Journal of King Saud University – Science 35 (2023) 102637

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Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

A note on approximate controllability of second-order impulsive stochastic Volterra-Fredholm integrodifferential system with infinite delay

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ARTICLE INFO

Article history:
Received 28 December 2021
Revised 26 February 2023
Accepted 4 March 2023
Available online 11 March 2023

2010 Mathematics Subject Classification:
34G20
35B41
47H04
60H20
93B05

Keywords:
Mild solution
Multivalued maps
Fixed point theorem
Stochastic differential system
Infinite delay

ABSTRACT

This manuscript examines the approximate controllability of a second-order stochastic Volterra-Fredholm integrodifferential system including delay and impulses. Primarily, by utilizing stochastic theory, the cosine family of operators, and the fixed point approach, we verify the existence of mild solutions for the given system. In particular, we establish a new set of sufficient requirements for the approximate controllability of the system. On the condition that the associated linear system is approximately controllable, the outcome is derived. In addition, we extend our system with nonlocal conditions. To demonstrate the theory of the primary outcomes, an example is shown.
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1. Introduction

In the field of mathematical control theory, controllability is one of the key ideas. Both finite and infinite dimensional systems have a powerful impact on the notion of controllability. Exact controllability and approximate controllability are the two basic notions of controllability in infinite-dimensional settings. The notion of exact controllability rarely applies to infinite-dimensional control problems. So, it is crucial to investigate approximate controllability. Mahmudov and Denker (2000), Mahmudov (2001) examined several kinds of controllability and established the required and adequate conditions for the controllability notions. In recent days, studying the numerical solution of differential systems is seeking great attention from many researchers. Further, Al-Smadi et al. (2014); Arqub and Rashaideh (2018) discussed the numerical solution of differential equations for periodic boundary value problems. Momani et al. (2020), the authors analyzed convergence for Lienard's equation involving the Atangana-Baleanu-Caputo model and established the piecewise optimal fractional reproducing kernel solution. Moreover, by using the Atangana-Baleanu fractional approach, Momani et al. (2020) introduced the reproducing kernel algorithm for the numerical solution of the Van der Pol damping model.

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Peer review under responsibility of King Saud University.

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https://doi.org/10.1016/j.jksus.2023.102637
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The theory of impulsive differential equations is currently receiving a lot of interest from researchers and has become a significant topic of study in recent years. Chang (2007) used the fixed point approach of Schauder along with the semigroup operator to provide an adequate requirement for the controllability of impulsive differential systems involving delay. Further, Sivasankaran et al. (2011) established the existence of global solutions for second-order impulsive differential systems via the Alternative fixed point approach of Leray–Schauder. Many biological processes use differential systems with impulsive conditions. Applications include thresholds, bursting rhythm models in biology and medicine, and frequency modulated systems. Consult the books (Bainov and Simeonov, 1993; Laksmikantham et al., 1989) for more information.

On the other hand, second-order differential systems have received much greater attention since they are utilized to analyze a variety of real-world issues. Sometimes it is advantageous to deal directly with second-order differential systems instead of converting them to first-order systems. It may be utilized to model a variety of physical processes. Hernández et al. (2009) investigated the existence of mild solutions for impulsive second-order neutral differential systems involving infinite delay via the concept of the cosine family of operators. By utilizing the fixed point approach of Bohnenblust-Karlin, the authors have established the approximate controllability of second-order differential systems in (Mahmudov et al., 2016). Recently, Vijayakumar et al. (2021b) examined the approximate controllability of second-order impulsive neutral differential systems consisting of Sobolev-type, impulses, neutral functions, and infinite delay via the fixed point theorems along with the cosine function of operators.

Stochastic differential systems are used in the development and analysis of mechanical, electrical, control engineering, and physical sciences, in addition to producing more realistic models. Ren and Sun (2002) verified the existence and uniqueness of mild solutions for second-order stochastic evolution systems consisting of neutral functions involving infinite delay via the successive approximation technique. Later, Sakthivel et al. (2010) investigated the approximate controllability of second-order stochastic equations involving impulsive effects by means of Hölder's inequality, stochastic theory, and the fixed point approach. For more specifics, refer to the book (Mao, 1997) and the research papers (Mahmudov and Mckibben, 2006; Ren et al., 2011; Revathi et al., 2016; Yan, 2015).

Moreover, Volterra-Fredholm's integrodifferential systems play an important role in the study of physics and biology. Chang and Chalishajar (2008) formulated the requirements for the controllability of Volterra-Fredholm-type integrodifferential systems by utilizing the fixed point theorem of Bohnenblust-Karlin along with the semigroup operator. Further, Muthukumar and Balasubramaniam (2011) used the Banach fixed point approach to discuss the approximate controllability of stochastic Volterra-Fredholm type integrodifferential equations. Recently, Vijayakumar et al. (2021a) verified the approximate controllability outcomes for fractional Volterra-Fredholm integrodifferential systems consisting of the Sobolev type recently via fractional calculus, the cosine family of operators, and the fixed point technique.

Inspired by the above articles, this manuscript is concerned with studying the approximate controllability of second-order stochastic Volterra-Fredholm integrodifferential system involving impulsive effects and infinite delay of the form

$$\begin{aligned} dz'(\varpi) &\in [Az(\varpi) + \mathscr{B}u(\varpi)]d\varpi \\ &\quad + G\left(\varpi, z_\varpi, \int_0^\varpi l(\varpi,\mu,z_\mu)d\mu, \int_0^{\mathfrak{p}} m(\varpi,\mu,z_\mu)d\mu\right)dW(\varpi), \qquad (1.1) \\ &\quad \varpi \in \mathscr{V} = [0,\mathfrak{p}],\ \varpi \neq \varpi_\imath,\ \imath = 1,2,\ldots,\jmath, \\ z(\varpi) &= \alpha(\varpi) \in L^2(\Omega,\mathscr{B}_\vartheta),\ \varpi \in \mathscr{V}_1 = (-\infty,0], \quad z'(0) = z_1 \in \mathscr{U}, \end{aligned}$$

In the preceding system (1.1)–(1.3), $z(\cdot)$ takes value in $\mathscr{U}$, where $\mathscr{U}$ is a separable real Hilbert space with $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. The histories z_ϖ from $(-\infty,0]$ into $\mathscr{B}_\vartheta$ presented as $z_\varpi = z(\varpi+\theta), \theta \leqslant 0$, connect with an abstract phase space $\mathscr{B}_\vartheta$. The infinitesimal generator of a strongly continuous cosine family on $\mathscr{U}$ is A that maps from $\mathscr{D}(A) \subset \mathscr{U}$ into $\mathscr{U}$. Consider another Hilbert space $\mathbb{V}$, which has an inner product $\langle\cdot,\cdot\rangle_{\mathbb{V}}$ and a norm $\|\cdot\|_{\mathbb{V}}$. The nonempty, closed, bounded and convex multi-valued map $G : \mathscr{V} \times \mathscr{B}_\vartheta \times \mathscr{U} \times \mathscr{U} \to \mathscr{N}\left(L_2^0\right)$. Here $D = \{(\varpi,\mu) \in \mathscr{V}\times\mathscr{V} : \mu \leqslant \varpi\}, l, m : D \times \mathscr{B}_\vartheta \to \mathscr{U}$ are continuous.

Let $z_0 = \alpha \in L^2(\Omega,\mathscr{B}_\vartheta)$ and z_1 be the $\mathscr{U}$-valued $\mathfrak{I}_0$-measurable random variable independent of the Wiener process $\{W(\varpi) : \varpi \geqslant 0\}$ with a finite second moment. The control function $u(\cdot)$ is takes values in $L^2(\mathscr{V},\mathscr{X})$ and $\mathscr{X}$ is a separable Hilbert space. In addition, $\mathscr{B} : \mathscr{X} \to \mathscr{U}$ is a bounded linear operator. $J_\imath, \overline{J_\imath} : \mathscr{U} \to \mathscr{N}(\mathscr{U})$ are multi-valued maps with closed graph. In addition, consider $0 = \varpi_0 < \varpi_1 < \varpi_2 < \cdots < \varpi_\jmath < \varpi_{\jmath+1} = \mathfrak{p}$ is the prefixed points. The jumps at the points ϖ_j belongs to $(0,\mathfrak{p})$ are given by $\Delta z|_{\varpi=\varpi_\imath} = z(\varpi_\imath^+) - z(\varpi_\imath^-), \Delta z'|_{\varpi=\varpi_\imath} = z'(\varpi_\imath^+) - z'(\varpi_\imath^-), \forall$ $\imath = 1,2,\ldots,\jmath$. Here $z(\varpi_\imath^+), z(\varpi_\imath^-), z'(\varpi_\imath^+)$ and $z'(\varpi_\imath^-)$ represent the right and left limits of $z(\varpi)$ at $\varpi = \varpi_\imath$ and $z'(\varpi)$ at $\varpi = \varpi_\imath$, respectively. For our convenience, we denote $\mathscr{W}_1(\varpi) = \int_0^\varpi l(\varpi,\mu,z_\mu)d\mu$ and $\mathscr{W}_2(\varpi) = \int_0^{\mathfrak{p}} m(\varpi,\mu,z_\mu)d\mu$.

The manuscript presentation plan is in the following way: We quickly give a few key facts and terminologies linked with our study in Section 2, which is used in the whole analysis of our work. Section 3 is designated for consideration of the approximate controllability. Section 4, continues our investigation of the system (1.1)–(1.3) with nonlocal circumstances. We offer an application that is presented to illustrate the concept of the primary outcomes in Section 5.

2. Preliminaries

To discuss our primary outcomes, we now introduce some basic concepts, key terms, and facts.

Make the assumption that $(\Omega,\mathfrak{I},\mathbb{P})$ is a complete probability space fitted with $\{\mathfrak{I}_\varpi\}_{\varpi\in[0,\mathfrak{p}]}$ being a normal filtration. The expectation with respect to the measure $\mathbb{P}$ is represented as $E(\cdot)$. Consider the separable Hilbert spaces $\mathscr{U},\mathbb{V}$ and $\{W(\varpi),\varpi \geqslant 0\}$ denote a Wiener process with the bounded liner covariance operator $\mathscr{Q} \ni Tr(\mathscr{Q}) < \infty$. Let us consider that there is a system $\{e_{\breve{\kappa}}\}_{\breve{\kappa}=1}^\infty$ belongs to $\mathbb{V}$, with complete orthonormal and a bounded sequence of number $\{\lambda_{\breve{\kappa}}\}_{\breve{\kappa}=1}^\infty \geqslant 0 \ni$

$$\mathscr{Q}e_{\breve{\kappa}} = \lambda_{\breve{\kappa}}e_{\breve{\kappa}},\ \breve{\kappa} = 1,2,\ldots$$

and a sequence $\beta_{\breve{\kappa}}$ of independent Brownian motions such that

$$\langle W(\varpi), e\rangle = \sum_{\breve{\kappa}=1}^\infty \sqrt{\lambda_{\breve{\kappa}}}\langle e_{\breve{\kappa}}, e\rangle\beta_{\breve{\kappa}}(\varpi),\ \varpi \in \mathscr{V},$$

and $\mathfrak{I} = \mathfrak{I}_\varpi^W, \mathfrak{I}_\varpi^W$ is the σ-algebra referring W.

The space of all bounded operators from $\mathbb{V}$ into $\mathscr{U}$ with the norm $\|\cdot\|$ is represented by $L(\mathbb{V},\mathscr{U})$. Consider $L_2^0 = L_2(\mathscr{Q}^{1/2}\mathbb{V};\mathscr{U})$ is the space containing all Hilbert–Schmidt operators with the norm

$$\left\langle \theta,\theta^*\right\rangle_{L_2^0} = \|\theta\mathscr{Q}^{1/2}\|^2 = Tr(\theta\mathscr{Q}\theta^*)$$

and θ is known as a $\mathscr{Q}$-Hilbert–Schmidt operator from $\mathscr{Q}^{1/2}\mathbb{V} \to \mathscr{U}$. Consider $L_2(\Omega,\mathfrak{I}_{\mathfrak{p}},\mathscr{U})$ is the space containing $\mathfrak{I}_{\mathfrak{p}}$-measurable square integrable random variables along with values in $\mathscr{U}$. Let $L_2^{\mathfrak{I}}(\mathscr{V},\mathscr{U})$ represents the Banach space of all $\mathfrak{I}_\varpi$-adapted, $\mathscr{U}$-valued measurable square integrable systems on $\mathscr{V}\times\Omega$. Suppose that the Banach space of continuous function from $[0,\mathfrak{p}]$ into $L_2(\Omega,\mathfrak{J}_\varpi,\mathscr{U})$ is $\mathscr{C}([0,\mathfrak{p}];L_2(\Omega,\mathfrak{J}_\varpi,\mathscr{U}))$ equipped with $\sup_{\varpi\in\mathscr{V}}E\|z(\varpi)\|^2_{\mathscr{U}}<\infty$. The family of all $\mathfrak{J}_0$-measurable is represented by $L_2^0(\Omega,\mathscr{U})$, $\mathscr{U}$-valued random variables $z(0)$.

Provided that $\mathfrak{z}:\mathscr{V}\to\mathscr{U}$ is a measurable function, then it is Bochner integrable if $\|\mathfrak{z}\|$ is Lebesgue. Let $L^1(\mathscr{V},\mathscr{U})$ be the Banach space of measurable functions supplied along with

$$\|\mathfrak{z}\|_{L^1}=\int_{\mathscr{V}}\|\mathfrak{z}(\varpi)\|d\varpi.$$

Consider the subsequent representations:

$$\mathscr{N}(\mathscr{U})=\{z\in\mathscr{N}(\mathscr{U}):\mathscr{U}\neq\varnothing\},$$
$$\mathscr{N}_{cl}(\mathscr{U})=\{z\in\mathscr{N}(\mathscr{U}):z\text{ is closed}\},$$
$$\mathscr{N}_{bd}(\mathscr{U})=\{z\in\mathscr{N}(\mathscr{U}):z\text{ is bounded}\},$$
$$\mathscr{N}_{cv}(\mathscr{U})=\{z\in\mathscr{N}(\mathscr{U}):z\text{ is convex}\},$$
$$\mathscr{N}_{cp}(\mathscr{U})=\{z\in\mathscr{N}(\mathscr{U}):z\text{ is compact}\}.$$

Assume that $\mathscr{U}_d$ maps from $\mathscr{N}(\mathscr{U})\times\mathscr{N}(\mathscr{U})$ into $R^+\cup\{\infty\}$ presented as

$$\mathscr{U}_d(\mathbb{X},\mathbb{Y})=\max\left\{\sup_{a\in\mathbb{X}}d(a,\mathbb{Y}),\sup_{b\in\mathbb{Y}}d(\mathbb{X},b)\right\}.$$

In the above $d(\mathbb{X},b)=\inf_{a\in\mathbb{X}}d(a,b),\ d(a,\mathbb{Y})=\inf_{b\in\mathbb{Y}}d(a,b)$.

Thus, $(\mathscr{N}_{bd,cl}(\mathscr{U}),\mathscr{U}_d)$ and $(\mathscr{N}_{cl}(\mathscr{U}),\mathscr{U}_d)$ are metric space and generalized metric space respectively.

The abstract phase space $\mathscr{B}_\vartheta$ is now represented.

Consider a continuous function ϑ maps from $(-\infty,0]$ into $(0,+\infty)$ with $\ell=\int_{-\infty}^0\vartheta(\varpi)d\varpi<+\infty$. For any $c>0$, we introduce

$$\mathscr{B}_\vartheta=\left\{\alpha:(-\infty,0]\to\mathscr{U},\ \left(E\|\alpha(\theta)\|^2\right)^{1/2}\text{ is bounded and measurable function on } [-c,0]\text{ and }\int_{-\infty}^0\vartheta(\mu)\sup_{\theta\in[\mu,0]}\left(E\|\alpha(\theta)\|^2\right)^{1/2}d\mu<+\infty\right\}.$$

Suppose that $\mathscr{B}_\vartheta$ is endowed with

$$\|\alpha\|_{\mathscr{B}_\vartheta}=\int_{-\infty}^0\vartheta(\mu)\sup_{\mu\leq\theta\leq0}\left(E\|\alpha(\theta)\|^2\right)^{1/2}d\mu,\ \alpha\in\mathscr{B}_\vartheta,$$

next, $\left(\mathscr{B}_\vartheta,\|\cdot\|_{\mathscr{B}_\vartheta}\right)$ denotes a Banach space (Li and Liu, 2007; Ren and Sun, 2002).

Consider

$$\mathscr{B}'_\vartheta=\{z:(-\infty,c]\to\mathscr{U}:\ z_j\in\mathscr{C}(\mathscr{V}_\imath,\mathscr{U}),\text{ and there exists } z(\varpi_\imath^+)\text{ and } z(\varpi_\imath^-)\text{ with } z(\varpi_\imath^-)=z(\varpi_\imath),\ z_0=\alpha\in\mathscr{B}_\vartheta,\ \imath=0,1,2,\ldots,\jmath\},$$

where $z_\imath$ is the restriction of z to $\mathscr{V}_\imath=(\varpi_\imath,\varpi_{\imath+1}]$, $\imath=0,1,2,\ldots,\jmath$. Set the seminorm $\|\cdot\|_{\mathfrak{p}}$ belongs to $\mathscr{B}'_\vartheta$ given as

$$\|z\|_{\mathfrak{p}}=\|z_0\|_{\mathscr{B}_\vartheta}+\sup_{\mu\in[0,\varpi]}\left(E\|z(\mu)\|^2\right)^{\frac12},\ z\text{ in }\mathscr{B}'_\vartheta.$$

Lemma 2.1. *(SeeLi and Liu, 2007) "Assume that $z\in\mathscr{B}'_\vartheta$, then for $\varpi\in\mathscr{V}$, $z_\varpi\in\mathscr{B}_\vartheta$. Moreover,*

$$\ell\left(E\|z(\varpi)\|^2\right)^{\frac12}\leqslant\|z_\varpi\|_{\mathscr{B}_\vartheta}\leqslant\|z_0\|_{\mathscr{B}_\vartheta}+\ell\sup_{0\leqslant\mu\leqslant\varpi}\left(E\|z(\mu)\|^2\right)^{\frac12},$$

where $\ell=\int_{-\infty}^0\vartheta(\varpi)d\varpi<\infty$."

Definition 2.2.

(Fattorini, 1985; Kisyński, 1972; Travis and Webb, 1978) "$\{\mathbb{C}(\varpi)\}_{\varpi\in R}\subset L(\mathscr{U})$ is said to be a strong continuous cosine family provided that

(*i*) $\mathbb{C}(0)=I$,
(*ii*) $\mathbb{C}(\varpi)z$ is continuous in ϖ on $\mathbb{R}$ for any$z\in\mathscr{U}$,
(*iii*) $\mathbb{C}(\varpi+\mu)+\mathbb{C}(\varpi-\mu)=2\mathbb{C}(\varpi)\mathbb{C}(\mu)$ for all $\mu,\varpi\in R$.

Consider the sine family $\{\mathbb{S}(\varpi)\}_{\varpi\in R}\subset L(\mathscr{U})$ is given as

$$\mathbb{S}(\varpi)z=\int_0^\varpi\mathbb{C}(\mu)zd\mu,\ \varpi\in R,\ z\in\mathscr{U}.$$

The generator $A:\mathscr{U}\to\mathscr{U}$ of $\{\mathbb{C}(\varpi)_{\varpi\in R}\subset L(\mathscr{U})\}$ is given by

$$Az=\frac{d^2}{d\varpi^2}\mathbb{C}(\varpi)z|_{\varpi=0},\ \forall\ z\in\mathscr{D}(A).$$

In the above $\mathscr{D}(A)=\{z\in\mathscr{U}:\mathbb{C}(\cdot)z\in\mathscr{C}^2(R,\mathscr{U})\}$.

It is known that the infinitesimal generator A is a closed densely defined operator on $\mathscr{U}$. Such cosine and corresponding sine families and their generators fulfill the subsequent properties.".

Lemma 2.3. *(Fattorini, 1985; Kisyński, 1972; Travis and Webb, 1978) "Suppose that A is the infinitesimal generator of a cosine family of operators $\{\mathbb{C}(\varpi):\varpi\in R\}$. Then the subsequent hold:*

(1) there exists $\mathbb{M}\geqslant1$ and $\beta_1\geqslant0$ such that $\|\mathbb{C}(\varpi)\|\leqslant\mathbb{M}e^{\beta_1|\varpi|}$ and hence, $\|\mathbb{S}(\varpi)\|\leqslant\mathbb{M}e^{\beta_1|\varpi|}$,
(2) $A\int_\mu^\gamma\mathbb{S}(\mu)zd\mu=[\mathbb{C}(\gamma)-\mathbb{C}(\mu)]z,\forall\ 0\leqslant\mu\leqslant\gamma<\infty$,
(3) there exists $N^*\geqslant1$ such that $\|\mathbb{S}(\mu)-\mathbb{S}(\gamma)\|\leqslant N^*\left|\int_\mu^\gamma e^{\beta_1|\mu|}d\mu\right|,\forall\ 0\leqslant\mu\leqslant\gamma<\infty$."

Both $\{\mathbb{C}(\varpi)\}_{\varpi\in\mathscr{V}}$ and $\{\mathbb{S}(\varpi)\}_{\varpi\in\mathscr{V}}$ are uniformly bounded, according to the uniform boundedness principle paired with Lemma 2.3. From Deimling (1992, 1997), we present some facts related to multi-valued maps.

Remark 2.4. "The multi-valued map Ψ fulfills the subsequent characteristics:

- Provided that $\Psi(z)$ is convex (closed), for all $z\in\mathscr{U}$, next a multivalued mapping $\Psi:\mathscr{U}\to\mathscr{N}(\mathscr{U})$ is convex (closed). Ψ is bounded on bounded sets provided

$$\Psi(\mathbb{D})=\bigcup_{z\in\mathbb{D}}\Psi(z)$$

is bounded in $\mathscr{U}$, for any bounded set $\mathbb{D}$ of $\mathscr{U}$, i.e,

$$\sup_{z\in\mathbb{D}}\{\sup\{\|x\|:x\in\Psi(z)\}\}<\infty.$$

- Ψ is named u.s.c on $\mathscr{U}$, if for any z_0 in $\mathscr{U}$, the non-empty set $\Psi(z_0)$ is a closed subset of $\mathscr{U}$, and if for each open set $\mathbb{D}$ of $\mathscr{U}$ arresting $\Psi(z_0)$, $\exists$ an open neighborhood M of z_0 with $\Psi(M)\subseteq\mathbb{D}$.
- Provided that $\Psi(\mathbb{D})$ is relatively compact for all bounded subset $\mathbb{D}\subseteq\mathscr{U}$, then is Ψ completely continuous.
- Provided that Ψ has a closed graph and is completely continuous with nonempty values, then Ψ is u.s.c, that is, $z^{\check{\jmath}}$ tends to z^*, $v^{\check{\jmath}}$ tends to v^*, $v^{\check{\jmath}}$ belongs to $\Psi z^{\check{\jmath}}$ imply u^* belongs to Ψz^*. If there is z belongs to $\mathscr{U}$ such that z belongs to $\Psi(z)$, then Ψ has a fixed point.
- A multi-valued map $\Psi:\mathscr{V}$ into $\mathscr{N}_{cl,cv,bd}$ is called measurable provided that $\forall\ z\in\mathscr{U}$, the function $\mathscr{O}$ maps from $\mathscr{V}$ into R^+, denoted by

$$\mathscr{O}(\varpi)=d(z,\Psi(\varpi))=\inf\{\|z-x\|:x\in\Psi(\varpi)\},$$

is measurable."

Definition 2.5. (Deimling, 1992; Hu and Papageorgiou, 1997) " The multi-valued function $G : \mathscr{V} \times \mathscr{B}_\vartheta \times \mathscr{U} \times \mathscr{U} \to \mathscr{N}(\mathscr{U})$ is called L^2-Caratheodory provided that

(*i*) $\varpi \mapsto G(\varpi, z, x, y)$ is measurable for each $(z, x, y) \in \mathscr{B}_\vartheta \times \mathscr{U} \times \mathscr{U}$;
(*ii*) $(z, x, y) \mapsto G(\varpi, z, x, y)$ is u.s.c for almost all $\varpi \in \mathscr{V}$;
(*iii*) for each $\bar{p} > 0 \ \exists \ \hbar_{\bar{p}} \in L^1(\mathscr{V}, R^+)$ such that

$$\|G(\varpi, z, x, y)\|^2 = \sup\left\{E\|\mathfrak{z}\|^2 : \mathfrak{z} \in G(\varpi, z, x, y)\right\} \leqslant \hbar_{\bar{p}}(\varpi),$$
for almost everywhere $\varpi \in \mathscr{V}$.

To provide approximate controllability, let us define the operators:

$$\Upsilon_0^{\mathfrak{p}} = \int_0^{\mathfrak{p}} \mathbb{S}(\mathfrak{p} - \mu)\mathscr{B}\mathscr{B}^*\mathbb{S}^*(\mathfrak{p} - \mu)d\mu : \mathscr{U} \to \mathscr{U},$$
$$\mathscr{R}(\eta, \Upsilon_0^{\mathfrak{p}}) = \left(\eta I + \Upsilon_0^{\mathfrak{p}}\right)^{-1} : \mathscr{U} \to \mathscr{U}.$$

In the above $\mathscr{B}^*$ and $\mathbb{S}^*(\mathfrak{p})$ represents adjoints of $\mathscr{B}$ and $\mathbb{S}(\mathfrak{p})$. The linear operator $\Upsilon_0^{\mathfrak{p}}$ is bounded, as we can easily deduce.

To prove the approximate controllability of (1.1)–(1.3), we required the accompanying assumption:

H_0: $\eta\mathscr{R}(\eta, \Upsilon_0^{\mathfrak{p}}) \to 0$ as $\eta \to 0^+$ belongs to the strong operator topology.

From Mahmudov and Denker (2000), the assumption **H_0** is equivalent to the fact that the linear control problem

$$\frac{d}{d\varpi}z'(\varpi) \in Az(\varpi) + (\mathscr{B}u)(\varpi), \quad \varpi \in [0, \mathfrak{p}], \tag{2.1}$$
$$z(0) = z_0, \quad z'(0) = z_1, \tag{2.2}$$

is approximately controllable on $\mathscr{V}$.

Lemma 2.6. Lasota and Opial (1965, Lasota and Opial).*"Suppose $\mathscr{V}$ is a compact real interval, and $\mathscr{U}$ is a Hilbert space. Consider G is an L^2-Caratheodory multi-valued map $\mathbb{T}_{G,z} \neq \varnothing$ and let $\mathscr{G}$ be a linear continuous mapping from $L^2(\mathscr{V}, \mathscr{U}) \to \mathscr{C}(\mathscr{V}, \mathscr{U})$. Next, the operator*

$$\mathscr{G} \circ \mathbb{T}_G : \mathscr{C}(\mathscr{V}, \mathscr{U}) \to \mathscr{N}_{cp,cv}(\mathscr{U}), \ z \to (\mathscr{G} \circ \mathbb{T}_G)(z) = \mathscr{G}(\mathbb{T}_{G,z}),$$

is a closed graph operator in $\mathscr{C}(\mathscr{V}, \mathscr{U}) \times \mathscr{C}(\mathscr{V}, \mathscr{U})$, where $\mathbb{T}_{G,z}$ is known as the selectors set from G, is denoted by

$$\mathbb{T}_{G,z} = \left\{\mathfrak{z} \in L^2\left([0, \mathfrak{p}], L_2^0\right) : \mathfrak{z}(\varpi) \text{ belongs to } G(\varpi, z_\varpi, \mathscr{W}_1(\varpi), \mathscr{W}_2(\varpi)),\right.$$
for a.e. $\varpi \in \mathscr{V}\}$.

Lemma 2.7. Dhage (2006, Dhage).*"Let $\mathscr{F}_1 : \mathscr{U} \to \mathscr{N}_{cl,cv,bd}(\mathscr{U})$ and $\mathscr{F}_2 : \mathscr{U} \to \mathscr{N}_{cp,bd}(\mathscr{U})$ be two multi-valued operators defined on a Hilbert space $\mathscr{U}$. If $\mathscr{F}_1$ is a contraction and $\mathscr{F}_2$ is completely continuous. Then, either the operator inclusion $\lambda z \in \mathscr{F}_1 z + \mathscr{F}_2 z$ has a solution when $\lambda = 1$ or the set $\mathscr{M} = \{z \in \mathscr{U} : z \in \lambda\mathscr{F}_1 z + \lambda\mathscr{F}_2 z$ for some $\lambda \in (0, 1)\}$ is unbounded."*

3. Approximate controllability outcomes

The approximate controllability of (1.1)–(1.3) is the primary subject of this section. Let $\mathscr{V}_0 = (-\infty, \mathfrak{p}]$ and before starting the main result, we provide the mild solution of (1.1)–(1.3).

Definition 3.1. A stochastic process $z : \mathscr{V}_0 \times \Omega \to \mathscr{U}$ is said to be a mild solution of (1.1)–(1.3) provided that

(*i*) $z(\varpi)$ is measurable and adapted to $\mathfrak{I}_\varpi$, for each $\varpi \geqslant 0$.
(*ii*) $z(\varpi) \in \mathscr{U}$ has *càdlàg* paths on $\varpi \in \mathscr{V}$ a.s., $\forall \ 0 \leqslant \mu < \varpi \leqslant \mathfrak{p}$, and $\exists \ \mathfrak{z} \in \mathbb{T}_{G,z}$ and the impulsive conditions $\Delta z|_{\varpi=\varpi_\imath} = J_\imath(z(\varpi_\imath^-)), \ \Delta z'|_{\varpi=\varpi_\imath} = \overline{J}_\imath(z(\varpi_\imath^-)) \ (\imath = 1, 2, \ldots, \jmath)$ such that the subsequent integral equation hold

$$\begin{aligned} z(\varpi) = {} & \mathbb{C}(\varpi)\alpha(0) + \mathbb{S}(\varpi)z_1 + \int_0^\varpi \mathbb{S}(\varpi - \mu)\mathfrak{z}(\mu)dW(\mu) \\ & + \int_0^\varpi \mathbb{S}(\varpi - \mu)\mathscr{B}u(\mu)d\mu \\ & + \sum_{0<\varpi_\imath<\varpi} \mathbb{C}(\varpi - \varpi_\imath)J_\imath(z(\varpi_\imath^-)) \\ & + \sum_{0<\varpi_\imath<\varpi} \mathbb{S}(\varpi - \varpi_\imath)\overline{J}_\imath(z(\varpi_\imath^-)), \quad \varpi \in \mathscr{V}. \end{aligned} \tag{3.1}$$

(*iii*) $z_0 = \alpha \in \mathscr{B}_\vartheta$ on $\mathscr{V}_1$ and $z'(0) = z_1 \in \mathscr{U}$ fulfilling $z_0, z_1 \in L_2^0(\Omega, \mathscr{U})$.

The subsequent assumptions were made for the problem analysis (1.1)–(1.3):

H_1: A is the infinitesimal generator of a strongly continuous cosine family $\{\mathbb{C}(\varpi) : \varpi \in [0, \mathfrak{p}]\}$ on $\mathscr{U}$ and $\{\mathbb{S}(\varpi) : \varpi \in [0, \mathfrak{p}]\}$ fulfills $\|\mathbb{C}(\varpi)\|^2 \leqslant P_1, \|\mathbb{S}(\varpi)\|^2 \leqslant P_2$ $\forall \ \varpi \geqslant 0$ for some positive constants P_1 and P_2.
H_2: The operator $\mathbb{C}(\varpi), \ \varpi \geqslant 0$ is compact.
H_3: The multi-valued maps with closed graph $J_\imath, \overline{J}_\imath : \mathscr{U} \to \mathscr{N}_{cl,cp}(\mathscr{U}), \ \imath = 1, 2, \ldots, \jmath$ and $\exists$ positive constants $P_\imath, \overline{P}_\imath$ such that

$$\mathscr{U}_d^2(J_\imath(y), J_\imath(x)) \leqslant P_\imath\|y - x\|^2, \ y, x \in \mathscr{U},$$
$$\mathscr{U}_d^2(\overline{J}_\imath(y), \overline{J}_\imath(x)) \leqslant \overline{P_\imath}\|y - x\|^2, \ y, x \in \mathscr{U}.$$

H_4: $G : \mathscr{V} \times \mathscr{B}_\vartheta \times \mathscr{U} \times \mathscr{U} \to \mathscr{N}\left(L_2^0\right)$ is L^2-Caratheodory function fulfills:
For every $z \in \mathscr{B}_\vartheta, G(\cdot, z, x, y)$ is measurable and the function $G(\varpi, \cdot, \cdot, \cdot)$ is u.s.c $\forall \ \varpi \in \mathscr{V}$. $\forall$ fixed (z, x, y) in $\mathscr{B}_\vartheta \times \mathscr{U} \times \mathscr{U}$, the set

$$\mathbb{T}_{G,z} = \left\{\mathfrak{z} \in L^2\left([0, \mathfrak{p}], L_2^0\right) : \mathfrak{z}(\varpi) \text{ belongs to } G(\varpi, z_\varpi, \mathscr{W}_1(\varpi), \mathscr{W}_2(\varpi))\right\},$$

for almost everywhere $\varpi \in \mathscr{V}$ and which is nonempty.
H_5: The functions l, m maps from $D \times \mathscr{B}_\vartheta$ into $\mathscr{U}$ are continuous that fulfills the following conditions:
(*i*) $\exists$ positive constants P_l, P_m such that $\forall \ \varpi, \mu \in \mathscr{V}, \widehat{x}, \widehat{y} \in \mathscr{B}_\vartheta$

- $E\mathscr{U}_d^2(\int_0^\varpi l(\varpi, \mu, \widehat{x})d\mu, \int_0^\varpi l(\varpi, \mu, \widehat{y})d\mu) \leqslant P_l\|\widehat{x} - \widehat{y}\|_{\mathscr{B}_\vartheta}^2$.
- $E\mathscr{U}_d^2(\int_0^{\mathfrak{p}} m(\varpi, \mu, \widehat{x})d\mu, \int_0^{\mathfrak{p}} m(\varpi, \mu, \widehat{y})d\mu) \leqslant P_m\|\widehat{x} - \widehat{y}\|_{\mathscr{B}_\vartheta}^2$.

(*ii*) The continuous functions $\breve{\mathscr{P}}_l, \breve{\mathscr{P}}_m : \mathscr{V} \to [0, \infty)$, such that

- $E\|\int_0^\varpi l(\varpi, \mu, \widehat{x})d\mu\|^2 \leqslant \breve{\mathscr{P}}_l(\varpi)\|\widehat{x}\|_{\mathscr{B}_\vartheta}^2, \ \varpi \in \mathscr{V}, \ \widehat{x} \in \mathscr{B}_\vartheta$.
- $E\|\int_0^{\mathfrak{p}} m(\varpi, \mu, \widehat{x})d\mu\|^2 \leqslant \breve{\mathscr{P}}_m(\varpi)\|\widehat{x}\|_{\mathscr{B}_\vartheta}^2, \ \varpi \in \mathscr{V}, \ \widehat{x} \in \mathscr{B}_\vartheta$.

H_6:
(*i*) $\exists$ a constant P_G such that

$$E\mathscr{U}_d^2(G(\varpi, \widehat{x}_1, \widehat{x}_2, \widehat{x}_3), G(\varpi, \widehat{y}_1, \widehat{y}_2, \widehat{y}_3)) \leqslant P_G\left[\|\widehat{x}_1 - \widehat{y}_1\|_{\mathscr{B}_\vartheta}^2 + E\|\widehat{x}_2 - \widehat{y}_2\|^2 + E\|\widehat{x}_3 - \widehat{y}_3\|^2\right],$$

where $\varpi \in \mathscr{V}, \ \widehat{x}_1, \widehat{y}_1 \in \mathscr{B}_\vartheta, \ \widehat{x}_i, \widehat{y}_i \in \mathscr{U}, \ i = 2, 3$.
(*ii*) $\exists$ an integrable function $\mathfrak{m}_\mathfrak{z} : \mathscr{V} \to [0, \infty)$ such that

$$E\|G(\varpi, \widehat{x}_1, \widehat{x}_2, \widehat{x}_3)\|^2 = \sup\left\{E\|\mathfrak{z}\|^2 : \mathfrak{z} \in G\left(\varpi, \widehat{x}_1, \widehat{x}_2, \widehat{x}_3\right)\right\} \leqslant \mathfrak{m}_\mathfrak{z}(\varpi)\Theta\left(\|\widehat{x}_1\|_{\mathscr{B}_\vartheta}^2 + E\|\widehat{x}_2\|^2 + E\|\widehat{x}_3\|^2\right),$$

for a.e $\varpi \in \mathscr{V}$ and $\widehat{x}_1 \in \mathscr{B}_\vartheta$, $(\widehat{x}_2, \widehat{x}_3) \in \mathscr{U} \times \mathscr{U}$, where Θ maps from R^+ into $(0, \infty)$ is a continuous nondecreasing function with

$$\Theta\left(\breve{\mathscr{P}}_l(\varpi)\|\hat{x}\|^2_{\mathscr{B}_\vartheta}\right) \leqslant \breve{\mathscr{P}}_l(\varpi)\Theta\left(\|\hat{x}\|^2_{\mathscr{B}_\vartheta}\right),$$
$$\Theta\left(\breve{\mathscr{P}}_m(\varpi)\|\hat{x}\|^2_{\mathscr{B}_\vartheta}\right) \leqslant \breve{\mathscr{P}}_m(\varpi)\Theta\left(\|\hat{x}\|^2_{\mathscr{B}_\vartheta}\right),$$

for each $\varpi \in \mathscr{V}$, $\widehat{x} \in \mathscr{B}_\vartheta$.

H₇: The following inequality holds

$$\mathbb{P}_4 \int_0^{\mathfrak{p}} \mathfrak{m}_{\mathfrak{z}}(\mu)\left(1 + \breve{\mathscr{P}}_l(\mu) + \breve{\mathscr{P}}_m(\mu)\right)d\mu \leqslant \int_c^\infty \frac{1}{\Theta(\wp)} d\wp,$$

where

$$\begin{aligned}\overline{P} = 6\Big\{&P_1 E\|\alpha(0)\|^2 + P_2 E\|z_1\|^2 + 6(\mathfrak{p}/\eta)^2 P_{\mathscr{B}}^2 P_2^2 \Big[E\|E\widehat{z}_{\mathfrak{p}} + \textstyle\int_0^{\mathfrak{p}} \widehat{\phi}(\zeta)dW(\zeta)\|^2 hfill \\ &+ P_1 E\|\alpha(0)\|^2 + P_2 E\|z_1\|^2 + 2\jmath P_1 \sum_{\iota=1}^{\jmath} E\|J_\iota(0)\|^2 + 2\jmath P_2 \sum_{\iota=1}^{\jmath} E\|\overline{J}_\iota(0)\|^2\Big] \\ &+ 2\jmath P_1 \sum_{\iota=1}^{\jmath} E\|J_\iota(0)\|^2 + 2\jmath P_2 \sum_{\iota=1}^{\jmath} E\|\overline{J}_\iota(0)\|^2\Big\},\end{aligned} \tag{3.2}$$

$$\mathbb{K}_0 = 2\ell^2\left[(6\mathfrak{p}/\eta)^2 P_{\mathscr{B}}^2 P_2^2 + 6\right]\left[2\jmath P_1 \sum_{\iota=1}^{\jmath} P_\iota + 2\jmath P_2 \sum_{j=1}^{k} \overline{P_\iota}\right],$$

$$\begin{aligned}\mathbb{K}_1 = 2\ell^2\Big\{&(6\mathfrak{p}/\eta)^2 P_{\mathscr{B}}^2 P_2^3 Tr(\mathscr{Q}) \textstyle\int_0^{\mathfrak{p}} \mathfrak{m}_{\mathfrak{z}}(\zeta)\Theta(\delta(\zeta) + \breve{\mathscr{P}}_l(\zeta)\delta(\zeta) + \breve{\mathscr{P}}_m(\zeta)\delta(\zeta))d\zeta + \overline{P}\Big\} \\ &+ 2\|\alpha\|^2_{\mathscr{B}_\vartheta},\end{aligned} \tag{3.3}$$

$$\mathbb{K}_2 = 12\ell^2 P_2 Tr(\mathscr{Q}), \quad \mathbb{P}_3 = \mathbb{K}_1/(1-\mathbb{K}_0), \quad \mathbb{P}_4 = \mathbb{K}_2/(1-\mathbb{K}_0), \tag{3.4}$$

$$\begin{aligned}P_0 = {}&3\jmath P_2 \sum_{\iota=1}^{\jmath} \overline{P}_\iota + 3\jmath P_1 \sum_{\iota=1}^{\jmath} P_\iota + \tfrac{9}{\eta^2}\mathfrak{p}^3 P_2^3 P_{\mathscr{B}}^2 Tr(\mathscr{Q})\ell^2 P_G(1 + P_l + P_m) \\ &+ \tfrac{9}{\eta^2}\jmath P_1 P_2^2 P_{\mathscr{B}}^2 \mathfrak{p}^2 \sum_{\iota=1}^{\jmath} P_\iota + \tfrac{9}{\eta^2}\jmath P_2^3 P_{\mathscr{B}}^2 \mathfrak{p}^2 \sum_{\iota=1}^{\jmath} \overline{P_\iota},\end{aligned} \tag{3.5}$$

and

$P_{\mathscr{B}} = \|\mathscr{B}\|^2$.

To figure out the control function, the next lemma is required.

Lemma 3.2. *(Mahmudov, 2001) $\forall$ $\widehat{z}_{\mathfrak{p}}$ in $L_2\left(\mathfrak{J}_{\mathfrak{p}}, \mathscr{U}\right)$ $\exists$ $\widehat{\phi}(\cdot) \in L_2^{\mathfrak{J}}\left(\Omega, L^2\left(\mathscr{V}, L_2^0\right)\right) \ni \widehat{z}_{\mathfrak{p}} = E\widehat{z}_{\mathfrak{p}} + \int_0^{\mathfrak{p}} \widehat{\phi}(\mu)dW(\mu)$.*

Now, $\forall$ $\eta > 0$ and $\widehat{z}_{\mathfrak{p}} \in L^2\left(\mathfrak{J}_{\mathfrak{p}}, \mathscr{U}\right)$, we define

$$\begin{aligned}u^\eta(\varpi, z) = {}&\mathscr{B}^*\mathbb{S}^*(\mathfrak{p} - \varpi) \\ &\Big[(\eta I + \Upsilon_0^{\mathfrak{p}})^{-1}(E\widehat{z}_{\mathfrak{p}} - \mathbb{C}(\mathfrak{p})\alpha(0) - \mathbb{S}(\mathfrak{p})z_1) + \textstyle\int_0^{\mathfrak{p}} \left(\eta I + \Upsilon_\mu^{\mathfrak{p}}\right)^{-1}\widehat{\phi}(\mu)dW(\mu) \\ &- \textstyle\int_0^{\mathfrak{p}} \left(\eta I + \Upsilon_\mu^{\mathfrak{p}}\right)^{-1}\mathbb{S}(\mathfrak{p} - \mu)\mathfrak{z}(\mu)dW(\mu) - (\eta I + \Upsilon_0^{\mathfrak{p}})^{-1} \\ &\sum_{0<\varpi_\iota<\mathfrak{p}} \mathbb{C}(\mathfrak{p} - \varpi_\iota)J_\iota(z(\varpi_\iota^-)) - (\eta I + \Upsilon_0^{\mathfrak{p}})^{-1} \sum_{0<\varpi_\iota<\mathfrak{p}} \mathbb{S}(\mathfrak{p} - \varpi_\iota)\overline{J}_\iota(z(\varpi_\iota^-))\Big],\end{aligned} \tag{3.7}$$

where $\mathfrak{z} \in \mathbb{T}_{G,z} = \left\{\mathfrak{z} \in L^2\left([0, \mathfrak{p}], L_2^0\right) : \mathfrak{z}(\varpi) \text{ belongs to } G(\varpi, z_\varpi, \mathscr{W}_1(\varpi), \mathscr{W}_2(\varpi))\right\}$, for almost everywhere $\varpi \in \mathscr{V}$.

The primary outcomes of the manuscript is the subsequent theorem.

Theorem 3.3. *Suppose that* **H₀**-**H₇** *are fulfilled. Provided that $P_0 < 1$ and $\mathbb{K}_0 < 1$, next $\forall$ $\eta > 0$, the second-order stochastic differential inclusions (1.1)–(1.3) has at least one mild solution on $[0, \mathfrak{p}]$.*

Proof. System (1.1)–(1.3) is converted into a fixed point system in order to demonstrate the existence of mild solutions. $\forall$ $\eta > 0$, we examine the operator $\mathscr{F}^\eta : \mathscr{B}'_\vartheta \to \mathscr{N}\left(\mathscr{B}'_\vartheta\right)$ determine as $\mathscr{F}^\eta z$ the family of $\varphi \in \mathscr{B}'_\vartheta \ni$

$$\varphi(\varpi) = \begin{cases} \alpha(\varpi), \quad \varpi \in (-\infty, 0], \\ \mathbb{C}(\varpi)\alpha(0) + \mathbb{S}(\varpi)z_1 + \int_0^\varpi \mathbb{S}(\varpi - \mu)\mathfrak{z}(\mu)dW(\mu) \\ + \int_0^\varpi \mathbb{S}(\varpi - \mu)\mathscr{B}u^\eta(\mu, z)d\mu + \sum_{0<\varpi_\iota<\varpi} \mathbb{C}(\varpi - \varpi_\iota)J_\iota(z(\varpi_\iota)) \\ + \sum_{0<\varpi_\iota<\varpi} \mathbb{S}(\varpi - \varpi_\iota)\overline{J}_\iota(z(\varpi_\iota)), \quad \varpi \in [0, \mathfrak{p}], \end{cases} \tag{3.8}$$

where $\mathfrak{z} \in \mathbb{T}_{G,z}$.

For $\alpha \in \mathscr{B}_\vartheta$, we now present $\widehat{\alpha}$ as

$$\widehat{\alpha}(\varpi) = \begin{cases} \alpha(\varpi), \quad \varpi \in (-\infty, 0], \\ \mathbb{C}(\varpi)\alpha(0) + \mathbb{S}(\varpi)z_1, \quad \varpi \in [0, \mathfrak{p}], \end{cases} \tag{3.9}$$

then $\widehat{\alpha}(\varpi) \in \mathscr{B}'_\vartheta$. Assume $z(\varpi) = x(\varpi) + \widehat{\alpha}(\varpi), -\infty < \varpi \leqslant \mathfrak{p}$. We now conclude that x fulfills $x_0 = 0$ and

$$\begin{aligned}x(\varpi) = {}&\textstyle\int_0^\varpi \mathbb{S}(\varpi - \mu)\mathfrak{z}(\mu)dW(\mu) \\ &+ \textstyle\int_0^\varpi \mathbb{S}(\varpi - \mu)\mathscr{B}\mathscr{B}^*\mathbb{S}^*(\mathfrak{p} - \mu)\left(\eta I + \Upsilon_0^{\mathfrak{p}}\right)^{-1}\Big[E\widehat{z}_{\mathfrak{p}} + \int_0^{\mathfrak{p}} \widehat{\phi}(\zeta)dW(\zeta) \\ &- \mathbb{C}(\mathfrak{p})\alpha(0) - \mathbb{S}(\mathfrak{p})z_1 - \textstyle\int_0^{\mathfrak{p}} \mathbb{S}(\mathfrak{p} - \zeta)\mathfrak{z}(\zeta)dW(\zeta) \\ &- \sum_{0<\varpi_\iota<\mathfrak{p}} \mathbb{C}(\mathfrak{p} - \varpi_\iota)J_\iota\left(x(\varpi_\iota^-) + \widehat{\alpha}(\varpi_\iota^-)\right) \\ &- \sum_{0<\varpi_\iota<\mathfrak{p}} \mathbb{S}(\mathfrak{p} - \varpi_\iota)\overline{J}_\iota\left(x(\varpi_\iota^-) + \widehat{\alpha}(\varpi_\iota^-)\right)\Big]d\mu \\ &+ \sum_{0<\varpi_\iota<\varpi} \mathbb{c}(\varpi - \varpi_\iota)J_\iota\left(x(\varpi_\iota^-) + \widehat{\alpha}(\varpi_\iota^-)\right) \\ &+ \sum_{0<\varpi_\iota<\varpi} \mathbb{S}(\varpi - \varpi_\iota)\overline{J}_\iota\left(x(\varpi_\iota^-) + \widehat{\alpha}(\varpi_\iota^-)\right), \quad \varpi \in [0, \mathfrak{p}],\end{aligned}$$

iff x fulfills

$$\begin{aligned}z(\varpi) = {}&\mathbb{C}(\varpi)\alpha(0) + \mathbb{S}(\varpi)z_1 + \textstyle\int_0^\varpi \mathbb{S}(\varpi - \mu)\mathfrak{z}(\mu)dW(\mu) \\ &+ \textstyle\int_0^\varpi \mathbb{S}(\varpi - \mu)\mathscr{B}\mathscr{B}^*\mathbb{S}^*(\mathfrak{p} - \mu)\left(\eta I + \Upsilon_0^{\mathfrak{p}}\right)^{-1} \\ &\times\Big[E\widehat{z}_{\mathfrak{p}} + \textstyle\int_0^{\mathfrak{p}} \widehat{\phi}(\zeta)dW(\zeta) - \mathbb{C}(\mathfrak{p})\alpha(0) \\ &- \mathbb{S}(\mathfrak{p})z_1 - \textstyle\int_0^{\mathfrak{p}} \mathbb{S}(\mathfrak{p} - \zeta)\mathfrak{z}(\zeta)dW(\zeta) \\ &- \sum_{0<\varpi_\iota<\mathfrak{p}} \mathbb{C}(\mathfrak{p} - \varpi_\iota)J_\iota\left(x(\varpi_\iota^-) + \widehat{\alpha}(\varpi_\iota^-)\right) \\ &- \sum_{0<\varpi_\iota<\mathfrak{p}} \mathbb{S}(\mathfrak{p} - \varpi_\iota)\overline{J}_\iota\left(x(\varpi_\iota^-) + \widehat{\alpha}(\varpi_\iota^-)\right)\Big]d\mu \\ &+ \sum_{0<\varpi_\iota<\varpi} \mathbb{C}(\varpi - \varpi_\iota)J_\iota\left(x(\varpi_\iota^-) + \widehat{\alpha}(\varpi_\iota^-)\right) \\ &+ \sum_{0<\varpi_\iota<\varpi} \mathbb{S}(\varpi - \varpi_\iota)\overline{J}_\iota\left(x(\varpi_\iota^-) + \widehat{\alpha}(\varpi_\iota^-)\right),\end{aligned}$$

$\varpi \in [0, \mathfrak{p}]$, and $z(\varpi) = \alpha(\varpi), \varpi \in (-\infty, 0]$.

Let $\mathscr{B}''_\vartheta = \{x \in \mathscr{B}'_\vartheta : x_0 = 0 \in \mathscr{B}_\vartheta\}$. For any $x \in \mathscr{B}''_\vartheta$, we obtain

$$\|x\|_{\mathfrak{p}} = \|x_0\|_{\mathscr{B}_\vartheta} + \sup_{\mu\in[0,\mathfrak{p}]}\left(E\|x(\mu)\|^2\right)^{1/2} = \sup_{\mu\in[0,\mathfrak{p}]}\left(E\|x(\mu)\|^2\right)^{1/2},$$

hence $\left(\mathscr{B}''_\vartheta, \|\cdot\|_{\mathfrak{p}}\right)$ is a Banach space. Fix

$$B_{\bar{p}} = \left\{x \in \mathscr{B}''_\vartheta : \|x\|^2_{\mathfrak{p}} \leqslant \bar{p}\right\},$$

for some positive constant $\bar{p}$, then $B_{\bar{p}}$ is subset of $\mathscr{B}''_\vartheta$ is uniformly bounded, and $\forall$ $x \in B_{\bar{p}}$. From Lemma 2.1, one can obtain

$$\begin{aligned}\|x_\varpi+\widehat{\alpha}_\varpi\|^2_{\mathscr{B}_\vartheta} &\leqslant 2\left(\|x_\varpi\|^2_{\mathscr{B}_\vartheta}+\|\widehat{\alpha}_\varpi\|^2_{\mathscr{B}_\vartheta}\right)\\ &\leqslant 4\left(\ell^2\sup_{\mu\in[0,\varpi]}E\|x(\mu)\|^2+\|x_0\|^2_{\mathscr{B}_\vartheta}+\ell^2\sup_{\mu\in[0,\varpi]}E\|\widehat{\alpha}(\mu)\|^2+\|\widehat{\alpha}_0\|^2_{\mathscr{B}_\vartheta}\right)\\ &\leqslant 4\ell^2\left(\bar{\mathfrak{p}}+2\left(P_1E\|\alpha(0)\|^2+P_2E\|z_1\|^2\right)\right)+4\|\alpha\|^2_{\mathscr{B}_\vartheta}=\bar{\mathfrak{p}}'.\end{aligned} \tag{3.10}$$

Define $\widehat{\mathscr{F}}:\mathscr{B}''_\vartheta\to\mathscr{N}\left(\mathscr{B}''_\vartheta\right)$ provided by $\widehat{\mathscr{F}}x$, the set of $\widehat{\varphi}\in\mathscr{B}''_\vartheta$ such that

$$\widehat{\varphi}(\varpi)=\begin{cases}0, \quad \varpi\in(-\infty,0],\\ \int_0^\varpi \mathbb{S}(\varpi-\mu)\mathfrak{z}(\mu)dW(\mu)\\ +\int_0^\varpi \mathbb{S}(\varpi-\mu)\mathscr{B}\mathscr{B}^*\mathbb{S}^*(\mathfrak{p}-\mu)\left(\eta I+\Upsilon_0^{\mathfrak{p}}\right)^{-1}\Big[E\widehat{z}_{\mathfrak{p}}\\ +\int_0^{\mathfrak{p}}\widehat{\phi}(\zeta)dW(\zeta)-\mathbb{C}(\mathfrak{p})\alpha(0)-\mathbb{S}(\mathfrak{p})z_1-\int_0^{\mathfrak{p}}\mathbb{S}(\mathfrak{p}-\zeta)\mathfrak{z}(\zeta)dW(\zeta)\\ -\sum\limits_{0<\varpi_\iota<\mathfrak{p}}\mathbb{C}(\mathfrak{p}-\varpi_\iota)J_\iota\left(x(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right)\\ -\sum\limits_{0<\varpi_\iota<\mathfrak{p}}\mathbb{S}(\mathfrak{p}-\varpi_\iota)\overline{J_\iota}\left(x(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right)\Big]d\mu\\ +\sum\limits_{0<\varpi_\iota<\varpi}\mathbb{C}(\varpi-\varpi_\iota)J_\iota\left(x(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right)\\ +\sum\limits_{0<\varpi_\iota<\varpi}\mathbb{S}(\varpi-\varpi_\iota)\overline{J_\iota}\left(x(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right), \quad \varpi\in\mathscr{V}.\end{cases} \tag{3.11}$$

Consider the multi-valued functions $\widehat{\mathscr{F}}_1$ and $\widehat{\mathscr{F}}_2$ presented as

$$\begin{aligned}\widehat{\mathscr{F}}_1x(\varpi)&=\int_0^\varpi \mathbb{S}(\varpi-\mu)\mathscr{B}u^\eta\left(\mu,x+\widehat{\alpha}\right)d\mu\\ &+\sum_{0<\varpi_\iota<\varpi}\mathbb{C}(\varpi-\varpi_\iota)J_\iota\left(x(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right)\\ &+\sum_{0<\varpi_\iota<\varpi}\mathbb{S}(\varpi-\varpi_\iota)\overline{J_\iota}\left(x(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right),\ \varpi\in\mathscr{V},\end{aligned} \tag{3.12}$$

$$\widehat{\mathscr{F}}_2x(\varpi)=\int_0^\varpi \mathbb{S}(\varpi-\mu)\mathfrak{z}(\mu)dW(\mu),\ \varpi\in\mathscr{V}. \tag{3.13}$$

It's obvious that $\widehat{\mathscr{F}}=\widehat{\mathscr{F}}_1+\widehat{\mathscr{F}}_2$. The issue of determining mild solutions of (1.1)–(1.3) is shortened to define the solution $x\in\widehat{\mathscr{F}}_1(x)+\widehat{\mathscr{F}}_2(x)$. We going to prove that $\widehat{\mathscr{F}}_1$ and $\widehat{\mathscr{F}}_2$ fulfill the assumptions of Lemma 2.7.

Step 1. Prove that $\widehat{\mathscr{F}}_1$ is a contraction.

Assume that $y,\widehat{y}\in\mathscr{B}''_\vartheta$. By our assumptions, Lemma 2.1 and Hölder's inequality, and since $\|y_0\|^2_{\mathscr{B}_\vartheta}=0$ and $\|\widehat{y}_0\|^2_{\mathscr{B}_\vartheta}=0, \forall\ \varpi\in\mathscr{V}$, we get

$$\begin{aligned}&E\mathscr{U}_d^2\left(\widehat{\mathscr{F}}_1y(\varpi),\widehat{\mathscr{F}}_1\widehat{y}(\varpi)\right)\\ &\leqslant 3E\mathscr{U}_d^2\Bigg(\sum_{0<\varpi_\iota<\varpi}\mathbb{S}(\varpi-\varpi_\iota)\overline{J_\iota}\left(y(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right),\\ &\sum_{0<\varpi_\iota<\varpi}\mathbb{S}(\varpi-\varpi_\iota)\overline{J_\iota}\left(\widehat{y}(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right)\Bigg)\\ &+3E\mathscr{U}_d^2\Bigg(\sum_{0<\varpi_\iota<\varpi}\mathbb{C}(\varpi-\varpi_\iota)J_\iota\left(y(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right),\\ &\sum_{0<\varpi_\iota<\varpi}\mathbb{C}(\varpi-\varpi_\iota)J_\iota\left(\widehat{y}(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right)\Bigg)\\ &+3E\mathscr{U}_d^2\Bigg(\int_0^\varpi\mathbb{S}(\varpi-\mu)\mathscr{B}\mathscr{B}^*\mathbb{S}^*(\mathfrak{p}-\mu)\left(\eta I+\Upsilon_0^{\mathfrak{p}}\right)^{-1}\\ &(\times)\Big\{-\int_0^{\mathfrak{p}}\mathbb{S}(\mathfrak{p}-\zeta)G\Big(\zeta,y_\zeta+\widehat{\alpha}_\zeta,\int_0^{\mathfrak{p}}l(\zeta,\xi,y_\xi+\widehat{\alpha}_\xi)d\xi,\ \int_0^{\mathfrak{p}}m(\zeta,\xi,y_\xi+\widehat{\alpha}_\xi)d\xi\Big)dW(\zeta)\\ &-\sum_{0<\varpi_\iota<\mathfrak{p}}\mathbb{C}(\mathfrak{p}-\varpi_\iota)J_\iota\left(y(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right)-\sum_{0<\varpi_\iota<\mathfrak{p}}\mathbb{S}(\mathfrak{p}-\varpi_\iota)\overline{J_\iota}\left(y(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right)\Big\}d\mu,\\ &\int_0^\varpi\mathbb{S}(\varpi-\mu)\mathscr{B}\mathscr{B}^*\mathbb{S}^*(\mathfrak{p}-\mu)\left(\eta I+\Upsilon_0^{\mathfrak{p}}\right)^{-1}\\ &(\times)\Big\{-\int_0^{\mathfrak{p}}\mathbb{S}(\mathfrak{p}-\zeta)G\Big(\zeta,\widehat{y}_\zeta+\widehat{\alpha}_\zeta,\int_0^{\mathfrak{p}}l(\zeta,\xi,\widehat{y}_\xi+\widehat{\alpha}_\xi)d\xi,\int_0^{\mathfrak{p}}m(\zeta,\xi,\widehat{y}_\xi+\widehat{\alpha}_\xi)d\xi\Big)dW(\zeta)\\ &-\sum_{0<\varpi_\iota<\mathfrak{p}}\mathbb{C}(\mathfrak{p}-\varpi_\iota)J_\iota\left(\widehat{y}(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right)-\sum_{0<\varpi_\iota<\mathfrak{p}}\mathbb{S}(\mathfrak{p}-\varpi_\iota)\overline{J_\iota}\left(\widehat{y}(\varpi_\iota^-)+\widehat{\alpha}(\varpi_\iota^-)\right)\Big\}d\mu\Bigg)\\ &\leqslant P_0\sup_{\mu\in\mathscr{V}}E\|y(\mu)-\widehat{y}(\mu)\|^2.\end{aligned}$$

Taking supremum over ϖ, we get

$$\mathscr{U}_d^2\left(\widehat{\mathscr{F}}_1(y),\widehat{\mathscr{F}}_1(\widehat{y})\right)\leqslant P_0\|y-\widehat{y}\|^2,$$

where $P_0<1$. Hence, $\widehat{\mathscr{F}}_1$ is a contraction.

Step 2. $\widehat{\mathscr{F}}_2$ is completely continuous and has compact, convex values.

Claim 1. $\widehat{\mathscr{F}}_2(x)$ is convex $\forall\ x\in\mathscr{B}''_\vartheta$.

In particular, provided that $\widehat{\varphi}_1,\widehat{\varphi}_2\in\widehat{\mathscr{F}}_2(x)$, next $\exists\ \mathfrak{z}_1,\mathfrak{z}_2\in\mathbb{T}_{G,x}$ $\ni$ for each $\varpi\in\mathscr{V}$, one can get

$$\widehat{\varphi}_i(\varpi)=\int_0^\varpi\mathbb{S}(\varpi-\mu)\mathfrak{z}_i(\mu)dW(\mu), \quad i=1,2.$$

Let ψ belongs to $[0,1]$. Then $\forall\ \varpi\in\mathscr{V}$, one can get

$$\begin{aligned}\left(\psi\widehat{\varphi}_1+(1-\psi)\widehat{\varphi}_2\right)(\varpi)=&\int_0^\varpi\mathbb{S}(\varpi-\mu)\\ &\times[\beta\mathfrak{z}_1(\mu)+(1-\beta)\mathfrak{z}_2(\mu)]dW(\mu).\end{aligned}$$

We can easily show $\mathbb{T}_{G,x}$ is convex, hence G has convex values. Therefore, $\psi\mathfrak{z}_1+(1-\psi)\mathfrak{z}_2\in\mathbb{T}_{G,x}$. Consequently,

$$\left(\psi\widehat{\varphi}_1+(1-\psi)\widehat{\varphi}_2\right)(\varpi)\in\widehat{\mathscr{F}}_2.$$

Claim 2. In $\mathscr{B}''_\vartheta,\widehat{\mathscr{F}}_2$ maps bounded sets into itself.

Absolutely, it is necessary to prove thatthere exists $\mathscr{T}>0\ni\forall$ $\widehat{\varphi}\in\widehat{\mathscr{F}}_2(x),\ x\in B_{\bar{\mathfrak{p}}}=\left\{x\in\mathscr{B}''_\vartheta:\|x\|^2_{\mathfrak{p}}\leqslant\bar{\mathfrak{p}}\right\}$, we obatin $\|\widehat{\varphi}\|^2_{\mathfrak{p}}\leqslant\mathscr{T}$.

Provided that $\widehat{\varphi}\in\widehat{\mathscr{F}}_2(x)$, next $\exists\ \mathfrak{z}\in\mathbb{T}_{G,x}$ such that $\forall\ \varpi\in\mathscr{V}$,

$$\widehat{\varphi}(\varpi)=\int_0^\varpi\mathbb{S}(\varpi-\mu)\mathfrak{z}(\mu)dW(\mu).$$

Therefore, by the assumption $\forall\ \varpi\in\mathscr{V}$, we obtain

$$\begin{aligned}E\|\widehat{\varphi}(\varpi)\|^2 &= E\|\int_0^\varpi\mathbb{S}(\varpi-\mu)\mathfrak{z}(\mu)dW(\mu)\|^2\\ &\leqslant P_2Tr(\mathscr{Q})\int_0^\varpi E\|\mathfrak{z}(\mu)\|^2d\mu\\ &\leqslant P_2Tr(\mathscr{Q})\|\hbar_{\bar{\mathfrak{p}}'}\|^2_{L^1}\\ &= \mathscr{T}.\end{aligned}$$

Then $\forall\ \widehat{\varphi}\in\widehat{\mathscr{F}}_2(x)$, we get $\|\widehat{\varphi}\|^2_{\mathfrak{p}}\leqslant\mathscr{T}$.

Claim 3. $\widehat{\mathscr{F}}_2$ maps bounded sets into equicontinuous sets of $\mathscr{B}''_\vartheta$.

$$\widehat{\varphi}(\varpi)=\int_0^\varpi\mathbb{S}(\varpi-\mu)\mathfrak{z}(\mu)dW(\mu).$$

Let $0<\nu_1<\nu_2\leqslant\mathfrak{p}$. Then, we obtain $\forall\ x\in B_{\bar{\mathfrak{p}}}$ and $\widehat{\varphi}\in\widehat{\mathscr{F}}_2(x)$ $\exists$ $\mathfrak{z}\in\mathbb{T}_{G,x}$ such that

$$\begin{aligned}E\|\widehat{\varphi}(\nu_2)-\widehat{\varphi}(\nu_1)\|^2 &\leqslant 2E\|\int_0^{\nu_2}[\mathbb{S}(\nu_2-\mu)-\mathbb{S}(\nu_1-\mu)]\mathfrak{z}(\mu)dW(\mu)\|^2\\ &\quad+2E\|\int_{\nu_1}^{\nu_2}\mathbb{S}(\nu_1-\mu)\mathfrak{z}(\mu)dW(\mu)\|^2\\ &\leqslant 2Tr(\mathscr{Q})\int_0^{\nu_2}\|\mathbb{S}(\nu_2-\mu)-\mathbb{S}(\nu_1-\mu)\|^2E\|\mathfrak{z}(\mu)\|^2d\mu\\ &\quad+2Tr(\mathscr{Q})\int_{\nu_1}^{\nu_2}\|\mathbb{S}(\nu_1-\mu)\|^2E\|\mathfrak{z}(\mu)\|^2d\mu.\end{aligned}$$

The RHS of the above result is independent of $x\in B_{\bar{\mathfrak{p}}}$ and $E\|\widehat{\varphi}(\nu_2)-\widehat{\varphi}(\nu_1)\|^2\to0$ as $\nu_2-\nu_1\to0\ \forall\ x\in B_{\bar{\mathfrak{p}}}$. As a result, the uniform operator topology is determined by the compactness of $\mathbb{S}(\varpi)$ and $\mathbb{C}(\varpi)$ for $\varpi>0$. Therefore, the set $\left\{\widehat{\mathscr{F}}_2(x):x\in B_{\bar{\mathfrak{p}}}\right\}$ is equicontinuous.

Claim 4. $\widehat{\mathscr{F}}_2$ is a compact multi-valued map.

According to the preceding claims, $\widehat{\mathscr{F}}_2B_{\bar{\mathfrak{p}}}$ is a uniformly bounded and equicontinuous, as previously stated. $\widehat{\mathscr{F}}_2$ maps $B_{\bar{\mathfrak{p}}}$ into a precompact set belongs to $\mathscr{B}''_\vartheta$ by referring Arzela-Ascoli theorem, i.e., $\forall$ fixed $\varpi\in\mathscr{V},\Phi(\varpi)=\left\{\widehat{\mathscr{F}}_2x(\varpi):x\in B_{\bar{\mathfrak{p}}}\right\}$ is precompact belongs to $\mathscr{U}$.

Obviously, $\Phi(0)=\left\{\widehat{\mathscr{F}}(0)\right\}$. Let $\varpi>0$ be fixed and for $0<\epsilon<\varpi$, determine

$$\widehat{\mathscr{F}}_2^{\epsilon}x(\varpi)=\int_0^{\varpi-\epsilon}\mathbb{S}(\varpi-\mu)\mathfrak{z}(\mu)dW(\mu).$$

Because $\mathbb{S}(\varpi)$ is compact, $\Phi_{\epsilon}(\varpi)=\left\{\widehat{\mathscr{F}}_2^{\epsilon}x(\varpi):x\in B_{\bar{\mathfrak{p}}}\right\}$ is precompact in $x\ \forall\ \epsilon, 0<\epsilon<\varpi$. Additionally,

$$\begin{aligned}E\|\widehat{\mathscr{F}}_2x(\varpi)-\widehat{\mathscr{F}}_2^{\epsilon}x(\varpi)\|^2 &= E\|\textstyle\int_0^{\varpi}\mathbb{S}(\varpi-\mu)\mathfrak{z}(\mu)dW(\mu)\\ &\quad-\textstyle\int_0^{\varpi-\epsilon}\mathbb{S}(\varpi-\mu)\mathfrak{z}(\mu)dW(\mu)\|^2\\ &\leqslant E\|\textstyle\int_{\varpi-\epsilon}^{\varpi}\mathbb{S}(\varpi-\mu)\mathfrak{z}(\mu)dW(\mu)\|^2\\ &\leqslant Tr(\mathscr{Q})\textstyle\int_{\varpi-\epsilon}^{\varpi}\|\mathbb{S}(\varpi-\mu)\|^2E\|\mathfrak{z}(\mu)\|^2d\mu\\ &\leqslant P_2Tr(\mathscr{Q})\textstyle\int_{\varpi-\epsilon}^{\varpi}\hbar_{\bar{\mathfrak{p}}'}(\mu)d\mu.\end{aligned}$$

Therefore,

$$E\|\widehat{\mathscr{F}}_2x(\varpi)-\widehat{\mathscr{F}}_2^{\epsilon}x(\varpi)\|^2\to 0,\ \text{when}\ \epsilon\to 0^+.$$

In addition there are precompact sets are arbitrary close to $\left\{\widehat{\mathscr{F}}_2x(\varpi):x(\cdot)\in B_{\bar{\mathfrak{p}}}\right\}$. Therefore, $\widehat{\mathscr{F}}_2$ is compact multi-valued map.

Claim 5. $\widehat{\mathscr{F}}_2$ has a closed graph.

Assume $x_{\check{\mathfrak{q}}}$ tends to x_* as $\check{\mathfrak{q}}$ tends to ∞, $\widehat{\varphi}_{\check{\mathfrak{q}}}\in\widehat{\mathscr{F}}_2x_{\check{\mathfrak{q}}}$ and $\widehat{\varphi}_{\check{\mathfrak{q}}}$ tends to $\widehat{\varphi}_*$ as $\check{\mathfrak{q}}$ tends to ∞. Now, we prove $\widehat{\varphi}_*\in\widehat{\mathscr{F}}_2x_*$. Because $\widehat{\varphi}_{\check{\mathfrak{q}}}\in\widehat{\mathscr{F}}_2x_{\check{\mathfrak{q}}}\ \exists\ \mathfrak{z}_{\check{\mathfrak{q}}}\in\mathbb{T}_{G,x_{\check{\mathfrak{q}}}}$ such that

$$\widehat{\varphi}_{\check{\mathfrak{q}}}(\varpi)=\int_0^{\varpi}\mathbb{S}(\varpi-\mu)\mathfrak{z}_{\check{\mathfrak{q}}}(\mu)dW(\mu),\ \varpi\in\mathscr{V}.$$

We must demonstrate that $\exists\ \mathfrak{z}_*\in\mathbb{T}_{G,x_*}\ \ni$

$$\widehat{\varphi}_*(\varpi)=\int_0^{\varpi}\mathbb{S}(\varpi-\mu)\mathfrak{z}_*(\mu)dW(\mu),\ \varpi\in\mathscr{V}.$$

Now $\forall\ \varpi\in\mathscr{V}$, because $\mathfrak{z}$ is continuous, we obtain

$$\begin{aligned}&\left\|\left(\widehat{\varphi}_{\check{\mathfrak{q}}}(\varpi)-\int_0^{\varpi}\mathbb{S}(\varpi-\mu)\mathfrak{z}_{\check{\mathfrak{q}}}(\mu)dW(\mu)\right)\right.\\ &\left.-\left(\widehat{\varphi}_*(\varpi)-\int_0^{\varpi}\mathbb{S}(\varpi-\mu)\mathfrak{z}_*(\mu)dW(\mu)\right)\right\|^2\\ &\to 0\ \text{as}\ \check{\mathfrak{q}}\to\infty.\end{aligned}$$

Assume that $\mathscr{G}:L^2(\mathscr{V},\mathscr{U})\to\mathscr{C}(\mathscr{V},\mathscr{U})$, which is linear continuous operator,

$$(\mathscr{G}\mathfrak{z})(\varpi)=\int_0^{\varpi}\mathbb{S}(\varpi-\mu)\mathfrak{z}(\mu)dW(\mu).$$

As a result, referring Lemma 2.6, $\mathscr{G}\circ\mathbb{T}_G$ is a closed graph operator. Consequently, referring to $\mathscr{G}$, we obtain

$$\widehat{\varphi}_{\check{\mathfrak{q}}}(\varpi)-\int_0^{\varpi}\mathbb{S}(\varpi-\mu)\mathfrak{z}_{\check{\mathfrak{q}}}(\mu)dW(\mu)\in\ \mathscr{G}\left(\mathbb{T}_{G,x_{\check{\mathfrak{q}}}}\right).$$

Because $x_{\check{\mathfrak{q}}}\to x_*$, for some $x_*\in\mathbb{T}_{G,x_*}$, from Lemma 2.6,

$$\widehat{\varphi}_*(\varpi)-\int_0^{\varpi}\mathbb{S}(\varpi-\mu)\mathfrak{z}_*(\mu)dW(\mu)\in\ \mathscr{G}(\mathbb{T}_{G,x_*}).$$

Thus $\widehat{\mathscr{F}}_2$ has a closed graph.

Thus, $\widehat{\mathscr{F}}_2$ is a completely continuous multi-valued map with convex closed, upper semicontinuous, compact values.

Step 3. We will demonstrate that the family

$$\mathscr{M}=\left\{z\in\mathscr{U}:\lambda z\in\widehat{\mathscr{F}}_1z+\widehat{\mathscr{F}}_2z,\ \forall\ \lambda\in(0,1)\right\},$$

is bounded.

Consider $z\in\mathscr{U}$ and $\mathfrak{z}\in\mathbb{T}_{G,x}\ni$ for ϖ belongs to $\mathscr{V}$ we obtain

$$\begin{aligned}z(\varpi)=\ &\lambda\mathbb{C}(\varpi)\alpha(0)+\lambda\mathbb{S}(\varpi)z_1+\lambda\textstyle\int_0^{\varpi}\mathbb{S}(\varpi-\mu)\mathfrak{z}(\mu)dW(\mu)\\ &+\lambda\textstyle\int_0^{\varpi}\mathbb{S}(\varpi-\mu)\mathscr{B}u^{\eta}(\mu,z)d\mu\\ &+\lambda\sum_{0<\varpi_\imath<\varpi}\mathbb{C}(\varpi-\varpi_\imath)J_\imath(z(\varpi_\imath^-))\\ &+\lambda\sum_{0<\varpi_\imath<\varpi}\mathbb{S}(\varpi-\varpi_\imath)\overline{J}_\imath(z(\varpi_\imath^-)),\ \forall\ \lambda\in(0,1).\end{aligned}$$

For $\eta>0$, by applying $\mathbf{H_0}-\mathbf{H_7}$ and the Hölder's inequality, we obtain

$$\begin{aligned}E\|z(\varpi)\|^2\ &\leqslant\ 6\Big\{E\|\mathbb{C}(\varpi)\alpha(0)\|^2+E\|\mathbb{S}(\varpi)z_1\|^2\\ &+E\|\textstyle\int_0^{\varpi}\mathbb{S}(\varpi-\mu)\mathscr{B}u^{\eta}(\mu,z)d\mu\|^2\\ &+E\|\textstyle\int_0^{\varpi}\mathbb{S}(\varpi-\mu)\mathfrak{z}(\mu)dW(\mu)\|^2\\ &+E\|\sum_{0<\varpi_\imath<\varpi}\mathbb{C}(\varpi-\varpi_\imath)J_\imath(z(\varpi_\imath^-))\|^2\\ &+E\|\sum_{0<\varpi_\imath<\varpi}\mathbb{S}(\varpi-\varpi_\imath)\overline{J}_\imath(z(\varpi_\imath^-))\|^2\Big\}\\ &\leqslant 6\Big\{P_1E\|\alpha(0)\|^2+P_2E\|z_1\|^2\\ &+\tfrac{6\mathfrak{p}^2}{\eta^2}P_{\mathscr{B}}^2P_2^2\Big[E\|E\widehat{z}_{\mathfrak{p}}+\textstyle\int_0^{\mathfrak{p}}\widehat{\phi}(\zeta)dW(\zeta)\|^2+P_1E\|\alpha(0)\|^2\\ &+P_2E\|z_1\|^2+P_2Tr(\mathscr{Q})\textstyle\int_0^{\mathfrak{p}}\mathfrak{m}_{\mathfrak{z}}(\zeta)\Theta\Big(\|z_\zeta\|^2+\breve{\mathscr{P}}_l(\zeta)\|z_\zeta\|^2\\ &+\breve{\mathscr{P}}_m(\zeta)\|z_\zeta\|^2\Big)d\zeta\\ &+2\jmath P_1\sum_{\imath=1}^{\jmath}P_\imath\|z(\varpi_\imath^-)\|^2+2\jmath P_1\sum_{\imath=1}^{\jmath}E\|J_\imath(0)\|^2\\ &+2\jmath P_2\sum_{\imath=1}^{\jmath}\overline{P_\imath}\|z(\varpi_\imath^-)\|^2\\ &+2\jmath P_2\sum_{\imath=1}^{\jmath}E\|\overline{J}_\imath(0)\|^2\Big]+P_2Tr(\mathscr{Q})\textstyle\int_0^{\varpi}\mathfrak{m}_{\mathfrak{z}}(\mu)\Theta\\ &\Big(\|z_\mu\|^2+\breve{\mathscr{P}}_l(\mu)\|z_\mu\|^2+\breve{\mathscr{P}}_m(\mu)\|z_\mu\|^2\Big)d\mu\\ &+2\jmath P_1\sum_{\imath=1}^{\jmath}P_\imath\|z(\varpi_\imath^-)\|^2+2\jmath P_1\sum_{\imath=1}^{\jmath}E\|J_\imath(0)\|^2\\ &+2\jmath P_2\sum_{\imath=1}^{\jmath}\overline{P_\imath}\|z(\varpi_\imath^-)\|^2+2\jmath P_2\sum_{\imath=1}^{\jmath}E\|\overline{J}_\imath(0)\|^2\Big\}.\end{aligned}\tag{3.14}$$

Now, we assume the function δ determined by

$$\delta(\varpi)=\sup_{0\leqslant\mu\leqslant\varpi}E\|z(\mu)\|^2,\quad\varpi\in[0,\mathfrak{p}].$$

By Lemma 2.1 and (3.14), one can obtain

$$E\|z(\varpi)\|^2\leqslant 2\ell^2\sup_{0\leqslant\mu\leqslant\varpi}E\|z(\mu)\|^2+2\|\alpha\|^2_{\mathscr{B}_\vartheta}.$$

Therefore, we get

$$\begin{aligned}\delta(\varpi)\ &\leqslant 2\|\alpha\|^2_{\mathscr{B}_\vartheta}+2\ell^2\{\overline{P}+6P_2Tr(\mathscr{Q})\textstyle\int_0^{\varpi}\mathfrak{m}_{\mathfrak{z}}(\mu)\Theta(\delta(\mu)+\breve{\mathscr{P}}_l(\mu)\delta(\mu)+\breve{\mathscr{P}}_m(\mu)\delta(\mu))d\mu\\ &+\left(\tfrac{6\mathfrak{p}}{\eta}\right)^2P_{\mathscr{B}}^2P_2^2[P_2Tr(\mathscr{Q})\textstyle\int_0^{\mathfrak{p}}\mathfrak{m}_{\mathfrak{z}}(\zeta)\Theta(\delta(\zeta)+\breve{\mathscr{P}}_l(\zeta)\delta(\zeta)+\breve{\mathscr{P}}_m(\zeta)\delta(\zeta))d\zeta\\ &+2\jmath P_1\sum_{\imath=1}^{\jmath}P_\imath\delta(\varpi)+2\jmath P_2\sum_{\imath=1}^{\jmath}\overline{P_\imath}\delta(\varpi)\Big]+12\jmath P_1\sum_{\imath=1}^{\jmath}P_\imath\delta(\varpi)+12\jmath P_2\sum_{\imath=1}^{\jmath}\overline{P_\imath}\delta(\varpi)\Big\}.\end{aligned}$$

In the above $\overline{P}$ is given in (3.2). Thus, we get

$$\begin{aligned}\delta(\varpi)\ &\leqslant\mathbb{P}_3+\mathbb{P}_4\textstyle\int_0^{\varpi}\mathfrak{m}_{\mathfrak{z}}(\mu)\Theta(\delta(\mu)+\breve{\mathscr{P}}_l(\mu)\delta(\mu)+\breve{\mathscr{P}}_m(\mu)\delta(\mu))d\mu,\\ &\leqslant\mathbb{P}_3+\mathbb{P}_4\textstyle\int_0^{\varpi}\mathfrak{m}_{\mathfrak{z}}(\mu)(1+\breve{\mathscr{P}}_l(\mu)+\breve{\mathscr{P}}_m(\mu))\Theta(\delta(\mu))d\mu,\end{aligned}\tag{3.15}$$

where $\mathbb{P}_3$ and $\mathbb{P}_4$ are given in (3.4). Take the right hand of (3.15) as $\psi(\varpi)$. Next, $\psi(0) = \mathbb{P}_3, \delta(\varpi) \leqslant \psi(\varpi),\ 0 \leqslant \varpi \leqslant \mathfrak{p}$ and

$$\psi'(\varpi) \leqslant \mathbb{P}_4 \mathfrak{m}_\mathfrak{z}(\varpi)\big(1 + \breve{\mathscr{P}}_l(\varpi) + \breve{\mathscr{P}}_m(\varpi)\big)\Theta(\psi(\varpi)).$$

The above inequality implies that

$$\int_{\psi(0)}^{\psi(\varpi)} \frac{1}{\Theta(\wp)} d\wp \leqslant \mathbb{P}_4 \int_0^{\mathfrak{p}} \mathfrak{m}_\mathfrak{z}(\mu)\big(1 + \breve{\mathscr{P}}_l(\mu) + \breve{\mathscr{P}}_m(\mu)\big)d\mu \leqslant \int_{\mathfrak{p}}^{\infty} \frac{1}{\Theta(\wp)} d\wp.$$

Therefore, the above inequality demonstrate that there exists $\mathscr{H} \ni \psi(\varpi) \leqslant \mathscr{H},\ \varpi \in \mathscr{V}$ and thus $\|z_\varpi\|^2_{\mathscr{B}_\vartheta} \leqslant \delta(\varpi) \leqslant \psi(\varpi),\ \varpi \in \mathscr{V}$, where $\mathscr{H}$ depends only on $\mathfrak{p}$ and on the functions $\mathfrak{m}_\mathfrak{z}(\cdot), \breve{\mathscr{P}}_l(\cdot), \breve{\mathscr{P}}_m(\cdot)$ and $\Theta(\cdot)$. This shows that the set $\mathscr{M}$ is bounded.

We conclude from Lemma 2.7 that $\widehat{\mathscr{F}}_1 + \widehat{\mathscr{F}}_2$ has a fixed point, which is the mild solution of the system (1.1)–(1.3). This completes the proof.

Definition 3.4. "The state value of (1.1)–(1.3) at the terminal time $\mathfrak{p}$ is $z_\mathfrak{p}(z_0, u)$ relating to u and the initial value α. Describe the family

$$\mathscr{R}(\mathfrak{p}, z_0) = \Big\{z_\mathfrak{p}(z_0; u)(0) : u(\cdot) \text{ in } L^2(\mathscr{V}, \mathscr{X})\Big\},$$

which is called the reachable set of (1.1)–(1.3) at the terminal time $\mathfrak{p}$ and its closure in $\mathscr{U}$ is denoted by $\overline{\mathscr{R}(\mathfrak{p}, z_0)}$. If $\overline{\mathscr{R}(\mathfrak{p}, z_0)} = \mathscr{U}$, then the stochastic inclusions (1.1)–(1.3) is said to be approximately controllable on $[0, \mathfrak{p}]$."

Theorem 3.5. *Assume that* $\mathbf{H_0}$-$\mathbf{H_7}$ *are fulfilled and* $\mathfrak{z}$ *is uniformly bounded. Furthermore, provided that* $\{\mathbb{S}(\varpi)\}_{\varpi \geqslant 0}$ *and* $\{\mathbb{C}(\varpi)\}_{\varpi \geqslant 0}$ *are compact, next the system (1.1)–(1.3) is approximately controllable on* $\mathscr{V}$.

Proof. Suppose that $z^\eta(\cdot)$ is a fixed point of $\widehat{\mathscr{F}}$ in $B_{\bar{\mathfrak{p}}}$. According to Theorem 3.3, any fixed point of $\widehat{\mathscr{F}}$ is a mild solution of (1.1)–(1.3). This means that $\exists\ z^\eta \in \widehat{\mathscr{F}}(z^\eta)$; i.e., by the stochastic Fubini theorem $\exists\ \mathfrak{z} \in \mathbb{T}_{G,z^\eta}$ such that

$$\begin{aligned} z^\eta(\mathfrak{p}) &= \hat{z}_\mathfrak{p} - \eta\big(\eta I + \Upsilon_0^\mathfrak{p}\big)^{-1}\big(E\hat{z}_\mathfrak{p} - \mathbb{C}(\mathfrak{p})\alpha(0) - \mathbb{S}(\mathfrak{p})z_1\big) \\ &+ \eta \textstyle\int_0^\mathfrak{p} \Big(\eta I + \Upsilon_\mu^\mathfrak{p}\Big)^{-1} \hat{\phi}(\mu)dW(\mu) \\ &- \eta \textstyle\int_0^\mathfrak{p} \Big(\eta I + \Upsilon_\mu^\mathfrak{p}\Big)^{-1} \mathbb{S}(\mathfrak{p}-\mu)\mathfrak{z}^\eta(\mu)dW(\mu) \\ &- \eta\Big(\eta I + \Upsilon_\mu^\mathfrak{p}\Big)^{-1} \sum_{0<\varpi_\imath<\mathfrak{p}} \mathbb{C}(\mathfrak{p}-\varpi_\imath)J_\imath(z^\eta(\varpi_\imath^-)) \\ &- \eta\Big(\eta I + \Upsilon_\mu^\mathfrak{p}\Big)^{-1} \sum_{0<\varpi_\imath<\mathfrak{p}} \mathbb{S}(\mathfrak{p}-\varpi_\imath)\overline{J}_\imath(z^\eta(\varpi_\imath^-)). \end{aligned}$$

It follows from the characteristics of $\mathfrak{z}\ \exists\ \mathscr{C} > 0 \ni$

$$\|\mathfrak{z}\Big(\mu, z_\mu^\eta\Big)\|^2 \leqslant \mathscr{C},$$

in $\mathscr{V} \times \Omega$. Next, there is a subsequences, still stand for $\{\mathfrak{z}^\eta(\mu)\}$ and $\{J_\imath(z^\eta(\varpi_\imath^-)), \overline{J}_\imath(z^\eta(\varpi_\imath^-))\}$ which converges weakly to, say, $\{\mathfrak{z}(\mu)\}$ belongs to $L^2\Big([0,\mathfrak{p}], L_2^0\Big)$ and $\{J_\imath(w), \overline{J}_\imath(w)\}$ belongs to $\mathscr{U}$, respectively. The compactness of $\mathbb{S}(\varpi)$ and $\mathbb{C}(\varpi), \varpi > 0$, implies that

$$\begin{aligned} &\mathbb{S}(\mathfrak{p}-\varpi_\imath)\overline{J}_\imath(z^\eta(\varpi_\imath^-)) \to \mathbb{S}(\mathfrak{p}-\varpi_\imath)\overline{J}_\imath(w), \\ &\mathbb{S}(\mathfrak{p}-\mu)\mathfrak{z}^\eta(\mu) \to \mathbb{S}(\mathfrak{p}-\mu)\mathfrak{z}(\mu), \\ &\mathbb{C}(\mathfrak{p}-\varpi_\imath)J_\imath(z^\eta(\varpi_\imath^-)) \to \mathbb{S}(\mathfrak{p}-\varpi_\imath)J_\imath(w). \end{aligned}$$

On the other hand, ϖ belongs to $\mathscr{V}$, $\eta\Big(\eta I + \Upsilon_\mu^\mathfrak{p}\Big)^{-1}$ tends to 0 strongly as $\eta \to 0$ and $\|\eta\Big(\eta I + \Upsilon_\mu^\mathfrak{p}\Big)^{-1}\| \leqslant 1$. Therefore, using the Lebesgue dominated convergence theorem,

$$\begin{aligned} E\|z^\eta(\mathfrak{p}) - \quad \hat{z}_\mathfrak{p}\|^2 &\leqslant 6\|\eta\big(\eta I + \Upsilon_0^\mathfrak{p}\big)^{-1}\big(E\hat{z}_\mathfrak{p} - \mathbb{C}(\mathfrak{p})\alpha(0) - \mathbb{S}(\mathfrak{p})z_1\big)\|^2 \\ &+6E\Big(\textstyle\int_0^\mathfrak{p} \|\eta\Big(\eta I + \Upsilon_\mu^\mathfrak{p}\Big)^{-1}\hat{\phi}(\mu)\|^2_{L_2^0}d\mu\Big) \\ &+6E\Big(\textstyle\int_0^\mathfrak{p} \|\eta\Big(\eta I + \Upsilon_\mu^\mathfrak{p}\Big)^{-1}\mathbb{S}(\mathfrak{p}-\mu)[\mathfrak{z}^\eta(\mu) - \mathfrak{z}(\mu)]\|^2_{L_2^0}d\mu\Big) \\ &+6E\Big(\textstyle\int_0^\mathfrak{p} \|\eta\Big(\eta I + \Upsilon_\mu^\mathfrak{p}\Big)^{-1}\mathbb{S}(\mathfrak{p}-\mu)\mathfrak{z}(\mu)\|^2_{L_2^0}d\mu\Big) \\ &+6E\left(\|\eta\Big(\eta I + \Upsilon_\mu^\mathfrak{p}\Big)^{-1}\sum_{0<\varpi_\imath<\mathfrak{p}}\mathbb{C}(\mathfrak{p}-\varpi_\imath)J_\imath(z^\eta(\varpi_\imath^-))\|\right)^2 \\ &+6E\left(\|\eta\Big(\eta I + \Upsilon_\mu^\mathfrak{p}\Big)^{-1}\sum_{0<\varpi_\imath<\mathfrak{p}}\mathbb{S}(\mathfrak{p}-\varpi_\imath)\overline{J}_\imath(z^\eta(\varpi_\imath^-))\|\right)^2 \\ &\longrightarrow 0 \text{ as } \eta \longrightarrow 0^+. \end{aligned} \tag{3.16}$$

So, $z^\eta(\mathfrak{p}) \to \hat{z}_\mathfrak{p}$ as $\eta \to 0^+$ and, indicating that the problem (1.1)–(1.3) is approximate controllable. The proof is complete.

4. Nonlocal conditions

The nonlocal condition has far more influence on physics than the classical initial condition: $z(0) = z_0$. Byszewski (1991); Byszewski and Akca (1997) introduced the Cauchy problem involving nonlocal conditions for a semilinear system to verify the existence and uniqueness of solutions. Later, Huan (2015) studied the controllability of nonlocal second-order impulsive neutral stochastic integrodifferential systems involving impulsive effects, neutral functions, infinite delay, and nonlocal circumstances in Hilbert spaces via the fixed point theorem along with the help of the cosine family of operators. Recently, the approximate controllability of second-order stochastic equations with nonlocal circumstances has been investigated by Arora and Sukavanam (2015) through the fixed point approach of Sadovskii. For more information, look at the published papers (Henríquez and Hernández, 2006; Henríquez et al., 2014; Mophou and N'Guerekata, 2009; N'Guerekata, 2009; Slama and Boudaoui, 2017).

Consider the form of the second-order impulsive stochastic Volterra-Fredholm delay integrodifferential system with nonlocal conditions

$$\begin{aligned} &d(z'(\varpi)) \in [Az(\varpi) + \mathscr{B}u(\varpi)]d\varpi \\ &+G\big(\varpi, z_\varpi, \textstyle\int_0^\varpi l(\varpi,\mu,z_\mu)d\mu, \int_0^\mathfrak{p} m(\varpi,\mu,z_\mu)d\mu\big)dW(\varpi), \\ &\varpi \in \mathscr{V} = [0,\mathfrak{p}],\ \varpi \neq \varpi_j,\ \imath = 1,2,\ldots,\jmath, \end{aligned} \tag{4.1}$$

$$z(0) - h(z_{\varpi_1}, z_{\varpi_2}, z_{\varpi_3}, \ldots, z_{\varpi_n}) = z_0 = \alpha \in L^2(\Omega, \mathscr{B}_\vartheta),\ z'(0) = z_1 \in \mathscr{U}, \tag{4.2}$$

$$\Delta z|_{\varpi=\varpi_\imath} = J_\imath\big(z(\varpi_\imath^-)\big), \quad \imath = 1,2,\ldots,\jmath, \tag{4.3}$$

$$\Delta z'|_{\varpi=\varpi_\imath} = \overline{J}_\imath\big(z(\varpi_\imath^-)\big), \quad \imath = 1,2,\ldots,\jmath, \tag{4.4}$$

where $h : \mathscr{B}_\vartheta^k \to \mathscr{B}_\vartheta$ is a given function which fulfills:

- $\mathbf{H_8}$: The continuous function h mapping from $\mathscr{B}_\vartheta^k$ into $\mathscr{B}_\vartheta$ and $P_h > 0$ such that

$$\|h(\widehat{u}_{\varpi_1},\widehat{u}_{\varpi_2},\widehat{u}_{\varpi_3},\ldots,\widehat{u}_{\varpi_n})(\varpi)-h(\widehat{v}_{\varpi_1},\widehat{v}_{\varpi_2},\widehat{v}_{\varpi_3},\ldots,\widehat{v}_{\varpi_n})\times(\varpi)\|^2 \leqslant P_h\|\widehat{u}-\widehat{v}\|^2_{\mathscr{B}_\vartheta},$$

$\forall\ \widehat{u},\widehat{v}\in\mathscr{B}_\vartheta,\ \varpi\in(-\infty,0]$ and assume
$\mathscr{P}_h=\sup\left\{\|h(\widehat{u}_{\varpi_1},\widehat{u}_{\varpi_2},\widehat{u}_{\varpi_3},\ldots,\widehat{u}_{\varpi_n})(\varpi)\|^2:\widehat{u}\in\mathscr{B}_\vartheta\right\}$.

For instance, $h(z_{\varpi_1},z_{\varpi_2},z_{\varpi_3},\ldots,z_{\varpi_n})$ can be represented as

$$h(z_{\varpi_1},z_{\varpi_2},z_{\varpi_3},\cdots,z_{\varpi_n})=\sum_{k=1}^{\mathfrak{n}}b_k z_{\varpi_k}.$$

In the above $b_k(k=1,2,\ldots,\mathfrak{n})$ are provided as constants and $0<\varpi_1<\varpi_2<\varpi_3<\cdots<\varpi_{\mathfrak{n}}<\mathfrak{p}$.

Definition 4.1. A stochastic process $z:\mathscr{V}_0\times\Omega\to\mathscr{U}$ is said to be a mild solution of (4.1)–(4.4) provided that

(*i*) $z(\varpi)$ is measurable and adapted to $\mathfrak{J}_\varpi,\ \forall\ \varpi\geqslant 0$.
(*ii*) $z(\varpi)\in\mathscr{U}$ has *càdlàg* paths on $\varpi\in\mathscr{V}$ a.s., $\forall\ 0\leqslant\mu<\varpi\leqslant\mathfrak{p}$, and $\exists\ \mathfrak{z}\in\mathbb{T}_{G,z}$ and the impulsive conditions $\Delta z|_{\varpi=\varpi_\imath}=J_\imath(z(\varpi_\imath^-)),\ \Delta z'|_{\varpi=\varpi_\imath}=\overline{J}_\imath(z(\varpi_\imath^-))\ (\imath=1,2,\ldots,\jmath)$ such that the subsequent integral equation holds

$$\begin{aligned} z(\varpi) &= \mathbb{C}(\varpi)\left[\alpha(0)+h(z_{\varpi_1},z_{\varpi_2},z_{\varpi_3},\ldots,z_{\varpi_n})(0)\right]\\ &+\mathbb{S}(\varpi)z_1+\textstyle\int_0^\varpi\mathbb{S}(\varpi-\mu)\mathfrak{z}(\mu)dW(\mu)\\ &+\textstyle\int_0^\varpi\mathbb{S}(\varpi-\mu)\mathscr{B}u(\mu)d\mu\\ &+\sum_{0<\varpi_\imath<\varpi}\mathbb{C}(\varpi-\varpi_\imath)J_\imath(z_{\varpi_\imath})\\ &+\sum_{0<\varpi_\imath<\varpi}\mathbb{S}(\varpi-\varpi_\imath)\overline{J}_\imath(z_{\varpi_\imath}),\ \varpi\in\mathscr{V}. \end{aligned}\tag{4.5}$$

(*iii*) $z(0)-h(z_{\varpi_1},z_{\varpi_2},z_{\varpi_3},\ldots,z_{\varpi_n})=z_0=\alpha\ \in\mathscr{B}_\vartheta$ on $\mathscr{V}_1$ and $z'(0)=z_1\in\mathscr{U}$ fulfilling $z_0,z_1,h\in L_2^0(\Omega,\mathscr{U})$.

Theorem 4.2. *Assume the assumptions* $\mathbf{H_0}$-$\mathbf{H_8}$ *are fulfilled. In addition, the second-order stochastic differential inclusion (4.1)–(4.4) is approximately controllable on* $[0,\mathfrak{p}]$.

5. An example

Consider the second-order impulsive differential inclusions with control function

$$\begin{aligned}&\partial\left(\tfrac{\partial w(\varpi,\bar{\varkappa})}{\partial\varpi}\right)\in[w_{\bar{\varkappa}\bar{\varkappa}}(\varpi,\bar{\varkappa})+\upsilon(\varpi,\bar{\varkappa})]\partial\varpi\\ &+\widehat{\mathscr{G}}\big(\varpi,\textstyle\int_{-\infty}^{\varpi}\widehat{p}_1(\mu-\varpi)w(\mu,\bar{\varkappa})d\mu,\\ &\textstyle\int_0^\varpi\int_{-\infty}^0\widehat{l}(\mu,\bar{\varkappa},\tau-\mu)\widehat{l}_1(w(\tau,\bar{\varkappa}))d\tau d\mu,\\ &\textstyle\int_0^{\mathfrak{p}}\int_{-\infty}^0\widehat{m}(\mu,\bar{\varkappa},\tau-\mu)\widehat{m}_1(w(\tau,\bar{\varkappa}))d\tau d\mu\big)dW(\varpi),\\ &\varpi\neq\varpi_\imath,\ \varpi\in[0,1],\ \bar{\varkappa}\in[0,\pi],\end{aligned}\tag{5.1}$$

$$w(\varpi,0)=\ w(\varpi,\pi)=0,\quad\varpi\in[0,1],\tag{5.2}$$

$$w(\varpi,\bar{\varkappa})=\ \alpha(\varpi,\bar{\varkappa}),\quad\bar{\varkappa}\in[0,\pi],$$

$$\varpi\in(-\infty,0],\quad\frac{\partial}{\partial\varpi}w(0,\bar{\varkappa})=w_1,\quad\bar{\varkappa}\in[0,\pi],\tag{5.3}$$

$$w(\varpi_\imath^+,\bar{\varkappa})-w(\varpi_\imath^-,\bar{\varkappa})=J_\imath(w(\varpi_\imath^-,\bar{\varkappa})),\quad\imath=1,2,\ldots,\jmath,\tag{5.4}$$

$$w'(\varpi_\imath^+,\bar{\varkappa})-w'(\varpi_\imath^-,\bar{\varkappa})=\overline{J}_\imath(w(\varpi_\imath^-,\bar{\varkappa})),\quad\imath=1,2,\ldots,\jmath,\tag{5.5}$$

where $\alpha(\varpi,\bar{\varkappa}),\widehat{p}_1,\widehat{l},\widehat{\mathscr{G}},\widehat{m}$ are continuous, $W(\varpi)$ stands for a $\mathscr{Q}$-Wiener process, and $\mathscr{U}=\mathbb{V}=L^2([0,\pi])$ with $\|\cdot\|_{\mathscr{U}}$. Introduce $A:\mathscr{D}(A)\subset\mathscr{U}\to\mathscr{U}$ from $Aw=w''$ along with

$$\mathscr{D}(A)=\{w\in\mathscr{U}:w,w'\text{ are absolutely continuous},\ w\in\mathscr{U},w(0)=0=w(\pi)\}.$$

In addition, A has discrete spectrum, the eigen values are $-\breve{\kappa}^2,\breve{\kappa}=1,2,\ldots$ along with the similar normalized eigen vectors $e_{\breve{\kappa}}(\mu)=\sqrt{\frac{2}{\pi}}\sin(\breve{\kappa}\mu)$, then

$$Aw=\sum_{\breve{\kappa}=1}^{\infty}(-\breve{\kappa}^2)\langle w,e_{\breve{\kappa}}\rangle e_{\breve{\kappa}},\ w\in\mathscr{D}(A).$$

A has the infinitesimal generator of a strong continuous cosine family

$$\mathbb{C}(\varpi)w=\sum_{\breve{\kappa}=1}^{\infty}\cos(\breve{\kappa}\varpi)\langle w,e_{\breve{\kappa}}\rangle e_{\breve{\kappa}},\ w\in\mathscr{U},$$

and sine family

$$\mathbb{S}(\varpi)w=\sum_{\breve{\kappa}=1}^{\infty}\frac{1}{\breve{\kappa}}\sin(\breve{\kappa}\varpi)\langle w,e_{\breve{\kappa}}\rangle e_{\breve{\kappa}},\ w\in\ \mathscr{U}.$$

Therefore, it is obvious that $\|\mathbb{C}(\varpi)\|\leqslant 1$ and $\|\mathbb{S}(\varpi)\|\leqslant 1\ \forall\ \varpi\in\mathbb{R}$.

Now, we introduce $\mathscr{B}_\vartheta$. Assume $\vartheta(\varpi)=e^{2\varpi},\ \varpi\leqslant 0$ then $\int_{-\infty}^0\vartheta(\mu)d\mu=\frac{1}{2}$ and let

$$\|\alpha\|_{\mathscr{B}_\vartheta}=\int_{-\infty}^0\vartheta(\mu)\sup_{\mu\leqslant\theta\leq 0}\left(E\|\alpha(\theta)\|^2\right)^{\frac{1}{2}}d\mu,\ \text{for all }\alpha\in\mathscr{B}_\vartheta,$$

therefore, $\left(\mathscr{B}_\vartheta,\|.\|_{\mathscr{B}_\vartheta}\right)$ is a Banach space.

Characterize an infinite-dimensional space $\mathbb{V}$ as

$$\mathbb{V}=\left\{u\text{ such that }u=\sum_{j=2}^{\infty}u_j v_j,\ \text{with}\ \sum_{j=2}^{\infty}\mathbb{V}_j^2<\infty\right\}.$$

The norm in $\mathbb{V}$ is determined as $\|u\|^2_{\mathbb{V}}=\sum_{j=2}^\infty\mathbb{V}_j^2$. Now, determine a continuous linear mapping $\mathscr{B}:\mathbb{V}\to\mathscr{U}$ as

$$\mathscr{B}u=2u_2v_1+\sum_{j=2}^{\infty}u_jv_j\text{ for }u=\sum_{j=2}^{\infty}u_jv_j\text{ in }\mathbb{V}.$$

Define $\mathscr{B}:\mathscr{X}\to\mathscr{U}$ is interpreted as $\mathscr{B}u(\varpi)(\bar{\varkappa})=\upsilon(\varpi,\bar{\varkappa}),0\leqslant\bar{\varkappa}\leqslant\pi$.

Here for $(\varpi,\alpha)\in[0,\mathfrak{p}]\times\mathscr{B}_\vartheta,\alpha(\rho)(\bar{\varkappa})=\alpha(\rho,\bar{\varkappa}),(\rho,\bar{\varkappa})\in(-\infty,0]\times[0,\pi]$.

Assume

$$\begin{aligned}&w(\varpi)\left(\bar{\varkappa}\right)=w(\varpi,y),\\ &l(\varpi,\mu,\mathscr{X})\left(\bar{\varkappa}\right)=\textstyle\int_{-\infty}^0\hat{l}(\varpi,y,\rho)\hat{l}_1(\mathscr{X}(\rho)(y))d\rho,\\ &m(\varpi,\mu,\mathscr{X})\left(\bar{\varkappa}\right)=\textstyle\int_{-\infty}^0\hat{m}\left(\varpi,\bar{\varkappa},\rho\right)\hat{m}_1\left(\mathscr{X}(\rho)\left(\bar{\varkappa}\right)\right)d\rho,\\ &G\varpi,\mathscr{X},\textstyle\int_0^\varpi l(\varpi,\mu,\mathscr{X})d\mu,\int_0^{\mathfrak{p}}m(\varpi,\mu,\mathscr{X})d\mu\left(\bar{\varkappa}\right)\\ &=\hat{\mathscr{G}}\left(\varpi,\textstyle\int_{-\infty}^\varpi\hat{p}_1(\rho)\alpha(\rho)d\rho,\int_0^\varpi l(\varpi,\mu,\mathscr{X})\left(\bar{\varkappa}\right)d\mu,\int_0^{\mathfrak{p}}m(\varpi,\mu,\mathscr{X})\left(\bar{\varkappa}\right)d\mu\right).\end{aligned}$$

Using the above discussion, we may convert (5.1)–(5.5) into (1.1)–(1.3). In addition, we conclude that every requirement of Theorem 3.5 is fulfilled, and we conclude that (5.1)–(5.5) is approximately controllable on $[0,\mathfrak{p}]$.

6. Conclusion

This manuscript examined the approximate controllability of a second-order impulsive stochastic Volterra-Fredholm integrodifferential system involving infinite delay in Hilbert spaces. Through the fixed point approach of Dhage due to multi-valued maps, stochastic theory, and the cosine function of operators, we have verified the existence of mild solutions for the given system. The sufficient condition for approximate controllability was formulated and proved. Following that, we have extended our system with the nonlocal condition. Finally, an application is shown to demonstrate the primary outcomes. In our upcoming work, we plan to estimate the present system involving mixed fBm. Following that, we will extend this valuable system to fractional differential inclusions of order $(2, 3)$ using the fractional calculus.

Funding Statement

The work of Yong-Ki Ma was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1F1A1048937). This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444).

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at https://doi.org/10.1016/j.jksus.2023.102637.

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