1}$

We will show that

b_{p+2} exist and $b_{p+2} < b_{p+1}$

From (7), we have

$$T_{p+1}(x) - T_{p+2}(x) = \frac{x}{2} T_p(x)$$

and, from lemma 1, we have for all $x \in (0, b_p)$

$T_p(x) > 0$

So

$T_{p+2}(x) < T_{p+1}(x)$

Since

$0 < b_{p+1} < b_p$

then

$T_{p+2}(b_{p+1}) < T_{p+1}(b_{p+1})$

As

$T_{p+1}(b_{p+1}) = 0$

then

$T_{p+2}(b_{p+1}) < 0$

Furthermore, we have

$T_{p+2}(0) = 1 > 0$

So

$T_{p+2}(0).T_{p+2}(b_{p+1}) < 0$

As the function T_p is continuous on $[0, b_{p+1}]$, then, from Intermediate value theorem, there exists$c \in (0, b_{p+1})$ such as

$T_{p+2}(c) = 0$

As b_{p+2} is the smallest real root of T_{p+2}, then
$T_{p+2}(b_{p+2}) = 0$ and $b_{p+2} \in (0, b_{p+1})$
So

b_{p+2} exists and $b_{p+2} < b_{p+1}$

This end the proof of Lemma 2.

In order to study the global convergence of the sequences (11), we must calculate the derivative of the iterative function F_p of f and study its sign.

Lemma 3. *Let $p \in \mathbb{N}$.The iterative function F_p of f relative to the sequence (11) is given by*:

$$F_p(x) = x - \frac{f(x)}{f^{'}(x)} W_p(L) \tag{14}$$

$$L = L_f(x) = \frac{f(x) f^{''}(x)}{f^{'}(x)^2} \tag{15}$$

and

$$W_p(L) = \frac{T_p(L)}{T_{p+1}(L)}, \quad T_p(L) = \sum_{k=0}^{\left[\frac{p+1}{2}\right]} R_k^p L^k, \quad R_k^p = \frac{(-1)^k (p-k+1)!}{2^k (p-2k+1)!k!}$$

The derivative $F_p^{'}$ is given by:

$$F_p^{'}(x) = 1 - \frac{1}{T_{p+1}^2(L)} \cdot \left[A_p(L) + C_p(L).L^2.L_{f'}(x)\right] \tag{16}$$

where

$$A_p(L) = T_{p+1}^2(L) - (p+3)\left(\frac{L}{2}\right)^{p+2} \tag{17}$$

and

$$T_{p+1}^2(L) = \sum_{k=0}^{2\left(\left[\frac{p}{2}\right]+1\right)} a_{p,k}.L^k \tag{18}$$

where

$$a_{p,k} = \sum_{i=max\left(0,k-\left[\frac{p}{2}\right]-1\right)}^{min\left(k,\left[\frac{p}{2}\right]+1\right)} \frac{(-1)^k (p+2-i)!(p+2-k+i)!}{2^k i!(k-i)!(p+2-2i)!(p+2-2k+2i)!} \tag{19}$$

and

$$C_p(L) = \sum_{k=0}^{2\left[\frac{p}{2}\right]} b_{p,k} L^k \tag{20}$$

where

$$b_{p,k} = \frac{(-1)^k (p+2)}{2^{k+1}} \sum_{i=sup\left(0,k-\left[\frac{p}{2}\right]\right)}^{\left[\frac{k}{2}\right]} \frac{(k-2i+1)^2 (p-i+1)!(p-k+i)!}{i!(k-i+1)!(p-2i+2)!(p-2k+2i)!} \tag{21}$$

Proof of lemma 3. Let $p \in \mathbb{N}$.From (14) and (15) we have

$$F_p^{'}(x) = 1 - \left(\frac{f(x)}{f^{'}(x)}\right)^{'}.W_p(L) - \left(\frac{f(x)}{f^{'}(x)}\right).L'.W_p^{'}(L) \tag{22}$$

where

$$\left(\frac{f(x)}{f^{'}(x)}\right)^{'} = 1 - L$$

$$\left(\frac{f(x)}{f^{'}(x)}\right).L' = L.\left[1 + L(L_{f'}(x) - 2)\right] \tag{23}$$

and

$$L' = (L_f(x))' \qquad L_{f'}(x) = \frac{f'(x)f'''(x)}{f''(x)^2}$$

$$W_p^{'}(L) = \frac{C_p(L)}{T_{p+1}^2(L)} \tag{24}$$

where

$$C_p(L) = T_p^{'}(L).T_{p+1}(L) - T_{p+1}^{'}(L).T_p(L) \tag{25}$$

Using (23) and (24), the formula (22) become

$$F_p^{'}(x) = 1 - \frac{1}{T_{p+1}^2(L)}\left[A_p(L) + L^2.C_p(L).L_{f'}(x)\right] \tag{26}$$

where

$$A_p(L) = (1-L).T_p(L).T_{p+1}(L) + L(1-2L).C_p(L) \tag{27}$$

and $C_p(L)$ is given by (25). By developing the calculation, we obtain

$$C_p(L) = \sum_{k=0}^{2\left[\frac{p}{2}\right]} b_{p,k} L^k \tag{28}$$

where

$$b_{p,k} = \sum_{i=sup\left(0,k-\left[\frac{p}{2}\right]\right)}^{\left[\frac{k}{2}\right]} (k-2i+1).\left[-R_i^p.R_{k-i+1}^{p+1} + R_i^{p+1}.R_{k-i+1}^p\right] \tag{29}$$

Using R_i^j given in (11), we obtain

$$-R_i^p.R_{k-i+1}^{p+1} + R_i^{p+1}.R_{k-i+1}^p = \frac{(-1)^k (p-i+1)!(p-k+i)!(k-2i+1)(p+2)}{2^{k+1}(k-i+1)!(p-2k+2i)!(p-2i+2)!i!} \tag{30}$$

Replacing (30) in (29), we obtain (21). Furthermore, we have:

$$T_{p+1}^2(L) = \left[\sum_{k=0}^{\left[\frac{p}{2}\right]+1} R_k^{p+1} L^k\right]^2 = \sum_{k=0}^{2\left(\left[\frac{p}{2}\right]+1\right)} a_{p,k}.L^k$$

where

$$a_{p,k} = \sum_{i=max\left(0,k-\left[\frac{p}{2}\right]-1\right)}^{min\left(k,\left[\frac{p}{2}\right]+1\right)} R_i^{p+1}.R_{k-i}^{p+1}$$

Using R_i^j given in (11), we obtain (19).
Developing the expression of $A_p(L)$ given by (27), we obtain

$$A_p(L) = -(p+3)\left(\frac{L}{2}\right)^{p+2} + \sum_{k=0}^{2\left(\left[\frac{p}{2}\right]+1\right)} \beta_{p,k}.L^k \tag{31}$$

where

$$\beta_{p,k} = \sum_{i=max\left(0,k-\left[\frac{p}{2}\right]-1\right)}^{min\left(k,\left[\frac{p}{2}\right]+1\right)} \left(\frac{-1}{2}\right)^k . \frac{(-1)^k (p-i+2)!(p-k+2+i)!}{2^k (p+2-2i)!(k-i)!(p+2-2k+2i)!i!} \tag{32}$$

We note that $\beta_{p,k} = a_{p,k}$. So, by using (18) and (31), we obtain (17).

This end the proof of Lemma 3.

The formulas (16)–(21) are of great importance because they give the derivatives $F_p^{'}$ for all the methods of the family (11). In the literature, these derivatives are known only for the two following cases (Ezquerro and Hernández (1997) in theorem 6.2):

- Halley's method $(p = 0)$

$$F_0^{'}(x) = \frac{L_f^2(x)(3 - 2L_{f'}(x))}{(2 - L_f(x))^2} \tag{33}$$

- Super Halley's method $(p = 1)$

$$F_1^{'}(x) = \frac{L_f^2(x)(L_f(x) - L_{f'}(x))}{2(1 - L_f(x))^2} \tag{34}$$

A simple calculation allows us to verify that our formulas give exactly the same expressions.

Now Let's look the sign of C_p that we will use later.

Lemma 4. *The polynomials C_p, defined in (20) and (21) for different values of the natural integer p, are all strictly positive on $(-\infty, b_{p+1})$, where b_{p+1} are the smallest real roots of the polynomials T_{p+1}.*

Proof. *Let $p \in \mathbb{N}$.We have for all $x \in (-\infty, b_{p+1})$:*

$$x_{n+1}^{p+2} = x_n - \frac{f(x_n)}{f'(x_n) + \frac{f''(x_n)}{2}\left(x_{n+1}^{p+1} - x_n\right)}$$

$$= x_n - \frac{f(x_n)}{f'(x_n)} \cdot \frac{1}{1 + \frac{f''(x_n)}{2f'(x_n)}\left(-\frac{f(x_n)}{f'(x_n)} W_{p+1}(L_n)\right)}$$

$$x_{n+1}^{p+2} = x_n - \frac{f(x_n)}{f'(x_n)} \cdot \frac{1}{1 - \frac{L_n}{2}.W_{p+1}(L_n)} = x_n - \frac{f(x_n)}{f'(x_n)}.W_{p+2}(L_n)$$

where

$$W_{p+2}(L_n) = \frac{1}{1 - \frac{L_n}{2}.W_{p+1}(L_n)}$$

So,

$$W_{p+2}(L_n) = \frac{1}{1 - \frac{L_n}{2}\cdot\left(\frac{1}{1-\frac{L_n}{2}.W_p(L_n)}\right)} = 1 - \frac{L_n}{L_n W_p(L_n) - 2 + L_n}$$

Using (15), we have

$$F_{p+2}(x) = x - \frac{f(x)}{f'(x)} W_{p+2}(L)$$

where $L \in \left]-\infty, b_{p+3}\right[$ and

$$W_{p+2}(L) = 1 - \frac{L}{LW_p(L) - 2 + L}$$

By deriving $F_{p+2}(x)$ given above, we find:

$$C_{p+2}(L) = \frac{1}{2}(T_{p+1}(L))^2 + \left(\frac{L}{2}\right)^2 C_p(L)$$

Then, developing the calculations, we obtain :

$$\begin{cases} C_p(L) = \frac{1}{2}\left[\left(\frac{L}{2}\right)^{2\left[\frac{p}{2}\right]} + \sum_{k=0}^{sup\left(0,\left[\frac{p}{2}\right]-1\right)} \left(\frac{L}{2}\right)^{2k}\left(T_{p-2k-1}(L)\right)^2\right], \quad p \in \mathbb{N}^* \\ C_0(L) = C_1(L) = \frac{1}{2} \end{cases} \tag{35}$$

Consequently, for all $p \in \mathbb{N}$ and for all $L \in (-\infty, b_{p+1})$, $C_p(L) > 0$
This end the proof of Lemma 4.
Now, we present a study on the global convergence of the $\{Bp\}$ family's methods (Hernández (1988); Ezquerro and Hernández (1997)).

3.3. Monotonic convergence of new sequences

Let $p \in \mathbb{N}$, C_p and L are defined by (20), (21) and (15). We consider the function g_p defined by:

$$g_p(L) = \frac{p+3}{2^{p+2}} \cdot \frac{L^p}{C_p(L)} \tag{36}$$

where $L \in (-\infty, b_{p+1})$

Theorem 3. *Let $p \in \mathbb{N}$, $f \in C^4[a, b]$, $f'(x) \neq 0$, $f''(x) \neq 0$, $L < b_{p+1}$ and*

$$L_{f'}(x) \le g_p(L) \tag{37}$$

On an interval $[a, b]$ containing the root α of f. The sequence (11) is decreasing (resp. increasing) and converges to α from any point $x_0 \in [a, b]$ checking $f(x_0)f'(x_0) > 0$ (*resp.* $f(x_0)f'(x_0) < 0$).

Proof. *Let $p \in \mathbb{N}$, $f \in C^4[a, b]$, $f'(x) \neq 0$, $f''(x) \neq 0$ and $L < b_{p+1}$ on $[a, b]$ containing α. Let's look for the condition on $L_{f'}$ for convergence to be monotonous.*

If

$$f(x_0)f'(x_0) > 0 \tag{38}$$

then

$$x_0 > \alpha$$

The mean Value Theorem gives

$$x_1^p - \alpha = F_p'(\beta)(x_0 - \alpha) \tag{39}$$

where $\beta \in (\alpha, x_0)$. Using (16) and (17), we have

$$F_p'(x) \ge 0 \text{ is equivalent to } C_p(L)L^2 L_{f'}(x) \le T_{p+1}^2(L) - A_p(L) \tag{40}$$

Using (17) and lemma 4, we obtain

$$L_{f'}(x) \le \frac{p+3}{2^{p+2}} \cdot \frac{L^p}{C_p(L)}$$

Thus, if the condition (37) is satisfied for all $L \in (-\infty, b_{p+1})$, then we have:

$$F_p'(x) \ge 0$$

for all $x \in [\alpha, b]$, especially

$$F_p'(\beta) \ge 0$$

Since $x_0 > \alpha$, then, from (39), we obtain

$$x_1^p \ge \alpha$$

By induction, we obtain that, for all $n \in \mathbb{N}$

$$x_n^p \ge \alpha$$

Furthermore, from (11), we have

$$x_1^p - x_0 = -W_p(L_0)\frac{f(x_0)}{f'(x_0)}$$

From lemma 1, the function T_p $(resp. T_{p+1})$ is strictly positive on $(-\infty, b_p)\left[resp. (-\infty, b_{p+1})\right]$. As

$$b_{p+1} < b_p \text{ and } L_0 < b_{p+1}$$

Then,

$$W_p(L_0) = \frac{T_p(L_0)}{T_{p+1}(L_0)} > 0$$

So

$$x_1^p < x_0$$

By induction we obtain for all $n \in \mathbb{N}$

$$x_{n+1}^p \le x_n^p$$

Thereby, the sequence (11) is decreasing and converges to a limit $r \in [\alpha, b]$, so

$$r = r - \frac{f(r)}{f'(r)} W_p(L_f(r))$$

Thus

$$f(r).W_p(L_f(r)) = 0$$

Since, from lemma 1, we have

$$T_p(L_f(r)) > 0$$

then

$W_p(L_f(r)) \neq 0$

so

$f(r) = 0$

As α is the unique root of f, then

$r = \alpha$.

This end the proof of theorem 3.

Corollary 2. *Let $p \in \mathbb{N}$, $f \in C^4[a,b]$, $f'(x) \neq 0$, $f''(x) \neq 0$, $0 \leq L < b_{p+1}$ and $L_{f'}(x) \leq 0$ on an interval $[a,b]$ containing the root α of f. The sequence (11) is decreasing (resp. increasing) and converges to α from any point $x_0 \in [a,b]$ checking $f(x_0)f'(x_0) > 0$ (rep.$f(x_0)f'(x_0) < 0$).*

Proof. *For $0 \leq L < b_{p+1}$, we have*

$L^p \geq 0$ and $C_p(L) > 0$

So

$$g_p(L) = \frac{p+3}{2^{p+2}} \cdot \frac{L^p}{C_p(L)} \geq 0$$

As, by hypothesis

$L_{f'}(x) \leq 0$

then

$L_{f'}(x) \leq g_p(L)$

By applying theorem 3, we obtain the thesis.

The formula (37) is of great importance because it gives the necessary conditions on $L_{f'}$ to ensure the monotonous convergence of all the methods of the new Halley's family (11). In the literature, these conditions are known only for the cases of Halley's method $(L_{f'} \leq 3/2)$ and Super Halley's method $(L_{f'} \leq L)$ (Hernández (1988) in theorem (i)).

*3.3.1. Convergence of the **Bp** methods*

Here, we treat the case where the convergence of (11) is ensured in any form: monotonic convergence, oscillating convergence or non-regular oscillation (between two successive iterations, it sometimes there is oscillation, sometimes no). This case is guaranteed if for all $x \in [a,b]$, we have

$-1 < F_p'(x) < 1$

We consider the functions h_p and k_p defined on $J^*_{p+1} = (-\infty, 0) \cup (0, b_{p+1})$ *by:*

$$h_p(L) = \frac{T^2_{p+1}(L) + (p+3)(L/2)^{p+2}}{L^2 C_p(L)} \text{ and}$$

$$k_p(L) = \frac{-T^2_{p+1}(L) + (p+3)(L/2)^{p+2}}{L^2 C_p(L)} \quad (41)$$

Where L is defined in (15), $(T_{p+1}(L))^2$ and $C_p(L)$ are given in (18) and (20).

Theorem 4. *Let $p \in \mathbb{N}$, $f \in C^4[a,b]$, $f'(x) \neq 0$, $f''(x) \neq 0$ and $L < b_{p+1}$ on an interval $I = [a,b]$ containing the root α of f, and*

$$k_p(L) < L_{f'}(x) < h_p(L) \quad (42)$$

for all $x \in I^* = [a, \alpha) \cup (\alpha, b]$. The sequence *(11)* converges to α from any point x_0 where

$a \leq F_p(x_0) \leq b$

Proof. *Let $\in \mathbb{N}$, $f \in C^4[a,b]$, $f'(x) \neq 0$, $f''(x) \neq 0$ and $L < b_{p+1}$ on $[a,b]$containing α. We treat the case where $\alpha < x_0 \leq b$. By the Mean Value Theorem, we have*:

$x_1^p - \alpha = F_p'(\lambda)(x_0 - \alpha)$

Where $\lambda \in (\alpha, x_0)$. From (16), we can prove that if the condition (42) is satisfied, then for all $x \in I^*$, we have

$-1 < F_p'(x) < 1$

As

$F_p'(\alpha) = 0$

then for all $x \in [a,b]$, we obtain

$-1 < F_p'(x) < 1$

Thus, there exists $M_p \in (0,1)$ such that, for all $x \in [a,b]$, we have

$\left|F_p'(x)\right| \leq M_p$

So

$|x_1^p - \alpha| \leq M_p |x_0 - \alpha|$

By induction, we get, for all $n \in \mathbb{N}$

$|x_n^p - \alpha| \leq (M_p)^n |x_0 - \alpha|$

In addition, since

$a \leq F_p(x_0) \leq b$

it follows that, for all $n \in \mathbb{N}$

$a \leq x_n^p \leq b$

and therefore, the sequence (11) converges to α.

4. Advantages of the new family

It is interesting to study the variation of the convergence speed of the sequences (11) as a function of the parameter p. For this, we will compare $\left(x_n^{p+1}\right) and (x_n^p)$.

Lemma 5. *Let $p \in \mathbb{N}$, $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ be defined respectively by $\left\{x_n^{p+1}\right\}$and $\{x_n^p\}$ given by (11). We have:*

$$u_{n+1} - v_{n+1} = -\frac{f(x_n)}{f'(x_n)} \left(\frac{\left(\frac{L_n}{2}\right)^{p+2}}{T_{p+1}(L_n).T_{p+2}(L_n)} \right) \qquad n \in \mathbb{N} \quad (43)$$

Proof. *We have*

$$u_{n+1} - v_{n+1} = F_{p+1}(x_n) - F_p(x_n)$$
$$= \frac{f(x_n)}{f'(x_n)} \left(\frac{T_{p+2}(L_n).T_p(L_n) - T^2_{p+1}(L)}{T_{p+1}(L_n).T_{p+2}(L_n)} \right)$$

Using (7), it follows that

$$T_{p+2}(L_n)T_p(L_n) - T^2_{p+1}(L) = \frac{L_n}{2}\left[T_{p+1}(L_n)T_{p-1}(L_n) - T^2_{p+1}(L)\right]$$

So

$$T_{p+2}(L_n)T_p(L_n) - T^2_{p+1}(L) = \left(\frac{L_n}{2}\right)^p \left[T_2(L_n)T_0(L_n) - (T_1(L_n))^2\right]$$
$$= -\left(\frac{L_n}{2}\right)^{p+2}$$

and (43) is completed.

Theorem 5. *Let $p \in \mathbb{N}$, $f \in C^4[a,b]$, $f'(x) \neq 0$, $f''(x) \neq 0$, $0 \le L < b_{p+2}$ and $L_{f'}(x) \le 0$ on an interval $[a,b]$, containing the root α of f. Starting from the same initial point $x_0 \in [a,b]$, the convergence's rate of the sequence $\left\{x_n^{p+1}\right\}$ given by (11) is higher than the one of $\{x_n^p\}$.*

Proof. *By induction, Let $p \in \mathbb{N}$ and we assume that the assumptions of theorem 5 are verified. If*

$f(x_0)f'(x_0) > 0$

then

$x_0 > \alpha$

From corollary 2 and lemma 2, if, for all $x \in [a,b]$, we have $L_{f'}(x) \le 0$ and $0 \le L < b_{p+2}$ then the sequence (v_n) and (u_n) are decreasing and converge to α from any point $x_0 \in [a,b]$.

Since $u_0 = v_0 = x_0$ then $u_0 \le v_0$

We have

$$u_1 - v_1 = -\frac{f(x_0)}{f'(x_0)}\left(\frac{\left(\frac{L_0}{2}\right)^{p+2}}{T_{p+1}(L_0)T_{p+2}(L_0)}\right)$$

and from lemma 1, for all $0 \le L_0 < b_{p+2}$, we have

$T_{p+1}(L_0) > 0$ and $T_{p+2}(L_0) > 0$

So

$u_1 \le v_1$

We assume that

$u_n \le v_n$

Since, F_{p+1} is increasing on $[a,b]$, we get

$F_{p+1}(u_n) \le F_{p+1}(v_n)$

In addition, we have:

$$F_{p+1}(v_n) - F_p(v_n) = -\frac{f(v_n)}{f'(v_n)}\left(\frac{\left(\frac{L_f(v_n)}{2}\right)^{p+2}}{T_{p+1}(L_f(v_n))T_{p+2}(L_f(v_n))}\right) \le 0$$

So

$F_{p+1}(u_n) \le F_p(v_n)$

thus

$u_{n+1} \le v_{n+1}$

This end the proof of theorem 5.

The theorem 5 announces a result of high importance: under some conditions, the convergence speed of the methods *Bp* improves if the parameter p increases. Thus, since p can take high values, then the convergence speed can always be improved with p. As the methods of Halley and Super Halley are obtained for $p = 0$ and $p = 1$, then their rate's convergence will be lower than the one of the other methods of our family.

5. Numerical results

Numerical computations reported here were carried out in MATLAB R2015b and the stopping criterion was taken as $|x_{n+1} - x_n| \le 10^{-15}$ *and* $|f(x_n)| \le 10^{-15}$.

5.1. Numerical Comparison between some methods of new family

We consider $f(x) = x^2 - 9x + 20$ *on*$[2,4]$ and we take $x_0 = 2.1$. The conditions of theorem 5 are satisfied for our methods $(B0, B2, B3, B5$ *and* $B6)$ given by (11) for $p = 0, 2, 3, 5$ *and* 6. In Table 1, we note that:

- All sequences are increasing and converge to the zero $\alpha = 4$ *of* f;
- The convergence speed of methods increases with the parameter p;
- Our methods $B2$, $B3$, $B5$ *and* $B6$ converge more rapidly than Halley's method $B0$.

This example confirms the importance of the theorem 5.

5.2. Comparison with other methods

The tests functions used in Table 3 are given in Table 2.

We indicate the number of iterations (NI) and the number of function evaluations (NOFE) required to meet the shutdown criterion.

On the left side of the Table 3, we compare our methods $B6$ and $B11$, given by (11) for $p = 6$ and 11, with Newton's method (N) defined by (1) (Sharma et al. (2012)) and some cubically convergent methods: Sharma (S) defined by (17)$(\alpha = 0.5)$, Jiang-Han's method (J) defined by (19) $(\alpha = 1)$ in Sharma et al. (2012), Chun's method (U) defined by (23) $(a_n = 1)$ in Chun (2007), Halley's method $(B0)$ defined by (2.3) before.

Table 1
Convergence's comparison of some methods *Bp*.

	B0	*B2*	*B3*	*B5*	*B6*
x_0	2.1	2.1	2.1	2.1	2.1
	3.608727895037079	3.58200315543224	3.91411497158407	3.96485646880613	3.977250615324
	3.977250615324702	3.99239178714579	3.99999975520541	3.99999999999823	4.0
	3.999988994594171	3.99999999674939	4.0	4.0	–
	4.0	4.0	–	–	–

Table 2
Test functions and their roots.

Test functions	Root (α)	Test functions	Root (α)
$f_1(x)=x^2 - 5x + 6$	3.00000000000000	$f_7(x) = xlnx$	1.000000000000000
$f_2(x) = 1+(x-3)e^x$	2.947530902542285	$f_8(x) = e^x - 4x^2$	0.714805912362777
$f_3(x) = (x-2)^2 - lnx$	3.057103549994738	$f_9(x) = (x-2)^4 - 1$	3.000000000000000
$f_4(x)=2\cosh x + 2cosx - 6$	1.85792082915019	$f_{10}(x) = 2sinx - 1$	0.5235987755982989
$f_5(x) = 0.5x^3 + 0.75x^2 - 3x - 1$	2.000000000000000	$f_{11}(x) = e^x - 3x^2$	0.910007572488709
$f_6(x) = x^{12} - 2x^3 - x + 1$	0.5903344367965851		−0.458962267536948

Table 3
Comparison with order methods.

Comparison with third order methods									Comparison with higher order methods						
f	x_0	NI							f	x_0	NOFE				
		N	S	U	J	$B0$	$B6$	$B11$			C	W	R	$B2$	$B8$
f_1	5	7	4	5	5	4	2	2	f_1	5	12	21	12	9	6
f_2	2.55	6	5	5	5	4	3	3	f_2	2.55	12	15	12	9	9
f_3	2.45	6	7	5	7	4	3	3	f_3	2.6	12	18	12	9	9
f_5	1.3	4	8	8	7	4	3	3	f_4	1.4	12	18	12	9	9
f_7	0.5	6	6	5	7	4	3	3	f_6	0.26	12	12	12	9	9
f_8	0.4	6	5	5	5	4	3	3	f_7	0.58	12	18	12	9	9
f_9	2.7	6	5	5	5	4	3	3	f_8	0.4	12	12	12	9	9
f_{10}	1.2	5	9	5	5	4	3	3	f_9	2.72	12	12	12	9	9
f_{11}	0.5	6	5	6	5	4	3	3	f_{11}	−0.1	12	9	12	9	9

On the right side of the same Table, we compare our methods $B2$ and $B8$, given by (11) for $p = 2$ and 8, with some higher order methods: Wang and Zhang (2014) (W) defined by (19) ($\gamma = \beta = -0.6$), a fourth-order method; Chun and Ham (2007) (C) defined by ((10)–(12)), a sixth-order methods; Noor et al. (2007) (R) defined by (Algorithm 2.4), a fifth-order method.

The results obtained for our methods are similar or better than those of other third and higher order methods, as they require the same number of iterations/number of function evaluations or less. These results are promising and show the efficiency and speed of the new family.

6. Conclusion

In this paper, we constructed a new Halley's family of third order iterative techniques for solving nonlinear equations with simple roots. The originality of this family lies first in the fact that these sequences are governed by one exceptional formula depending on a natural integer parameter p, and then, in the case where certain conditions are met, the convergence speed of its methods improves when the value of p increases. In addition, a study on the global convergence of the new methods has been carried out. Finally, the performance of our methods is compared with some methods of similar or higher order. The numerical results showed the robustness, efficiency and speed of the proposed family's techniques.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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