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Optimality conditions for parabolic systems with variable coefficients involving Schrödinger operators

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Abstract In this paper, we study the existence of a solution to $n \times n$ parabolic systems with variable coefficients involving Schrödinger operators defined on an unbounded domain of R^n. We then discuss the necessary and sufficient conditions of optimality for these systems.

1. Introduction

Today, optimal control problems of distributed systems, that include partial differential equations have many mechanical and technical sources and a variety of technological and scientific applications.

Indeed, many optimal control problems of elliptic systems involving Schrödinger operators of the distributed type have been studied, as in Serag (2000, 2004) and Serag and Qamlo (2005). Whereas some of these problems had positive weight functions (Serag, 2004; Serag and Qamlo, 2005), others had constant coefficients, e.g., (Serag, 2000).

The necessary and sufficient conditions of optimality for 2×2 parabolic and hyperbolic systems involving Schrödinger operators have already been discussed in (Bahaa, 2006; Qamlo, 2013).

In addition, optimal control problems for systems involving parabolic and hyperbolic operators with an infinite number of variables have been introduced in (Kotarski et al., 2002; Serag, 2007; Qamlo, 2008, 2009; Bahaa and El-Shatery, 2013).

Furthermore, time-optimal control of infinite order parabolic and hyperbolic systems has been studied in (Kowalewski and Krakowiak, 2008; Kowalewski, 2009).

Here, we discuss the following n × n parabolic systems with variable coefficients involving Schrödinger operators that are defined on an unbounded domain of R^n:

$$\begin{cases} \frac{\partial}{\partial t} Y + L_q Y = A(x)Y + F(x,t) & \text{in } Q, \\ Y(x) \to 0 \ as \ |x| \to \infty, & \\ Y(x) = 0 & \text{on } \sum, \\ y_i(x,0) = y_{i,0}(x) \quad \forall i = 1, \cdots n, & \text{in } \Omega, \end{cases} \tag{1}$$

Y and F are column matrices with elements y_i and f_i, repectively, In addition, $Q = \Omega \times (0,T)$ with boundary $\Sigma = \partial\Omega \times (0,T)$ and Ω is an unbounded domain of R^n with boundary $\partial\Omega$. and L_q is a $n \times n$ diagonal matrix of the Schrödinger operator $(-\Delta + q)$, the potential q is a positive function that tends to ∞ at infinity, and $A(x)$ is an $n \times n$ matrix of variable coefficients $a_{ij}(x)$ $(1 \leqslant i,j \leqslant n)$ that satisfy the following conditions:

there exist $r > 1$ and $k > 0$ such that

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$$a_{ij}(x) \in \left(0, \frac{k}{(1+|x|^2)^r}\right) \quad \forall i,j = 1,2,\ldots,n, \quad \forall x \in \Omega, \tag{2}$$

$$a_{ij}(x) \leqslant \sqrt{a_{ii}(x)a_{jj}(x)} \quad \forall i,j = 1,2,\ldots,n, \quad \forall x \in \Omega. \tag{3}$$

We first prove the existence and uniqueness of the state for system (1), and we then introduce the necessary and sufficient conditions of optimality for this system by a set of equations and inequalities.

2. Some facts and results

To prove our theorems, we recall certain results that are introduced in Djellit and Yechoui (1997) regarding the existence of the principal eigenvalue λ_q^+ of the following problem:

$$\begin{cases} (-\Delta + q)y = \lambda g(x)y & \text{in} \quad \Omega, \\ y \to 0 \quad \text{as} \quad |x| \to \infty, & \\ y = 0 & \text{on} \quad \Gamma, \end{cases} \tag{4}$$

where Ω is an unbounded connected open subset of R^n with boundary $\partial\Omega$ and both the potential q and the weight function $g(x)$ are measurable functions that tend to zero at infinity.

For $n > 2$, if $\exists\alpha > 0,\ \beta \geqslant 1, \alpha > \beta,\ \exists k > 0, c > 0$ such that

$$0 < g(x) \leqslant \frac{k}{(1+|x|^2)^\alpha}, \qquad 0 < q(x) \leqslant \frac{c}{(1+|x|^2)^\beta}, \tag{5}$$

where the eigenvalue problem (4) has a positive principal eigenvalue λ_1^+ that is simple and associated with a positive eigenfunction φ_q in V_+. Moreover λ_1^+ is characterized by:

$$\lambda_1^+ \int_\Omega g(x)|y|^2 \leqslant \int_\Omega (|\nabla y|^2 + q|y|^2), \tag{6}$$

where

$V_+ = \{y \in V(\Omega) : \int_\Omega g|u|^2 dx > 0\}$ and

$V(\Omega) = \{y \in D'(\Omega) : p_1 y \in L^2(\Omega), \nabla y \in L^2(\Omega)\};$

$p_\alpha = \rho^{2\alpha}(x), \alpha > 0, \rho(x) = (1+|x|^2)^{-1/2},$

Furthermore, $V(\Omega)$ is a Hilbert space with the inner product $(y,\psi)_V = \int_\Omega(\nabla y \cdot \nabla\psi + p_1 y \cdot \psi)dx$ and the corresponding norm $\|y\|_V = \left(\int_\Omega(|\nabla y|^2 + p_1|y|^2)dx\right)^{1/2}$ which is equivalent to $\|y\|_q = \left(\int_\Omega(|\nabla y|^2 + q|y|^2)dx\right)^{1/2}$.

Now, to study our system (1), we recall the introduced by Serag (2000):

$$L_g^2(\Omega) = \{y : \Omega \to R : \int_\Omega g(x)y^2 dx < \infty\},$$

with an inner product $(y,\psi)_g = \int_\Omega g(x)y\psi\ dx$. We then have the following embeddings:

$V(\Omega) \subseteq L_g^2(\Omega) \subseteq V'(\Omega),$

$V(\Omega) \subseteq L^2(\Omega) \subseteq V'(\Omega),$

and we introduce the space $L^2(0,T;V(\Omega))$ of measurable functions $t \to f(t)$ which is defined on the open interval $(0,T)$, as the variable $t \in (0,T)$ denotes the time, where $T < \infty$.

$L^2(0,T;V(\Omega))$ is a Hilbert space with the scalar product

$$(f(t),g(t))_{L^2(0,T;V(\Omega))} = \int_{(0,T)} (f(t),g(t))_{V(\Omega)} dt,$$

and the norm $\|f(t)\|_{L^2(0,T;V(\Omega))} = \left(\int_{(0,T)} \|f(t)\|^2_{V(\Omega)} dt\right)^{1/2} < \infty.$

Analogously, we can define the space $L^2\left(0,T;L_g^2(\Omega)\right) = L^2(Q),$

with the following scalar product:

$$(f(t),g(t))_{L^2(Q)} = \int_{(0,T)} (f(t),g(t))_{L_g^2(\Omega)} dt = \int_Q f(t)g(t) dx\ dt.$$

Then we have the following chain:

$L^2(0,T;V(\Omega)) \subseteq L^2(Q) \subseteq L^2(0,T;V'(\Omega)),$

and by the Cartesian product, we have

$(L^2(0,T;V(\Omega)))^n \subseteq (L^2(Q))^n \subseteq (L^2(0,T;V'(\Omega)))^n.$

In addition, we will use the following definition of M-matrices (Bermann and Plemmons, 1979; Serag, 2004).

Definition 1. A nonsingular matrix $\beta = (b_{ij})$ is an M-matrix if $b_{ij} < 0$ for $i \neq j, b_{ii} > 0$ and if the principal minors extracted from β are positive.

3. The scalar case

In this section, we consider the scalar case (i.e., a system that consists of one equation):

$$\begin{cases} \frac{\partial y(x)}{\partial t} + (-\Delta + q)y(x) = g(x)y(x) + f(x,t) & \text{in} \quad Q, \\ y(x) \to 0 \quad \text{as} \quad |x| \to \infty & \\ y(x) = 0 & \text{on} \quad \sum, \\ y(x,0) = y_0(x) & \text{in} \quad \Omega \end{cases} \tag{7}$$

Proposition 1. *For $f \in L^2(0,T;V'(\Omega))$ and $y_0(x) \in V(\Omega)$, there exists a unique solution $y \in L^2(0,T;V(\Omega))$ for system (7) if $1 < \lambda_1^+$.*

Proof. The continuous bilinear form:

$$\pi(t;y,\varphi) = \int_\Omega (\nabla y \nabla\varphi + qy\varphi)dx - \int_\Omega g(x)y\varphi dx \tag{8}$$

is obviously coercive on $V(\Omega)$.

In fact, we have:

$$\pi(t;y,y) = \int_\Omega (|\nabla y|^2 + q|y|^2)dx - \int_\Omega g(x)y^2 dx$$
$$= \int_\Omega (|\nabla y|^2 + (q+mg)|y|^2)dx - (1+m)$$
$$\int_\Omega g(x)y^2 dx, \quad m > 0,$$

Then, from (6):

$$\pi(t;y,y) \geqslant \left(1 - \frac{1+m}{\lambda_1^+ + m}\right)\int_\Omega (|\nabla y|^2 + (q+mg)|y|^2)dx,$$

that is,

$$\pi(t; y, y) \geqslant \left(1 - \frac{1+m}{\lambda_1^+ + m}\right)\|y\|_q^2, \tag{9}$$

which proves the coerciveness condition of the bilinear form (8) on $V(\Omega)$. Then, by the Lax-Milgram lemma, there exists a unique solution $y \in L^2(0,T;V(\Omega))$ for the system (7). Now, we can formulate the optimal control problem for system (7) as follows:

The space $L^2(Q)$ is the space of controls. For a control $u \in L^2(Q)$, the state $y(u) \in L^2(0,T;V(\Omega))$ of the system is given by the solution of the following problem:

$$\begin{cases} \frac{\partial y(u)}{\partial t} + (-\Delta + q)y(u) = g(x)y(u) + f(x,t) + u & \text{in} \quad Q, \\ y \to 0 \quad \text{as} \quad |x| \to \infty \\ y(u) = 0 & \text{on} \quad \sum, \\ y(x,0,u) = y_0(x) & \text{in} \quad \Omega \end{cases} \tag{10}$$

where $y(u) \in L^2(0,T;V(\Omega)), \frac{\partial y(u)}{\partial t} \in L^2(0,T;V'(\Omega))$.

The observation equation is given by $z(u) = y(u)$.

For a given $z_d \in L^2(Q)$, the cost function is given by

$$J(v) = \|y(v) - z_d\|^2_{L^2(Q)} + M\|v\|^2_{L^2(Q)}, \tag{11}$$

where M is a positive constant.

The control problem is to find $u \in U_{ad}$ such that $J(u) \leqslant J(v)$,

where U_{ad} is a closed convex subset of $L^2(Q)$.

The cost function (11) can be written as was performed by Lions (1971):

$$J(v) = a(v,v) - 2L(v) + \|y(0) - z_d\|^2_{L^2(Q)},$$

where $a(v,v)$ is a continuous coercive bilinear form and $L(v)$ is a continuous linear form on $L^2(0,T;V(\Omega))$ .Then using the general theory of Lions (1971), there exists a unique optimal control $u \in U_{ad}$ such that $J(u) = \inf J(v)$ for all $v \in U_{ad}$. Moreover, we have the following proposition that gives the necessary and sufficient conditions of optimality: □

Proposition 2. *Assume that (9) holds. If the cost function is given by (11), the optimal control $u \in L^2(Q)$ is then characterized by:*

$$\begin{cases} \frac{-\partial p(u)}{\partial t} + (-\Delta + q)p(u) - g(x)p(u) = y(u) - z_d & \text{in} \quad Q, \\ p \to 0 \quad \text{as} \quad |x| \to \infty, \\ p(u) = 0 & \text{on} \quad \sum, \\ p(x,T,u) = 0 & \text{in} \quad \Omega \end{cases} \tag{12}$$

where $p(u) \in L^2(0,T;V(\Omega)), \frac{\partial p(u)}{\partial t} \in L^2(0,T;V'(\Omega))$. *Furthermore, we have the inequality*

$$(p(u) + Mu, v - u)_{L^2Q} \geqslant 0 \qquad \forall v \in U_{ad}, \tag{13}$$

together with (10), where p(u) is the adjoint state.

4. The case of systems

4.1. Operator equation

In this section, we prove the existence and uniqueness of a solution to system (1), which can be written as follows:

$$\begin{cases} \frac{\partial y_i(x)}{\partial t} + (-\Delta + q)y_i(x) = \sum_{j=1}^n a_{ij}(x)y_j + f_i(x,t) & \text{in} \quad Q, \\ y_i \to 0 \quad \text{as} \quad |x| \to \infty, \\ y_i(x) = 0 & \text{on} \quad \sum, \\ y_i(x,0) = y_{i,0}(x) \quad \forall i = 1,2,3\cdots n. & \text{in} \quad \Omega, \end{cases}$$

We introduce the continuous bilinear form $\pi(t;y,\varphi)$: $(V(\Omega))^n \times (V(\Omega))^n \to R$ as follows:

$$\pi(t;y,\varphi) = \sum_{i=1}^n \int_\Omega (\nabla y_i \nabla \varphi_i + q y_i \varphi_i)dx - \sum_{j\neq i}^n \int_\Omega a_{ij}(x) y_j \varphi_i dx - \sum_{i=1}^n \int_\Omega a_{ii}(x) y_i \varphi_i dx, \tag{14}$$

Proposition 3. *If conditions (2) and (3) hold, then the bilinear form (14) is coercive on $(V(\Omega))^n$ if the matrix:*

$$\left(\Lambda_1^+(a_{ii}) - I\right) = \begin{bmatrix} \lambda_1^+(a_{11}) - 1 & -1 & \dots & -1 \\ -1 & \lambda_1^+(a_{22}) - 1 & \dots & -1 \\ \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \dots & \lambda_1^+(a_{nn}) - 1 \end{bmatrix} \tag{15}$$

is a nonsingular M-matrix (15); it is assumed that $\Lambda_1^+(a_{ii})$ is a diagonal matrix with elements $\lambda_1^+(a_{ii})$.

$\lambda_1^+(a_{ii})$ is the principal eigenvalue for the eigenvalue problem (4) when we replace the function $g(x)$ with $a_{ii}(x)$ in (4).

Proof.

$$\pi(t;y,y) = \sum_{i=1}^n \int_\Omega (|\nabla y_i|^2 + q|y_i|^2) - \sum_{j\neq i}^n \int_\Omega a_{ij}(x) y_i y_j - \sum_{i=1}^n \int_\Omega a_{ii}(x)|y_i|^2$$

$$\pi(t;y,y) = \sum_{i=1}^n \int_\Omega (|\nabla y_i|^2 + (q + a_{ii}(x))|y_i|^2) - \sum_{j\neq i}^n \int_\Omega a_{ij}(x) y_i y_j - 2\sum_{i=1}^n \int_\Omega a_{ii}(x)|y_i|^2.$$

Consider the following variational characterization of $\lambda_1^+(a_{ii})$:

$$\lambda_1^+(a_{ii}) \int_\Omega a_{ii}(x)|y|^2 \leqslant \int_\Omega (|\nabla y|^2 + q|y|^2), \tag{16}$$

By employing this characterization, we obtain the following:

$$\pi(t;y,y) \geqslant \sum_{i=1}^n \int_\Omega (|\nabla y_i|^2 + (q + a_{ii}(x))|y_i|^2) - \sum_{j\neq i}^n \int_\Omega a_{ij}(x) y_i y_j - 2\sum_{i=1}^n \frac{1}{\lambda_1^+(a_{ii}) + 1} \int_\Omega (|\nabla y_i|^2 + (q + a_{ii}(x))|y_i|^2).$$

Using (3), we obtain the following:

$$\pi(t;y,y) \geqslant \sum_{i=1}^n \int_\Omega (|\nabla y_i|^2 + (q + a_{ii}(x))|y_i|^2) - 2 \sum_{\substack{i=1 \\ j=i+1}}^n \int_\Omega \times \sqrt{a_{ii}(x) a_{jj}(x)} y_i y_j - 2\sum_{i=1}^n \frac{1}{\lambda_1^+(a_{ii}) + 1} \int_\Omega (|\nabla y_i|^2 + (q + a_{ii}(x))|y_i|^2).$$

By the Cauchy-Schwartz inequality and (16), we deduce:

$$\pi(t;y,y) \geqslant \sum_{i=1}^{n}\left(1-\frac{2}{\lambda_1^+(a_{ii})+1}\right)\int_\Omega(|\nabla y_i|^2+(q+a_{ii}(x))|y_i|^2) - \sum_{\substack{i=1\\ j=i+1}}^{n} \frac{2}{\sqrt{\lambda_1^+(a_{ii})+1}\sqrt{\lambda_1^+(a_{jj})+1}}.$$

$$\left(\int_\Omega(|\nabla y_i|^2+(q+a_{ii}(x))|y_i|^2)\right)^{1/2}\cdot\left(\int_\Omega(|\nabla y_j|^2+(q+a_{jj}(x))|y_j|^2)\right)^{1/2}$$
$$=\sum_{i=1}^{n}\left(\frac{\lambda_1^+(a_{ii})-1}{\lambda_1^+(a_{ii})+1}\right)\int_\Omega(|\nabla y_i|^2+(q+a_{ii}(x))|y_i|^2)$$
$$-\sum_{\substack{i=1\\ j=i+1}}^{n}\left(\frac{\int_\Omega|\nabla y_i|^2+(q+a_{ii})|y_i|^2}{\lambda_1^+(a_{ii})+1}+\frac{\int_\Omega|\nabla y_j|^2+(q+a_{jj})|y_j|^2}{\lambda_1^+(a_{jj})+1}\right)$$
$$+\sum_{\substack{i=1\\ j=i+1}}^{n}\left(\sqrt{\frac{\int_\Omega|\nabla y_i|^2+(q+a_{ii})|y_i|^2}{\lambda_1^+(a_{ii})+1}}-\sqrt{\frac{\int_\Omega|\nabla y_j|^2+(q+a_{jj})|y_j|^2}{\lambda_1^+(a_{jj})+1}}\right)^2.$$

We then have the following:

$$\geqslant\sum_{i=1}^{n}\left(\frac{\lambda_1^+(a_{ii})-1}{\lambda_1^+(a_{ii})+1}\right)\int_\Omega(|\nabla y_i|^2+(q+a_{ii}(x))|y_i|^2)$$
$$-\sum_{\substack{i=1\\ j=i+1}}^{n}\left(\frac{\int_\Omega|\nabla y_i|^2+(q+a_{ii})|y_i|^2}{\lambda_1^+(a_{ii})+1}+\frac{\int_\Omega|\nabla y_j|^2+(q+a_{jj})|y_j|^2}{\lambda_1^+(a_{jj})+1}\right)$$
$$=\sum_{i=1}^{n}\left(1-\frac{n+1}{\lambda_1^+(a_{ii})+1}\right)\int_\Omega(|\nabla y_i|^2+(q+a_{ii}(x))|y_i|^2).$$

From (15), we deduce that

$$\geqslant\sum_{i=1}^{n}\left(1-\frac{n+1}{\lambda_1^+(a_{ii})+1}\right)\int_\Omega(|\nabla y_i|^2+q|y_i|^2).$$

Hence

$$\pi(t;y,y)\geqslant \mathrm{C}\sum_{i=1}^{\mathrm{n}}\|y_i\|_q^2, \qquad (17)$$

which proves the coerciveness condition of the bilinear form (14) on $(V(\Omega))^n$. Thus, by the Lax-Milgram lemma, there exists a unique solution $y=\{y_1,y_2,\ldots,y_n\}\in(L^2(0,T;V(\Omega)))^n$ such that:

$$\left(\frac{\partial y}{\partial t},\varphi\right)+\pi(t;y,\varphi)=L(\varphi)\qquad\forall\varphi\in(L^2(0,T;V(\Omega)))^n,$$

where $L(\varphi)$ is a continuous linear form on $(L^2(0,T;V(\Omega)))^n$ that takes the following form:

$$L(\varphi)=\sum_{i=1}^{n}\int_Q f_i(x,t)\varphi_i(x)dxdt+\sum_{i=1}^{n}\int_\Omega y_{i,0}(x)\varphi_i(x,0)dx,$$

□

As a result, we have the following theorem:

Theorem 1. *Under hypotheses (2), (3) and (15), for a given* $f=\{f_1,f_2,\ldots,f_n\}\in(L^2(0,T;V'(\Omega)))^n$ *and* $y_{i,0}(x)\in V(\Omega)$, *there exists a unique solution* $y=\{y_1,y_2,\ldots,y_n\}\in(L^2(0,T;V(\Omega)))^n$ *to system (1).*

4.2. The control problem

In this section, using the theory of Lions (1971), we discuss the existence and characterization of the optimal control for system (1).

The space $(L^2(Q))^n$ is the space of controls. For a control $u=\{u_1,u_2,\ldots,u_n\}\in(L^2(Q))^n$, the state $y(u)=\{y_1(u),y_2(u),\ldots,y_n(u)\}\in(L^2(0,T;V(\Omega)))^n$ of system (1) is given by the solution of:

$$\begin{cases}\frac{\partial y_i(u)}{\partial t}+(-\Delta+q)y_i(u)=\sum_{j=1}^{n}a_{ij}(x)y_j(u)+f_i(x,t)+u_i & \text{in } Q,\\ y_i\to0 \quad\text{as}\quad |x|\to\infty, & \\ y_i(u)=0 & \text{on } \sum,\\ y_i(x,0,u)=y_{i,0}(u)\quad\forall i=1,2,3,\ldots,n & \text{in } \Omega,\end{cases} \qquad (18)$$

with $y(u)\in(L^2(0,T;V(\Omega)))^n,\frac{\partial y(u)}{\partial t}\in(L^2(0,T;V'(\Omega)))^n$.

The observation equation is given by $z(u)=\{z_1(u),z_2(u),\ldots,z_n(u)\}=y(u)=\{y_1(u),y_2(u),\ldots,y_n(u)\}$.

For a given $z_d=\{z_{d1},z_{d2},\ldots,z_{dn}\}$ in $(L^2(Q))^n$, the cost function is given by:

$$J(v)=\sum_{i=1}^{n}\int_{(0,T)}(y_i(v)-z_{di},y_i(v)-z_{di})_{L_g^2(\Omega\Omega)}dt + M\|v\|^2_{(L^2(Q))^n},\qquad \textit{where}\ \mathrm{M}\ \text{is a positive constant.} \qquad (19)$$

Thus, the control problem is to find $\inf J(v)$ over a closed convex subset U_{ad} of $(L^2(Q))^n$.

The cost function (19) can be written as in Lions (1971):

$$J(v)=a(v,v)-2L(v)+\|y(0)-z_d\|^2_{(L^2(Q))^n},$$

where $a(v,v)$ is a continuous coercive bilinear form and $L(v)$ is a continuous linear form on $(L^2(0,T;V(\Omega)))^n$. Then using the general theory of Lions (1971), there exists a unique optimal control $u\in U_{ad}$ such that $J(u)=\inf J(v)$ for all $v\in U_{ad}$. Moreover, we have the following theorem which gives the necessary and sufficient conditions of optimality.

Theorem 2. Assume that (2), (3) and (15) hold. If the cost function is given by (19), there exists a unique optimal control $u=\{u_1,u_2,\ldots u_n\}\in(L^2(Q))^n$ such that $J(u)\leqslant J(v)\quad\forall v\in U_{ad}$.

Moreover, this control is characterized by the following equations and inequalities:

$$\begin{cases}\frac{-\partial p_i(u)}{\partial t}+(-\Delta+q)p_i(u)-\sum_{j=1}^{n}a_{ji}(x)p_j(u)=y_i(u)-z_{di} & \text{in } Q,\\ p_i\to0\quad\text{as}\quad|x|\to\infty, & \\ p_i(u)=0 & \text{on } \sum,\\ p_i(x,T,u)=0\quad\forall i=1,2,3,\ldots,n & \text{in } \Omega,\end{cases} \qquad (20)$$

where $p(u)\in(L^2(0,T;V(\Omega)))^n,\frac{\partial p(u)}{\partial t}\in(L^2(0,T;V'(\Omega)))^n$.

$$\sum_{i=1}^{n}\int_{(0,T)}(p_i(u),v_i-u_i)_{L_g^2(\Omega)}dt+M(u,v-u)_{(L^2(Q))^n}\geqslant0\quad\forall v=(v_1,v_2,\cdots v_n)\in U_{ad}, \qquad (21)$$

The above equation can be combined with (18), where $p(u) = \{p_1(u), p_2(u), \dots p_n(u)\}$ is the adjoint state.

Proof. The optimal control $u = (u_1, u_2, \dots u_n) \in (L^2(Q))^n$ is characterized by Lions (1971):

$$\sum_{i=1}^{n} J'(u)(v_i - u_i) \geqslant 0 \qquad \forall v = (v_1, v_2, \dots, v_n) \quad \text{in} \quad U_{ad},$$

which is equivalent to

$$\sum_{i=1}^{n} (y_i(u) - z_{di}, y_i(v) - y_i(u))_{L^2(Q)} + M(u, v-u)_{(L^2(Q))^n} \geqslant 0.$$

This inequality can be written as follows:

$$\sum_{i=1}^{n} \int_{(0,T)} (y_i(u) - z_{di}, y_i(v) - y_i(u))_{L^2_g(\Omega)} dt + M(u, v-u)_{(L^2(Q))^n} \geqslant 0. \tag{22}$$

Now,

$$(p, Ay)_{(L^2(Q))^n} = \sum_{i=1}^{n} \int_{(0,T)} \left(p_i(u), \frac{\partial y_i(u)}{\partial t} + (-\Delta + q) y_i(u) - \sum_{j=1}^{n} a_{ij}(x) y_j(u) \right)$$

where

$$\begin{aligned} Ay(u) &= A\{y_1(u), y_2(u), \dots, y_n(u)\} \\ &= \Bigg\{ \tfrac{\partial y_1(u)}{\partial t} + (-\Delta + q) y_1(u) - \sum_{j=1}^{n} a_{1j}(x) y_j(u), \\ &\quad \tfrac{\partial y_2(u)}{\partial t} + (-\Delta + q) y_2(u) - \sum_{j=1}^{n} a_{2j}(x) y_j(u), \\ &\quad \vdots \\ &\quad \tfrac{\partial y_n(u)}{\partial t} + (-\Delta + q) y_n(u) - \sum_{j=1}^{n} a_{nj}(x) y_j(u) \Bigg\}. \end{aligned}$$

By using Green's formula

$$\begin{aligned} &(p, Ay)_{(L^2(Q))^n} \\ &= \sum_{i=1}^{n} \int_{(0,T)} \left(\frac{-\partial p_i(u)}{\partial t} + (-\Delta + q) p_i(u) - \sum_{j=1}^{n} a_{ji}(x) p_j(u), y_i \right)_{L^2_g(\Omega)} dt \\ &= (A^* p, y)_{(L^2(Q))^n}. \end{aligned}$$

Hence, we have

$$\begin{aligned} A^* p(u) &= A^* \{p_1(u), p_2(u), \cdots p_n(u)\} \\ &= \Bigg\{ \tfrac{-\partial p_1(u)}{\partial t} + (-\Delta + q) p_1(u) - \sum_{j=1}^{n} a_{j1}(x) p_j(u), \\ &\quad \tfrac{-\partial p_2(u)}{\partial t} + (-\Delta + q) p_2(u) - \sum_{j=1}^{n} a_{j2}(x) p_j(u), \\ &\quad \vdots \\ &\quad \tfrac{-\partial p_n(u)}{\partial t} + (-\Delta + q) p_n(u) - \sum_{j=1}^{n} a_{jn}(x) p_j(u) \Bigg\}. \end{aligned} \tag{23}$$

inasmuch as the adjoint equation takes the following form:

$$\frac{-\partial p(u)}{\partial t} + A^* p(u) = y(u) - z_d.$$

Therefore, from Theorem 1, we obtain a unique solution that satisfies $p(u) \in (L^2(0,T; V(\Omega)))^n$.

This result proves Eq. (20).

Now, Eq. (22) can be written as:

$$\sum_{i=1}^{n} \int_{(0,T)} \left(\frac{-\partial p_i(u)}{\partial t} + (-\Delta + q) p_i(u) - \sum_{j=1}^{n} a_{ji}(x) p_j(u), y_i(v) - y_i(u) \right)_{L^2_g(\Omega)} dt + M(u, v-u)_{(L^2(Q))^n} \geqslant 0.$$

Using Green's formula, we obtain:

$$\sum_{i=1}^{n} \int_{(0,T)} \left(p_i(u), \left(\frac{\partial}{\partial t} + (-\Delta + q) - \sum_{j=1}^{n} a_{ij} \right) y_i(v) - y_i(u) \right)_{L^2_g(\Omega)} dt + M(u, v-u)_{(L^2(Q))^n} \geqslant 0.$$

Furthermore, using (18), we have that

$$\sum_{i=1}^{n} \int_{(0,T)} (p_i(u), v_i - u_i)_{L^2_g(\Omega)} dt + M(u, v-u)_{(L^2(Q))^n} \geqslant 0,$$

which proves (21). □

Remarks 1.

(1) If $q = 0$ in system (1), we have some existence results for the elliptic operator that were discussed in Serag (1998b).
(2) If $q = 0$ and $n = 2$ in system (1), we obtain some results for the elliptic operator that were introduced in Serag (1998a).
(3) If $a_{ij}(x) = g(x)$ in system (1), we obtain some existence results for the elliptic operator that were obtained in Serag and Qamlo (2008).
(4) If $a_{ij}(x) = g(x)$ and $n = 2$ in system (1), we obtain some results for the elliptic operator that were obtained in Gali and Serag (1995).

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