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A note on fuzzy best approximation using Chebyshev's polynomials

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Abstract This paper deals with the best approximation for fuzzy valued functions using Chebyshev nodes. We prove a result on the best near-minimax approximation in the fuzzy sense. As an application, Runge's phenomenon is fuzzified in two different cases, i.e. the best approximation and the best near-minimax approximation.

1. Introduction

The problem of best polynomial approximation is a historical problem in applied mathematics. Since the best polynomial approximation problems appear in many different branches, it is important to study this problem from various view points. In this paper, the fuzzy best Chebyshev approximation is discussed. Combining the results in Powell (1981, Chapter 7) and the work of Abbasbany et al. (2007), we intend to introduce the best Chebyshev approximation (best near-minimax approximation) for fuzzy-valued functions. To do this, we use Chebyshev's nodes and naturally fuzzy Chebyshev's polynomials to introduce and compute fuzzy best near-minimax approximation.

The structure of the present paper is as follows. In Section 2, the basic concepts of our work are introduced. Section 3 contains a brief description of Chebychev's nodes, polynomials and approximation. In Section 4, we introduce the fuzzy best near-minimax approximation and a method for obtaining it. In Section 5, For a case study, the fuzzified Runge phenomenon is studied in details. Finally, we have come to conclusion in Section 6.

2. Preliminaries

Here, we recall the basic notations and concepts for fuzzy triangular numbers, fuzzy-valued functions and fuzzy-valued polynomials.

Definition 2.1 (Dubois and Prade, 1980, 2000). A fuzzy number is a map $u : \mathbb{R} \to I = [0, 1]$, which satisfies:

i. u is upper semi-continuous.
ii. $u(x) = 0$ outside some interval $[c, d] \subset \mathbb{R}$.
iii. There exist real numbers a, b such that $c \leqslant a \leqslant b \leqslant d$, where:

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- $u(x)$ is monotonic increasing on $[c, a]$,
- $u(x)$ is monotonic decreasing on $[b, d]$,
- $u(x) = 1, a \leqslant x \leqslant b$.

Let $F(\mathbb{R})$ be the set of all fuzzy numbers, and $TF(\mathbb{R})$ be the set of all triangular fuzzy numbers with membership function

$$\mu_{\tilde{a}}(x) = \begin{cases} 1 - \frac{a_s - x}{a_l}, & (a_s - a_l) \leqslant x \leqslant a_s, \\ 1 + \frac{a_s - x}{a_r}, & a_s \leqslant x \leqslant (a_s + a_r), \\ 0, & otherwise, \end{cases}$$

and denote by $\tilde{a} = (a_s, a_l, a_r)$, where a_l, a_r, and a_s are nonnegative real numbers as the left, right spread, and center of the fuzzy number, respectively (Dubois and Prade, 1980).

Also, α-cut of a fuzzy number $\tilde{a}$ is defined by Dubois and Prade (1980) and Abbasbany et al. (2007)

$$[\tilde{a}]^\alpha = \begin{cases} \{t \in \mathbb{R} | \mu_{\tilde{a}}(t) \geqslant \alpha\} & \alpha > 0, \\ \overline{\{t \in \mathbb{R} | \mu_{\tilde{a}}(t) > \alpha\}} & \alpha = 0. \end{cases}$$

Definition 2.2 (*Dubois and Prade, 1980*). For $\tilde{a} = (a_s, a_l, a_r)$ and $\tilde{b} = (b_s, b_l, b_r)$, belonging to $TF(\mathbb{R})$, we define a distance by

$$D(\tilde{a}, \tilde{b}) = |a_s - b_s| + |a_l - b_l| + |a_r - b_r|.$$

Remark 2.3. D is a meter. for more details see Abbasbany et al. (2007).

Definition 2.4. The norm of a triangular fuzzy number $\tilde{a} = (a_s, a_l, a_r)$, is

$$\|\tilde{a}\| = |a_s| + a_l + a_r.$$

Let $\tilde{a} = (a_s, a_l, a_r), \tilde{b} = (b_s, b_l, b_r) \in TF(\mathbb{R})$. Some results of applying fuzzy arithmetic on fuzzy numbers $\tilde{a}$ and $\tilde{b}$ are as follows:

- $k > 0, k.\tilde{a} = (ka_s, ka_l, ka_r)$,
- $k < 0, k.\tilde{a} = (ka_s, -ka_r, -ka_l)$,
- $\tilde{a} + \tilde{b} = (a_s + b_s, a_l + b_l, a_r + b_r)$,
- $\tilde{a} - \tilde{b} = (a_s - b_s, a_l + b_r, a_r + b_l)$.

A fuzzy number $\tilde{a} = (a_s, a_l, a_r) \in TF(\mathbb{R})$ is nonnegative if and only if $a_s - a_l \geqslant 0$, and this fuzzy number is non-positive, if and only if $a_s + a_r \leqslant 0$.

Denote by $\prod_N$, the set of all real-valued polynomials of degree at most N, and by $\prod_N^+$, the set of all nonnegative real-valued piecewise polynomials of degree at most N.

In this study, we employ a class of fuzzy valued polynomials on $TF(\mathbb{R})$, and we approximate a fuzzy function $\tilde{f} : \mathbb{R} \to TF(\mathbb{R})$, by such fuzzy valued polynomials (Abbasbany et al., 2007; Kaleva (1994)).

For each $\alpha \in [0, 1]$, the lower and upper spreads of a fuzzy function $\tilde{f}$, on its α-cut, are $[\tilde{f}]_-^\alpha$ and $[\tilde{f}]_+^\alpha$, respectively, such that, for all $x \in \mathbb{R}$

$$\left[\tilde{f}\right]_-^\alpha(x) = \inf\left\{t \in \mathbb{R} | t \in \left[\tilde{f}(x)\right]^\alpha\right\},$$
$$\left[\tilde{f}\right]_+^\alpha(x) = \sup\left\{t \in \mathbb{R} | t \in \left[\tilde{f}(x)\right]^\alpha\right\}.$$

Definition 2.5. A triangular fuzzy valued polynomial of degree at most N, is a function $\tilde{p} : \mathbb{R} \to TF(\mathbb{R})$, which

$$\tilde{p}(x) = (p(x), \underline{p}(x), \overline{p}(x)),$$

where $p(x) \in \prod_N$, also $\underline{p}(x)$ and $\overline{p}(x)$ are positive piecewise polynomials from degree at most N.

The set of all fuzzy valued polynomials is denoted by $\widetilde{\prod}_N$ (Abbasbany et al., 2007; Kaleva (1994)).

3. Chebyshev approximation

We now turn our attention to polynomial interpolation for $f(x)$ over $[-1, 1]$ based on the nodes $-1 \leqslant x_0 \leqslant x_1 \leqslant \cdots \leqslant x_N \leqslant 1$. Let $p_N(x)$ be the Lagrange interpolation polynomial of $f(x)$ (Mathew and Fink, 2004). Therefore, we have

$$f(x) = p_N(x) + E_N(x).$$

Here, the error function is $E_N(x) = Q(x)\frac{f^{(N+1)}(x)}{(N+1)!}$, where $Q(x) = (x - x_0)(x - x_1)\cdots(x - x_N)$. It is well-known that

$$|E_N(x)| \leqslant |Q(x)| \frac{\max_{-1 \leqslant x \leqslant 1}\left\{|f^{(N+1)}(x)|\right\}}{(N+1)!}.$$

Chebyshev studied how to minimize the upper bound for $|E_N(x)|$. One upper bound can be formed by taking the product of the maximum value of $|Q(x)|$ on $[-1, 1]$ and the maximum value $\left|\frac{f^{(N+1)}(x)}{(N+1)!}\right|$ on $[-1, 1]$. To minimize the statement $\max_{-1 \leqslant x \leqslant 1}|Q(x)|$, Chebyshev discovered that $x_0, x_1, \ldots, x_N$ should be chosen so that $Q(x) = (\frac{1}{2^N})T_{N+1}(x)$, where $T_{N+1}(x)$ is the polynomial which is generated by $N + 1$ Chebyshev's nodes. Moreover, we have the following well-known theorems and corollary (Philips and Taylor (1980); Powell (1981)).

Theorem 3.1. *Assume that N is fixed. Among all possible choices for $Q(x)$, and thus among all possible choices for the distinct nodes $\{x_k\}_{k=0}^N$, where $x_k \in [-1, 1], \forall k \in \{0, 1, \ldots, N\}$, the polynomial $T(x) = \frac{T_{N+1}(x)}{2^N}$ is the unique choice which has the following property*

$$\max_{-1 \leqslant x \leqslant 1}\{|T(x)|\} \leqslant \max_{-1 \leqslant x \leqslant 1}\{|Q(x)|\}.$$

Theorem 3.2. *If f is continuous on $[-1, 1]$, there is a unique minimax polynomial approximation of degree at most N for f on $[-1, 1]$.*

Corollary 3.3. If f is continuous on $[-1, 1]$, then the minimax approximation $p \in \prod_N$ to f is an interpolating polynomial on the $N + 1$ Chebyshev nodes of $[-1, 1]$.

4. The best near-minimax approximation of fuzzy-valued functions

In this section, we follow Abbasbany et al. (2007) and introduce the best minimax approximation of a fuzzy-valued function on a finite set of distinct points.

Let $\chi = \{x_0, x_1, x_2, \ldots, x_N\}$ be a set of $N + 1$ distinct points of $\mathbb{R}$, and $\tilde{f}_i = (f_{i_s}, f_{i_l}, f_{i_r}) \in TF(\mathbb{R})$ be the value of a fuzzy function $\tilde{f} : \mathbb{R} \to TF(\mathbb{R})$ at the point x_i, for $i = 0, 1, 2, \ldots, N$.

Definition 4.1 (*Abbasbany et al., 2007*). A fuzzy-valued polynomial $\tilde{p}^* \in \widetilde{\prod}_N$ is the best approximation to $\tilde{f}$ on $\chi = \{x_0, x_1, x_2, \ldots, x_N\}$, if

$$\max_{i=0,1,2,\ldots,N} D(\tilde{p}^*(x_i), \tilde{f}_i) = \min_{\tilde{p}\in\widetilde{\prod}_N} \left\{ \max_{i=0,1,2,\ldots,N} D(\tilde{p}(x_i), \tilde{f}_i) \right\}.$$

The problem is referred to the best near-minimax approximation (best Chebyshev approximation), as we use Chebyshev's nodes.

Suppose that $\tilde{\beta} = (\beta_s, \beta_l, \beta_r)$ is a fuzzy number, where $\beta_s \geqslant 0$, and $\|\tilde{\beta}\| = \max_{i=0,1,2,\ldots,N} D(\tilde{p}(x_i), \tilde{f}_i)$, that is

$$\beta_s + \beta_l + \beta_r = \max\left\{|p(x_i) - f_{i_s}| + |\underline{p}(x_i) - f_{i_l}| + |\overline{p}(x_i) - f_{i_r}|\right\}.$$

Taking

$$\begin{aligned} \beta_s &= \max_{i=0,1,2,\ldots,N} |p(x_i) - f_{i_s}|, \\ \beta_l &= \max_{i=0,1,2,\ldots,N} \left|\underline{p}(x_i) - f_{i_l}\right|, \\ \beta_r &= \max_{i=0,1,2,\ldots,N} |\overline{p}(x_i) - f_{i_r}|. \end{aligned} \tag{1}$$

Then, we have

$$\begin{aligned} \beta_s &\geqslant |p(x_i) - f_{i_s}|, \\ \beta_l &\geqslant \left|\underline{p}(x_i) - f_{i_l}\right|, \\ \beta_r &\geqslant |\overline{p}(x_i) - f_{i_r}|, \end{aligned}$$

for $i = 0, 1, 2, \ldots, N$.

Therefore, we get three independent linear programming problems to be solved:

$$\begin{cases} \min \beta_s \\ s.t. \\ \beta_s + \sum_{j=0}^{N} a_j x_i^j \geqslant f_{i_s}, \quad i = 0, 1, \ldots, N, \\ \beta_s - \sum_{j=0}^{N} a_j x_i^j \geqslant -f_{i_s}, \quad i = 0, 1, \ldots, N, \end{cases} \tag{2}$$

$$\begin{cases} \min \beta_l \\ s.t. \\ \beta_l + \sum_{j=0}^{N} \underline{a}_j x_i^j \geqslant f_{i_l} \quad i = 0, 1, \ldots, N, \\ \beta_l - \sum_{j=0}^{N} \underline{a}_j x_i^j \geqslant -f_{i_l} \quad i = 0, 1, \ldots, N, \end{cases} \tag{3}$$

and

$$\begin{cases} \min \beta_r \\ s.t. \\ \beta_r + \sum_{j=0}^{N} \overline{a}_j x_i^j \geqslant f_{i_r} \quad i = 0, 1, \ldots, N, \\ \beta_r - \sum_{j=0}^{N} \overline{a}_j x_i^j \geqslant -f_{i_r} \quad i = 0, 1, \ldots, N. \end{cases} \tag{4}$$

Using 3, 2 and 4, we find three crisp best approximations on data $(x_i, f_{i_s}), (x_i, f_{i_r})$ and (x_i, f_{i_l}) for $i = 0, 1, 2, \ldots, N$, and call them $p, \underline{p}$ and $\overline{p}$, respectively. Thus the best approximation to $\tilde{f}$ out of $\widetilde{\prod}_N$ on χ, at point x, is $\tilde{p}(x) = (p(x), \underline{p}(x), \overline{p}(x))$. Also the error of this approximation is $\tilde{\beta} = (\beta_s, \beta_l, \beta_r)$ (Abbasbany et al., 2007).

Remark 4.2. In general, we can use the following definition for the best approximation to $\tilde{f}$ out of $\widetilde{\prod}_N$ on χ, at point x,

$$\left(p(x), \max\left\{0, \underline{p}(x), -\overline{p}(x)\right\}, \max\left\{0, -\underline{p}(x), \overline{p}(x)\right\}\right). \tag{5}$$

Solving 3, 2 and 4, we have three independent polynomials $p, \underline{p}$ and $\overline{p}$ of degree at most N. But, applying 5, right and left spreads of fuzzy valued polynomial may be piecewise polynomials of degree at most N.

Now, we are ready to state the following theorems which may play dominant roles in the best approximation on Chebyshev's nodes.

Theorem 4.3. The best approximation of a fuzzy function based on the Chebyshev nodes exists and is unique.

Proof 1. The proof is an immediate consequence of Theorem 4.2.1, p. 99 of Abbasbany et al. (2007). □

Theorem 4.4. $\|\tilde{\beta}\| = 0$ if and only if the best approximation of a fuzzy-valued function based on the Chebyshev nodes is the best minimax approximation.

Proof 2. *Necessity*. $\|\tilde{\beta}\| = 0$ implies that $\beta_s = \beta_l = \beta_r = 0$. Therefore, from (1) we have

$$p(x_i) = f_{i_s}, \underline{p}(x_i) = f_{i_l}, \overline{p}(x_i) = f_{i_r},$$

for $i = 0, 1, \ldots, N$, where $\{x_0, x_1, x_2, \ldots, x_N\}$ are the Chebyshev nodes. Hence $\tilde{p}(x) = (p(x), \underline{p}(x), \overline{p}(x))$ is the fuzzy interpolating polynomial to $\tilde{f}$ on $\{x_0, x_1, x_2, \ldots, x_N\}$. The rest of the proof follows from Theorem 3.2 and Corollary 3.3, directly.*Sufficiency*. Suppose that the best approximation of a fuzzy-valued function based on the Chebyshev nodes is the best minimax approximation. Consequently, $\|\tilde{\beta}\| = 0$ is immediately derived from the uniqueness of minimax approximation. □

5. Runge's phenomenon

In the field of numerical analysis, Runge's phenomenon occurs when the polynomials with high degree are applied to approximate the Runge function (Runge, 1901). It was discovered by Runge (1901) when he studied the behavior of errors in the approximation of a certain functions by polynomials. Consider the function (Runng's function)

$$f(x) = \frac{b^2}{1 + (ax)^2} \tag{6}$$

where, $ab \neq 0$. If this function is approximated at equidistant points $x_i = -1 + i\frac{2}{N}, i = 0, 1, \ldots, N$, by polynomial $P_N(x)$, the resulting approximation oscillates toward the end of the interval. It can even be proven that the interpolation error tends toward infinity when the degree of the polynomial increases:

$$\lim_{N\to\infty} \left(\max_{-1\leqslant x\leqslant 1} |f(x) - P_N(x)| \right) = \infty. \tag{7}$$

However, the Weierstrass approximation theorem states that there is some sequence of approximating polynomials for which the error tends to zero. This shows that a high degree polynomial approximation at the equidistant points can be dangerous.

The oscillation can be minimized by using Chebyshev nodes instead of the equidistant nodes. In this case, the maximum error is guaranteed to diminish with increasing the degree of polynomial.

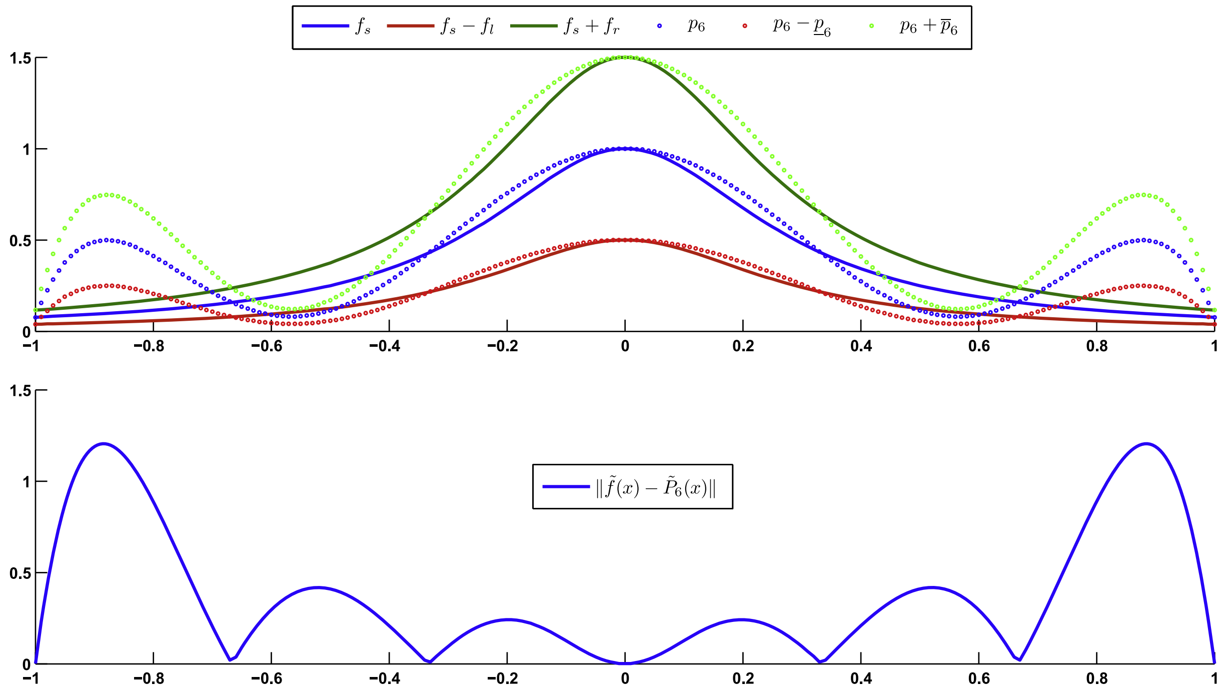

Figure 1 Results for Case 1.

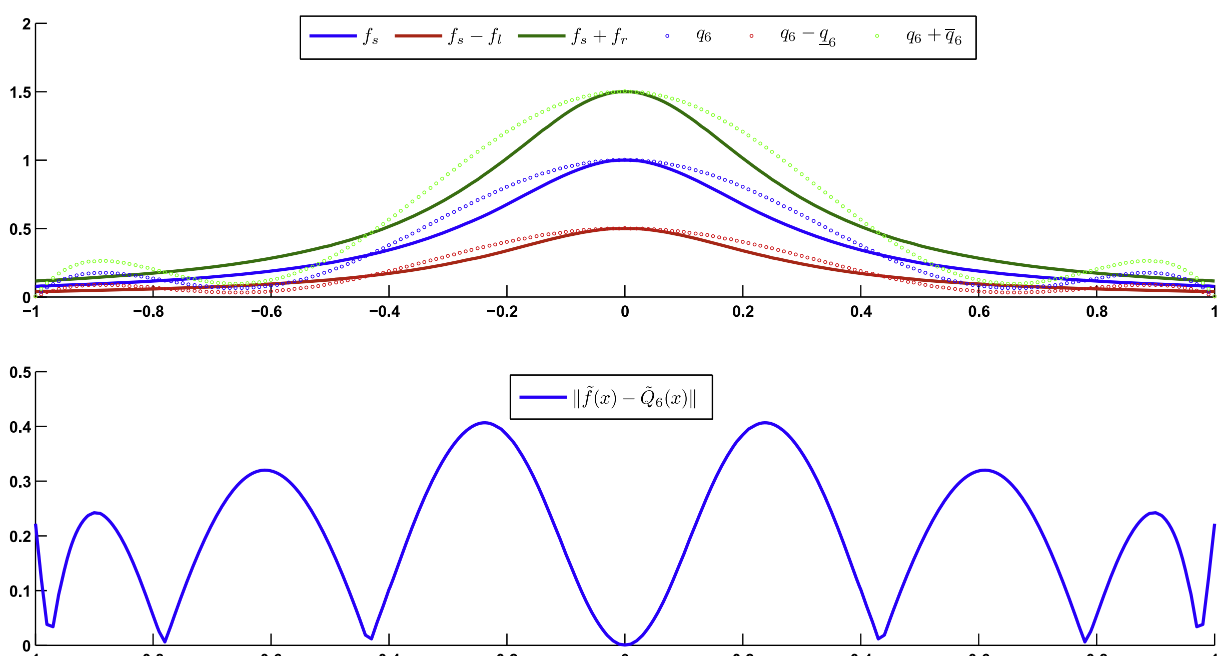

Figure 2 Results for Case 2.

In the following, we study this phenomenon numerically when the function $f(x)$ is a fuzzy-valued function. Consider the fuzzy-valued function

$$\tilde{f}(x) = \frac{\tilde{1}}{1 + 12x^2} \tag{8}$$

where $\tilde{1} = \left(1, \frac{1}{2}, \frac{1}{2}\right)$ and also, $\tilde{f}(x) = (f_s(x), f_l(x), f_r(x)) = \left(\frac{1}{1+12x^2}, \frac{0.5}{1+12x^2}, \frac{0.5}{1+12x^2}\right)$.

Now, the best approximation of function $\tilde{f}(x)$ is computed in two different cases for $N = 6$.

Case 1. Equidistant nodesLet $x_i = -1 + \frac{i}{3}, i = 0, 1, \ldots, 6$. Using (2)–(4), the best approximation, $\widetilde{P}_6 = \left(p_6, \overline{p}_6, \underline{p}_6\right)$, is obtained as follows

$$\begin{aligned} f_s(x) &\approx p_6(x) = -8.99x^6 + 14.7415x^4 - 6.6698x^2 + 1, \\ f_l(x) &\approx \underline{p}_6(x) = -4.4974x^6 + 7.3707x^4 - 3.3349x^2 + 0.5, \\ f_r(x) &\approx \overline{p}_6(x) = -4.4974x^6 + 7.3707x^4 - 3.3349x^2 + 0.5. \end{aligned} \tag{9}$$

The maximum absolute error, i.e. $\max_{-1\leqslant x\leqslant 1}\|\tilde{f}(x) - \widetilde{P}_6(x)\|$, is 1.1877. The results are illustrated in Fig. 1.

Case 2. Chebyshev nodesLet $x_i = \cos\left(\frac{(2i+1)\pi}{12}\right), i = 0, 1, \ldots, 6$. Using (2)–(4), the best approximation, $\widetilde{Q}_6 = (q_6, \overline{q}_6, \underline{q}_6)$, is obtained as follows

$$\begin{aligned} f_s(x) &\approx q_6(x) = -5.127x^6 + 9.4006x^4 - 5.27x^2 + 1, \\ f_l(x) &\approx \underline{q}_6(x) = -2.5638x^6 + 4.7003x^4 - 2.635x^2 + 0.5, \\ f_r(x) &\approx \overline{q}_6(x) = -2.5638x^6 + 4.7003x^4 - 2.635x^2 + 0.5. \end{aligned} \tag{10}$$

In this case, the maximum absolute error, i.e. $\max_{-1\leqslant x\leqslant 1}\|\tilde{f}(x) - \widetilde{Q}_6(x)\|$, is 0.4040. The results are illustrated in Fig. 2.

6. Conclusions

In this study, we proposed a method for finding the best near-minimax approximation of a fuzzy-valued function on a set of Chebyshev nodes. The existence and uniqueness of the best near-minimax approximation of a fuzzy-valued function were also proved. The best near-minimax approximation was compared to the best approximation (Abbasbany et al., 2007) by Runge's phenomenon in the fuzzy sense.

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