Journal of King Saud University – Science (2011) 23, 151–155

King Saud University
Journal of King Saud University – Science
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ORIGINAL ARTICLE

Solving a multi-order fractional differential equation using homotopy analysis method

H. Jafari [a,*], S. Das [b], H. Tajadodi [a]

[a] Department of Mathematics, University of Mazandaran, Babolsar, Iran
[b] Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi 221 005, India

Received 7 June 2010; accepted 28 June 2010
Available online 1 July 2010

KEYWORDS
Fractional differential equation;
Homotopy analysis method;
Caputo fractional derivative;
Riemann-Liouville fractional derivative

Abstract In this paper we have used the homotopy analysis method (HAM) to obtain solution of multi-order fractional differential equation. The fractional derivative is described in the Caputo sense. Some illustrative examples have been presented.

1. Introduction

Fractional differential equations have been found to be effective to describe some physical phenomena such as damping laws, rheology, diffusion processes, and so on. Several methods have been used to solve Fractional differential equations, such as Laplace transform method (Podlubny, 1999), Fourier transform method (Kemple and Beyer, 1997), Adomians decomposition method (ADM) (Daftardar-Gejji and Jafari, 2005; Daftardar-Gejji and Jafari, 2007; Jafari and Daftardar-Gejji, 2006), Homotopy analysis method (Liao, 2003; Momani and Odibat, 2008) and so on. For nonlinear FDE, however, one mainly resorts to numerical methods (Diethelm, 1997; Diethelm and Ford, 2002; Diethelm and Ford, 2004; Edwards et al., 2002). These numerical methods involve discretization of the variables, which gives rise to rounding off errors. Another drawback of numerical methods stems from the requirement of large computer memory.

In this paper, the homotopy analysis method (Liao, 1992) is applied to solve the multi-order fractional differential equation studied by Diethelm and Ford (2004):

$$D_*^{\alpha}y(t) = f\big(t, y(t), D_*^{\beta_1}y(t), \ldots, D_*^{\beta_n}y(t)\big), \quad y^{(k)}(0) = c_k, \quad k = 0, \ldots, m,$$

where $m < \alpha \leqslant m+1, 0 < \beta_1 < \beta_2 < \cdots < \beta_n < \alpha$ and D_*^{α} denotes Caputo fractional derivative of order α.

Liao (1992) employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method (HAM), Liao (1992, 2004, 2003). This method (HAM) (Liao, 2003) provides an effective procedure for explicit and numerical solutions of a

* Corresponding author. Tel.: +98 9112113567.
E-mail addresses: jafari@umz.ac.ir, jafari_h@math.com (H. Jafari), subir_das08@hotmail.com (S. Das), tajadodi@umz.ac.ir (H. Tajadodi).

1018-3647

Peer review under responsibility of King Saud University.
doi:10.1016/j.jksus.2010.06.023

Production and hosting by Elsevier

wide and general class of differential systems representing real physical problems. Based on homotopy of topology, the validity of the HAM is independent of whether or not there exist small parameters in the considered equation. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a possibility to analyze strongly nonlinear problems. Jafari and seifi have been solved diffusion-wave equation and partial differential equations and system of nonlinear fractional partial differential equations using homotopy analysis method (Jafari and Seifi, 2009; Jafari and Seifi, 2009). This method has been successfully applied to solve many types of nonlinear problems (Hayat et al., 2004; Momani and Odibat, 2008). These authors have discussed the analytical questions of existence and uniqueness of solutions and investigated how the solutions depend on the given initial data. Further they have presented an algorithm to convert the multi-order FDE into a system of FDE under some conditions and have developed numerical method to solve the system of FDE. In this paper we present an algorithm to convert the multi-order FDE into a system of FDE, without putting any of the restrictions. Thus our algorithm is valid in the most general case and yields fewer number of equations in a system compared to those in Diethelm–Ford algorithm. Consequently the solutions of the system of FDE have been obtained by employing the HAM approach. The paper has been organized as follows. Section 2 describes how to convert a multi-order FDE. In Section 3, HAM is developed to solve the system of FDE. Section 4 presents some illustrative examples. Discussion and conclusions are summarised in the Section 5.

2. Preliminaries

We enlist below some definitions (Luchko and Gorenflo, 1999; Podlubny, 1999) and basic results.

Definition 2.1. A real function $f(x), x>0$ is said to be in the space $C_\alpha, \alpha \in \Re$ if there exists a real number $p(>\alpha)$ such that $f(x)=x^p f_1(x)$ where $f_1(x)\in C[0,\infty)$. Clearly $C_\alpha \subset C_\beta$ if $\beta \leqslant \alpha$.

Definition 2.2. A function $f(x), x>0$ is said to be in the space $C_\alpha^m, m\in N\bigcup\{0\}$ if $f^{(m)}\in C_\alpha$.

Definition 2.3. The (left sided) Riemann-Liouville fractional integral of order $\mu>0$ of a function $f\in C_\alpha, \alpha \geqslant -1$ is defined as:

$$I^\mu f(t)=\frac{1}{\Gamma(\mu)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\mu}}d\tau, \quad \mu>0, \quad t>0, \qquad (1)$$
$$I^0 f(t)=f(t).$$

Definition 2.4. The (left sided) Riemann-Liouville fractional derivative of $f, f\in C_{-1}^m, m\in IN\bigcup\{0\}$ of order $\alpha>0$ is defined as:

$$D^\mu f(t)=\frac{d^m}{dt^m}I^{m-\mu}f(t), \quad m-1<\mu\leqslant m, \quad m\in IN. \qquad (2)$$

Definition 2.5. The (left sided) Caputo fractional derivative of $f, f\in C_{-1}^m, m\in IN\bigcup\{0\}$ is defined as:

$$D_*^\mu f(t)=\begin{cases}\left[I^{m-\mu}f^{(m)}(t)\right] & m-1<\mu<m, \quad m\in IN,\\ \frac{d^m}{dt^m}f(t) & \mu=m.\end{cases} \qquad (3)$$

Note that

(i) $I^\mu t^\gamma=\dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)}t^{\gamma+\mu}, \quad \mu>0, \quad \gamma>-1, \quad t>0.$

(ii) $I^\mu D_*^\mu f(t)=f(t)-\sum_{k=0}^{m-1}f^{(k)}(0+)\dfrac{t^k}{k!}, \quad m-1<\mu\leqslant m, \quad m\in IN.$

(iii) $D_*^\mu f(t)=D^\mu\left(f(t)-\sum_{k=0}^{m-1}f^{(k)}(0+)\dfrac{t^k}{k!}\right), \quad m-1<\mu\leqslant m, \quad m\in IN.$

(iv) $D^\beta I^\alpha f(t)=\begin{cases}I^{\alpha-\beta}f(t), & \text{if } \alpha>\beta,\\ f(t), & \text{if } \alpha=\beta,\\ D^{\beta-\alpha}f(t), & \text{if } \alpha<\beta.\end{cases}$

(v) $D_*^\alpha D^m f(t)=D_*^{\alpha+m}f(t), \quad m=0,1,2,\ldots,n-1<\alpha<n.$

Definition 2.6. The Mittag-Leffler function $E_\alpha(z)$ with $\alpha>0$ is defined by the following series representation valid in the whole complex plane (Mainardi, 1994):

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \quad \alpha>0, \quad z\in C. \qquad (4)$$

Lemma 2.7 Diethelm and Ford, 2002. *Let $y(t)\in C^k[0,T]$ for some $T>0$ and $k\in IN$ and let $q\notin IN$ be such that $0<q<k$. Then $D_*^q y(0)=0$.*

3. Multi-order FDE as a system of FDE

Daftardar-Gejji and Jafari have solved multi-order FDE by Adomian decomposition method (Daftardar-Gejji and Jafari, 2007):

$$D_*^\alpha y(t)=f\big(t,y(t),D_*^{\beta_1}y(t),\ldots,D_*^{\beta_n}y(t)\big),\ y^{(k)}(0)=c_k,$$
$$k=0,\ldots,m, \qquad (5)$$

where $m<\alpha\leqslant m+1$, $0<\beta_1<\beta_2<\cdots<\beta_n<\alpha$ and D_*^α denotes Caputo fractional derivative of order α. It should be noted that f can be non linear in general.

$$\alpha,\beta_j\in Q,\ \alpha-\beta_n\leqslant 1,\ \beta_j-\beta_{j-1}\leqslant 1, \forall j, \text{ and } 0\leqslant\beta\leqslant 1. \qquad (6)$$

In Daftardar-Gejji and Jafari (2007), it was proved that the Eq. (5) can be represented as a system of FDE, without any additional restrictions mentioned in Eq. (6). Here we present their approach. Set $y_1=y$ and define:

$$D_*^{\beta_1}y_1=y_2. \qquad (7)$$

Case (i) If $m-1\leqslant\beta_1<\beta_2\leqslant m$ then define

$$D_*^{\beta_2-\beta_1}y_2=y_3. \qquad (8)$$

Claim: $y_3=D_*^{\beta_2}y$. If $\beta_1=m-1$, then $D_*^{\beta_2-\beta_1}y_2=D_*^{\beta_2-(m-1)}y^{(m-1)}=D_*^{\beta_2}y_1$.

Hence the claim. If $\beta_1\notin IN$, then by Lemma 2.6, $D_*^{\beta_1}y_1(0)=0$ and as $\beta_2-\beta_1<1$,

$$\begin{aligned}D_*^{\beta_2-\beta_1}\left[D_*^{\beta_1}y_1\right]&=D^{\beta_2-\beta_1}\left[D_*^{\beta_1}y_1\right]=DI^{1+\beta_1-\beta_2}I^{m-\beta_1}y_1^{(m)}\\&=DI^{1+m-\beta_2}y_1^{(m)}=I^{m-\beta_2}y_1^{(m)}=D_*^{\beta_2}y_1\\&=D_*^{\beta_2}y.\end{aligned} \qquad (9)$$

Therefore $y_3=D_*^{\beta_2-\beta_1}y_2=D_*^{\beta_2}y$.

Case (ii) Consider $m-1\leqslant\beta_1<m\leqslant\beta_2$. If $\beta_1=m-1$, then define $D_*^{\beta_2-\beta_1}y_2=y_3$.

$D_*^{\beta_2-\beta_1}y_2 = D_*^{\beta_2-m+1}y_1^{(m-1)} = D_*^{\beta_2}y_1.$

If $m-1 < \beta_1 < m \leqslant \beta_2$, then define

$$D_*^{m-\beta_1}y_2 = y_3. \tag{10}$$

Claim: $y_3 = y^{(m)}$. As $\beta_1 \notin IN$, $D^{\beta_1}y_1(0) = y_2(0) = 0$ (in view of Lemma 2.6), and $0 < m-\beta_1 < 1$,

$$D_*^{m-\beta_1}y_2 = D^{m-\beta_1}y_2 = DI^{1+\beta_1-m}I^{m-\beta_1}y_1^{(m)} = DIy_1^{(m)} = y_1^{(m)} = y^{(m)}. \tag{11}$$

Hence $y_3 = y^{(m)}$. Further define:

$$D_*^{\beta_2-m}y_3 = y_4. \tag{12}$$

Claim $y_4 = D_*^{\beta_2}y$. As $y_4 = D_*^{\beta_2-m}y_3 = D_*^{\beta_2-m}y^{(m)} = D_*^{\beta_2}y$. And continuing similarly we can convert the initial value problem (5) into a system of FDE. The following example will illlustrate the method. Consider

$$D_*^{3.6}y = f\big(x, y, D_*^{1.2}y, D_*^{1.7}y, D_*^{2.1}y, D_*^{3.5}y\big), \tag{13}$$

where $y(0) = c_0$, $y'(0) = c_1$, $y''(0) = c_2$ and $y'''(0) = c_3$. This initial value problem can be viewed as the following system of FDE.

$$\begin{aligned}
&D_*^{1.2}y_1(x) = y_2(x), \quad y_1(0) = c_0, \quad y_1'(0) = c_1,\\
&D_*^{0.5}y_2(x) = y_3(x)\big(= D_*^{1.7}y(x)\big), \quad y_2(0) = 0,\\
&D_*^{0.3}y_3(x) = y_4(x)(= y''(x)), \quad y_3(0) = 0,\\
&D_*^{0.1}y_4(x) = y_5(x)\big(= D_*^{2.1}y(x)\big), \quad y_4(0) = c_2,\\
&D_*^{0.9}y_5(x) = y_6(x)\big(= y^{(3)}(x)\big), \quad y_5(0) = 0,\\
&D_*^{0.5}y_6(x) = y_7(x)\big(= D_*^{3.5}y(x)\big), \quad y_6(0) = c_3,\\
&D_*^{0.1}y_7(x) = f(x, y_1, y_2, y_3, y_5, y_7), \quad y_7(0) = 0,
\end{aligned} \tag{14}$$

where $y_1(x) = y(x)$.

The remark is in order

1. This algorithm is valid in the most general case, because we do not impose any of the restrictions on α, β_i as mentioned in Eq. (6). We let $\alpha, \beta_i \in \Re$, whereas in Diethelm and Ford (2004) $\alpha, \beta_i \in IQ$.

4. Homotopy analysis method and a system of FDE

We can present the multi-order Eq. (5) as system of fractional differential equations:

$$\begin{aligned}
&D^{\alpha_i}y_i(t) = y_{i+1}, \quad i = 1, 2, \ldots, n-1,\\
&D^{\alpha_n}y_i(t) = f(t, y_1, y_2, \ldots, y_n),\\
&y_i^{(k)}(0) = c_k^i, \quad 0 \leqslant k \leqslant m_i, \quad m_i < \alpha_i \leqslant m_i + 1, \quad 1 \leqslant i \leqslant n.
\end{aligned} \tag{15}$$

According to the HAM, we construct the so called zero-order deformation equations:

$$\begin{aligned}
&(1-q)D^{\alpha_i}[\varphi_i(t;q) - y_{i0}(t)] = qh_iH_i(t)[D^{\alpha_i}\varphi_i(t;q) - \varphi_{i+1}(t;q)],\\
&\qquad i = 1, 2, \ldots, n-1,\\
&(1-q)D^{\alpha_n}[\varphi_n(t;q) - y_{n0}(t)] = qh_iH_n(t)[D^{\alpha_i}\varphi_n(t;q) -\\
&\qquad f(t, \varphi_1, \varphi_2, \ldots, \varphi_n)],
\end{aligned} \tag{16}$$

where $q \in [0,1]$ is an embedding parameter, $h_i \neq 0$ are non-zero auxiliary parameters for $H_i(t) \neq 0$, $i = 1, 2, \ldots, n$ are non-zero auxiliary functions, $y_{i0}(t)$ are initial guess of $y_i(t)$, $\varphi_i(t;q)$ are unknown function, respectively. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when q = 0 and q = 1, it holds

$$\varphi_i(t;0) = y_{i0}(t), \varphi_i(t;1) = y_i(t), \quad i = 1, 2, \ldots, n, \tag{17}$$

respectively. Thus as q increases from 0 to 1 the solution $\varphi_i(t;q)$ varies from the initial guesses $y_{i0}(t)$ to the solution $y_i(t)$. Expanding $\varphi_i(t;q)$ in Taylor series with respect to q, we have

$$\varphi_i(t;q) = y_{i0}(t) + \sum_{m=1}^{\infty} y_{im}(t)q^m, \quad i = 1, 2, \ldots, n, \tag{18}$$

where

$$y_{im}(t) = \frac{1}{m!}\frac{\partial^m \varphi_i(t;q)}{\partial q^m}\Big|_{q=0}, \quad i = 1, 2, \ldots, n, \tag{19}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function are so properly chosen the series Eq. (18) converges at q = 1, then we have

$$y_i(t) = y_{i0}(t) + \sum_{m=1}^{\infty} y_{im}(t), \quad i = 1, 2, \ldots, n, \tag{20}$$

Define the vector

$\overrightarrow{y}_i = \{y_{i0}(t), y_{i1}(t), \ldots, y_{in}(t)\}.$

Differentiating Eq. (16) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we obtain the mth-order deformation equation for $i = 1, 2, \ldots, n$,

$$\begin{aligned}
&D^{\alpha_i}[y_{im}(t) - \chi_m y_{im-1}(t)] = h_iH_i(t)R_{im}(\overrightarrow{y}_{1m-1}, \ldots, \overrightarrow{y}_{nm-1}, t),\\
&\qquad i = 1, 2, \ldots, n-1,\\
&D^{\alpha_n}[y_{nm}(t) - \chi_m y_{nm-1}(t)] = h_iH_n(t)R_{nm}(\overrightarrow{y}_{1m-1}, \ldots, \overrightarrow{y}_{nm-1}, t),
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
&R_{im}(\overrightarrow{y}_{1m-1}, \ldots, \overrightarrow{y}_{nm-1}, t)\\
&\quad = \frac{1}{(m-1)!}\frac{\partial^{m-1}(D^{\alpha_i}\varphi_i(t;q) - \varphi_{i+1}(t;q))}{\partial q^{m-1}}\Big|_{q=0},\\
&R_{nm}(\overrightarrow{y}_{1m-1}, \ldots, \overrightarrow{y}_{nm-1}, t)\\
&\quad = \frac{1}{(m-1)!}\frac{\partial^{m-1}(D^{\alpha_i}\varphi_n(t;q) - f(t, \varphi_1, \varphi_2, \ldots, \varphi_n))}{\partial q^{m-1}}\Big|_{q=0}.
\end{aligned} \tag{22}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1,\\ 1, & m > 1. \end{cases}$$

Applying the Riemann-Liouville integral operator $I^{\alpha i}$ on both side of Eq. (21), we have

$$\begin{aligned}
y_{im}(t) &= \chi_m y_{im-1}(t) - \chi_m \sum_{j=0}^{n-1} y_{im-1}^j(0^+)\frac{t^j}{j!}\\
&\quad + I^{\alpha i}h_iH_i(t)R_{im}(\overrightarrow{y}_{1m-1}, \ldots, \overrightarrow{y}_{nm-1}, t),\\
y_{nm}(t) &= \chi_m y_{nm-1}(t) - \chi_m \sum_{j=0}^{n-1} y_{nm-1}^j(0^+)\frac{t^j}{j!}\\
&\quad + I^{\alpha i}h_iH_n(t)R_{nm}(\overrightarrow{y}_{1m-1}, \ldots, \overrightarrow{y}_{nm-1}, t).
\end{aligned} \tag{23}$$

In this way, it is easily to obtain y_{im} for $m \geqslant 1$ at Mth order, we have

$$y_i(t) = \sum_{m=0}^{M} y_{im}(t), \quad i = 1, 2, \ldots, n. \tag{24}$$

When $M \to \infty$, we get an accurate approximation of the original Eq. (15).

5. Illustrative Examples

To demonstrate the effectiveness of the method we consider here some multi-order FDE. We transform multi-order FDE to a system of FDE and solve the system of FDE using HAM.

(i) Consider the following initial value problem in case of the inhomogeneous Bagley–Torvik equation (Diethelm and Ford, 2002):

$$D_*^2 y(t) + D_*^{1.5} y(t) + y(t) = 1 + t, \quad y(0) = 1, \quad y'(0) = 1. \tag{25}$$

In view of the discussion in the Section 3, the Eq. (25) can be viewed as the following system of FDE:

$$D_*^{1.5} y_1 = y_2, \quad y_1(0) = y_1'(0) = 1,$$
$$D_*^{0.5} y_2 = -y_2 - y_1 + 1 + t, \quad y_2(0) = 0.$$

Using Eq. (23) we get the following scheme:

$$y_{10} = 1 + t, y_{20} = 0,$$
$$y_{1m} = \chi_m y_{1m} + h_1 I^{1.5}(D_*^{1.5} y_{1m-1} - y_{2m-1}), \quad m \geqslant 1,$$
$$y_{2m} = \chi_m y_{2m} + h_2 I^{0.5}(D_*^{0.5} y_{2m-1} + y_{2m-1} + y_{1m-1} - (1 - \chi_m)(1 + t)), \quad m \geqslant 1.$$

Thus we get:

$$y_{11} = h_1 I^{1.5}(D_*^{1.5} y_{10} - y_{20}) = 0,$$
$$y_{21} = h_2 I^{0.5}(D_*^{0.5} y_{20} + y_{20} + y_{10} - (1 + t)) = 0,$$

and hence

$$y_{1m} = 0, \quad y_{2m} = 0, \quad m \geqslant 1.$$

In view of the above terms, we find $y_1(t) = 1 + t$ and $y_2(t) = 0$. So $y(t) = 1 + t$ is the required solution of (25).

(ii) Consider the following initial value problem,

$$D_*^3 y(t) + D_*^{2.5} y(t) + y^2(t) = t^4, \quad y(0) = y'(0) = 0, \quad y''(0) = 2. \tag{26}$$

If we choose $y(t) = y_1$ and $D_*^{2.5} y(t) = y_2$, then Eq. (26) can be reduced to the following system of nonlinear FDE:

$$D_*^{2.5} y_1 = y_2, \quad y_1(0) = y_1'(0) = 0, \quad y''(0) = 2,$$
$$D_*^{0.5} y_2 = t^4 - y_2 - y_1^2, \quad y_2(0) = 0.$$

Applying $I^{2.5}$ and $I^{0.5}$ to both sides of above system and using HAM Eq. (23) we get the following scheme:

$$y_{10} = t^2, \quad y_{20} = 0,$$
$$y_{1m} = \chi_m y_{1m} + h_1 I^{2.5}(D_*^{2.5} y_{1m-1} - y_{2m-1}), \quad m \geqslant 1,$$
$$y_{2m} = \chi_m y_{2m} + h_2 I^{0.5}\left(D_*^{0.5} y_{2m-1} - t^4 + y_{2m-1} + \sum_{i=0}^{m-1} y_{1i} y_{1m-1-i}\right), \quad m \geqslant 1. \tag{27}$$

Thus

$$y_{11} = h_1 I^{2.5}(D_*^{2.5} y_{10}) = 0,$$
$$y_{21} = h_2 I^{0.5}(D_*^{0.5} y_{20} - t^4 + y_{20} + y_{10}^2) = 0.$$

So $y_{1m+1} = y_{2m+1} = 0$, for $m \geqslant 1$. Hence we find $y_1(t) = t^2$ and $y_2(t) = 0$. Therefore $y(t) = t^2$ is the required solution.

(iii) Consider the following equation,

$$D_*^{1.455} y(t) = -t^{0.1} \frac{E_{1.545}(-t)}{E_{1.445}(-t)} Exp(t) y(t) D_*^{0.555} y(t) + Exp(-2t) - \left[D_*^1 y(t)\right]^2. \tag{28}$$

Eq. (28) is equivalent to the following system of three FDE only.

$$D_*^{0.555} y_1(x) = y_2(x),$$
$$D_*^{0.445} y_2(x) = y_3(x) (= y'(x)),$$
$$D_*^{0.455} y_3(x) = -t^{0.1} \frac{E_{1.545}(-t)}{E_{1.445}(-t)} Exp(t) y_1(t) y_2(t) + Exp(-2t) - y_3^2(t), \tag{29}$$

where $y_1(t) = y(t)$. In view of Eq. (29) and HAM we get

$$y_{10} = y_1(0), \quad y_{20} = y_2(0), \quad y_{30} = y_3(0),$$
$$y_{1m} = \chi_m y_{1m} + h_1 I^{0.555}(D_*^{0.555} y_{1m-1} - y_{2m-1}),$$
$$y_{2m} = \chi_m y_{2m} + h_2 I^{0.445}(D_*^{0.445} y_{2m} - y_{3m-1}),$$
$$y_{3m} = \chi_m y_{3m} + h_3 I^{0.455}\left(D_*^{0.455} y_{3m-1} + t^{0.1} \frac{E_{1.545}(-t)}{E_{1.445}(-t)} Exp(t) y_{1m-1} y_{2m-1}(t) - Exp(-2t) + y_{3m-1}^2\right).$$

In Fig. 1, we draw exact solution ($y = y_1 = e^{-t}$) and solution obtained after 3 iterations $h_1 = -1.4$.

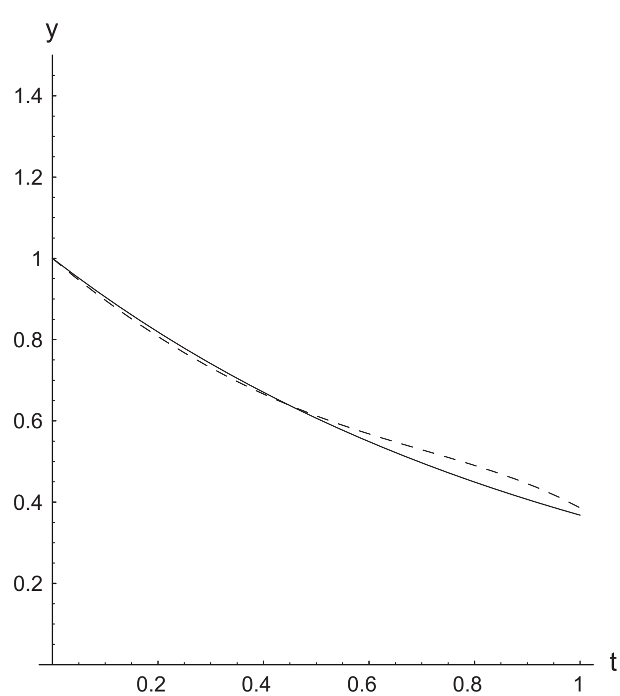

Figure 1 Exact solution (solid line) and HAM solution obtained after 3 iterations ($y \cong y_{13}$) (dashed line).

The computations presented here have been carried out with the help of Mathematica 5.

6. Discussion and conclusions

Homotopy analysis method consequently has been utilized to solve the system of fractional differential equations generated by a multi-order fractional differential equation. Thus it has been demonestrated that HAM proves useful in solving linear as well as non linear multi-order fractional differential equations.

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