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Progressive stress accelerated life test for inverse Weibull failure model: A parametric inference

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ABSTRACT

For devices with long life, a new and advanced type of stresses test called progressive stress partially accelerated life test (PSPALT) has been adopted to gain quickly information about the device's lifetime distribution. This article presents a PSPALT model when the inverse Weibull life distribution is considered. The progressive stress is assumed to be directly proportional to time. The statistical properties of the maximum likelihood (ML) estimators of the model parameters such as existence, uniqueness and invariance are studied. The mean squared errors of the ML estimators are calculated to evaluate their performances through a Monte Carlo simulation study.

1. Introduction

For highly reliable materials; components or devices, their lifetimes are usually longer under normal functioning circumstances. Therefore, it becomes time-consuming for getting enough failure time information or the cost of the experiment is high. Consequently, efficient methods of testing such materials are needed. Accelerated life tests (ALTs) or partially accelerated life tests (PALTs) are considered the most effective methods of testing units of long life span under unusual stresses such as voltage, temperature, humidity, strength, pressure, and so on. In this respect readers can refer, for example, to (Porter et al., 2019; He et al., 2018; Torre et al., 2018; Lan et al., 2018; Prudhomme et al., 2018; Ismail, 2020). In addition, interested readers can also refer to (Nelson, 1990) which is a comprehensible source for ALT.

ALTs generally assume a relationship between the stress level and the corresponding life distribution. This relationship is usually referred to as the time transformation function. Interested readers can refer, for example, to (Nelson, 1990; Nelson, 1980; Nelson, 1982). If such time transformation function is not known or can't be assumed, ALTs can't be applied and other tests, namely, PALTs come to be a good alternative of ALTs. Under ALTs, the test units are run only under high stress. But in PALTs the test items are run under both use- and high-stresses.

There are different ways to apply the stress. As indicated by (Yin and Sheng, 1987) the common types of stress are: constant, step, and progressive. Under constant-stress PALTs each item is run at either use condition or accelerated condition only. That is, each unit is run at a steady stress (constant-stress level) until the test is terminated or the unit fails. For more details, interested researchers can refer, for example, to (Ismail, 2019; Xin et al., 2020). Regarding to the step-stress PALTs, a test item is first run at use condition and, if it does not fail for a pre-specified time τ, then it is run at accelerated condition until it fails or the test is terminated. The progressive stress (PS) is similar to step stress, but the stress on units is a progressive (continuous) function. Statistical theory for progressive stress under ALTs has been studied by some authors, for example, (Yin and Sheng, 1987; Mann et al.,

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Nomenclature

Notation

a, b	parameters of function that relates rate parameter and stress
ALTs	accelerated life tests
K K1, K2 ML	slope of linear progressive stress (stress rate, a some pre-assigned positive constant) design and high stress rates maximum likelihood
MLEs	maximum likelihood estimates/estimators
MTTF	mean time to failure
n	number of test units (sample size)
na	number of failures at accelerated condition
nc	number of non-observed (censored) units
nu	number of failures at use condition
PALT(s)	partially accelerated life test(s)
PS	progressive stress
q	number of stress levels
S	stress
Si	stress of level i, i = 1, 2, . . ., q
SSPALTs	step-stress partially accelerated life tests
ti	observed lifetime of unit i tested under PSPALTs
WD	Weibull distribution
^	implies a maximum likelihood estimate
α	WD shape parameter
β	acceleration factor (β < 1)
λ	WD rate parameter (inverse of scale parameter)
η	censoring time
θ	WD scale parameter ($\theta = 1/\lambda$)
τ	stress change time
τi	time at which the stress goes from Si to Si + 1
ε	$\max_i (t_i - \tau_i)$

1974; Allen, 1959) discussed this kind of testing theoretically when the underlying life distributions are exponential. In addition, (Abdel-Hamid and AL-Hussaini, 2007) presented progressive stress under ALTs using finite mixture models. Also, (Abdel-Hamid and AL-Hussaini, 2011) considered progressive stress under ALTs when the lifetime of an item under use condition follows Weibull distribution (WD) with a scale parameter satisfying the inverse power law.

This article discusses PS under PALTs when the lifetime of test units under use condition follows the inverse Weibull distribution (IWD). The PS is assumed to be directly proportional to time. That is, the stress is a linearly increasing function in time. The objective of PALTs is to collect more failure data in a limited time without necessarily using high stresses to all test units. Moreover, as indicated by (Lin and Fei, 1991), testing time can be further shortened by progressive stress. They considered a nonparametric approach to progressive stress accelerated life tests (PSALTs). Except one article on PS under PALTs assuming Weibull distribution by (Ismail and Al-Babtain, 2015), all published works on PALTs had been considered under the two traditional types of stress: constant and step. For example, see (Goel, 1971; DeGroot and Goel, 1979; Bai and Chung, 1992; Bai et al., 1993; Ismail, 2004; Abdel-Hamid and Al-Hussaini, 2008; Aly and Ismail, 2008; Ismail, 2010; 2012a; 2012b; 2012c; 2013; Srivastava and Mittal, 2010; Tahir, 2003; Bhattacharyya and Soejoeti, 1989). Now, the present work will concentrate on PS under PALTs using the IWD. The idea of modeling PALTs with PS is a new one. In this article, the maximum likelihood (ML) estimators of the model parameters are obtained and their statistical properties such as existence, uniqueness and invariance are explored.

The rest of this article is organized as follows: Section 2 presents the test procedure, its assumptions and the used model. In Section 3, the ML equations under progressive stress PALTs (PSPALTs) are outlined to get the ML estimates of the model parameters. Section 4 discusses some statistical properties of the estimators. In Section 5, some simulation results about the performance of the ML estimators are provided. Section 6 concludes the article and presents some points for further research.

2. Test assumptions and model

2.1. Basic assumptions and test procedure

The following assumptions are used throughout the paper in the framework of PSPALTs:

For design stress, the lifetime distribution is assumed to be IWD.

Progressive stress is directly proportional to time (the stress is a linearly increasing function of time).

The testing is Type-I censored sample testing.

The rate parameter of IWD (inverse of the scale parameter) and stress are related as $\lambda = aS^b$. The stress can be as $S(t) = Kt$, where $a > 0$, $b > 0$, and $K > 0$. Progressive stress during stress level i ($i = 1, 2$) is expressed by $S_i(t) = K_i t$, with K_i pre-assigned positive constants, $K_1 < K_2$. That is, simple progressive stress test is assumed, ie, we have only two levels of stress which are design and high.

The test procedure is as follows. Let n units be tested under the progressive stress $S_i(t) = K_i t$, $i = 1, 2$ for a pre-assigned censoring time η. The n test units are tested under a linearly increasing stress condition. Each of the n test units is first run under use condition with stress rate K_1. If it does not fail by a pre-specified timeτ, it is run under accelerated condition with stress rate K_2 until it fails or it is censored.

2.2. The inverse Weibull distribution as a failure time model

The two-parameter IWD is considered in this article. It can be used to model a variety of failure characteristics; early failure, useful life and wear-out periods. It has a main role in numerous applications to describe and illustrate the degradation incidents of mechanical components. More details about this distribution were presented by (Nelson, 1990; Drapella, 1993; Jiang et al., 2001).

The probability density function (pdf) of a two-parameter IWD is given by:

$$f_T(t;\alpha,\theta) = \alpha\ \theta\ t^{-(\alpha+1)}\exp\{-\theta\ t^{-\alpha}\} \quad ;\ t > 0, \alpha > 0,\ \theta > 0, \tag{1}$$

The reliability function takes the form.

$$R(t) = 1 - \exp\{-\theta\ t^{-\alpha}\}, \tag{2}$$

and the corresponding failure rate function is given by:

$$h(t) = \frac{\alpha\ \theta\ t^{-(\alpha+1)}}{\exp\{\theta\ t^{-\alpha}\} - 1}. \tag{3}$$

Therefore, the cumulative distribution function (cdf) is given by

$$F(t) = \exp\{-\ (1/\ \lambda)t^{-\alpha}\}\ ,\ t > 0 \tag{4}$$

where λ is the rate parameter of IWD (inverse of the scale parameter).

Then, according to Yin and Sheng [10], the cdf with design stress rate k_1 based on assumption 4 is given by.

$$F(t) = \exp\{-\ [(1/a)k_1^{-b}t^{-(b+\alpha)}]\},\ t > 0 \tag{5}$$

The CDF, in (5), under linear PS is the Weibull distribution with new rate and shape parameters:

$$\breve{\lambda} = (1/a)k_1^{-b}\ and\ \breve{\alpha} = (b+\alpha) \tag{6}$$

3. Parameters estimation

This section considers the process of obtaining the ML estimates of the model parameters under PSPALTs using Type-I censored data. According to (DeGroot and Goel, 1979), the lifetime of a unit under step-stress PALTs (SSPALTs) can be written as.

$$Y = \begin{cases} T \text{ if } T \leqslant \tau \\ \tau + \beta^{-1}(T - \tau) \text{ if } T > \tau\ , \end{cases} \tag{7}$$

where T is the lifetime of the unit under use condition, τ is the stress change time and β is the acceleration factor; $\beta > 1$. This model is called the *tampered random variable* (TRV) model.

As mentioned by (Yin and Sheng, 1987), progressive stress is similar to step stress, but the stress on specimens is a progressive (continuous) function. In addition, they said that constant stress and step stress are particular cases of progressive stress.

According to (Yin and Sheng, 1987), for any progressive stress $s(t)$, there is a step stress;

$\tilde{s}(t) = s(\tau_i)\tau_{i-1} \leqslant t < \tau_i,\ \tau_0 = 0,\ i = 1,\ 2,\ \ldots,\ q$;where $\tau_i,\ i = 1,\ 2,\ \ldots,\ q$ are points in the domain of definition of $s(t)$ and represent the times at which the stress goes from S_i to S_{i+1}.

$\tilde{s}(t)$ is an approximation of $s(t)$. So, $s(t) = \lim_{\varepsilon\to 0} \tilde{s}(t)$ where $\varepsilon \equiv \max_i (t_i - \tau_i)$. That is, the maximum time difference between the needed time, t_i, to increase the stress and the time τ_i at which the stress goes from S_i to S_{i+1} will be very small or tends to zero.

Since the step stress $\tilde{s}(t)$ becomes progressive stress $s(t)$ when $\varepsilon \to 0$, the CDF,$\tilde{F}(t)$, under step stress converges to the CDF,$F(t)$, under progressive stress when $\varepsilon \to 0$. That is,$F(t) = \lim_{\varepsilon\to 0} \tilde{F}(t)$.

Progressive stress during stress level i $(i = 1, 2)$ is expressed by $S_i(t) = K_i t$, with K_i pre-assigned positive constants, $K_1 < K_2$. That is, based on the relationship $S_i(t) = K_i t$, $K_1 < K_2$, the units under use condition have stress rate (K_1) different from that (K_2) of the units under accelerated condition. Therefore, the pdf of Y under PSPALTs model can be given by.

$$f_Y(t) = \begin{cases} 0, & t \leqslant 0, \\ f_1(t) = \alpha\ (1/a)k_1^{-b}t^{-(b+\alpha+1)}\exp\{-\ (1/a)k_1^{-b}t^{-(b+\alpha)}\}, & 0 < t \leqslant \tau \\ f_2(t) = \alpha\ \beta\ (1/a)\ k_2^{-b}(\beta(t-\tau)+\tau)^{-\ (b+\alpha+1)}\exp\{-\ (1/a)k_2^{-b}(\beta(t-\tau)+\tau)^{-(b+\alpha)}\}, & t > \tau, \end{cases} \tag{8}$$

where $f_1(t)$ is the pdf under use condition and $f_2(t)$, obtained by the transformation-variable technique using $f_1(t)$ and the model given in (7), is the pdf under accelerated condition.

The observed values of the total lifetime under PSPALTs are given by:

$$t_{(1)} \leqslant \ldots\ \leqslant\ t_{(n_u)} \leqslant\ \tau\ \leqslant t_{(n_u+1)} \leqslant\ \ldots\ \leqslant\ t_{(n_u+n_a)} \leqslant \eta$$

where n_u and n_a denote the number of items failed at use and accelerated conditions, respectively, and $t_{(i)}$ is the order statistic realization of t_i based on *i.i.d* random variables. Let δ_{1i} and δ_{2i} be two indicator functions such that $\delta_{1i} \equiv I\ (t_i \leq \tau)$ and $\delta_{2i} \equiv I\ (\tau < t_i \leq \eta)$, $i = 1, \ldots, n$.

Since the total lifetimes $Y_1, \ldots, Y_n$ of n units are *i.i.d* r.v.'s, then the total likelihood function for them is given by:

$$\begin{aligned} L(a,b,\alpha,\beta) \propto & \prod_{i=1}^{n}\left[\alpha\ (1/a)k_1^{-b}t^{-(b+\alpha+1)}\exp\{-\ (1/a)k_1^{-b}t^{-(b+\alpha)}\}\right]^{\delta_{1i}} \\ & .\left[\alpha\ \beta\ (1/a)\ k_2^{-b}(\beta(t-\tau)+\tau)^{-\ (b+\alpha+1)}\exp\{-\ (1/a)k_2^{-b}(\beta(t-\tau)+\tau)^{-(b+\alpha)}\}\right]^{\delta_{2i}} \\ & .\left[1-\exp\{-\ (1/a)k_2^{-b}(\beta(\eta-\tau)+\tau)^{-(b+\alpha)}\}\right]^{\bar{\delta}_{1i}\bar{\delta}_{2i}} \end{aligned} \tag{9}$$

where $\bar{\delta}_{1i} = 1 - \delta_{1i}$ and $\bar{\delta}_{2i} = 1 - \delta_{2i}$.

The natural logarithm of the above likelihood function is given by.

$$\begin{aligned} ln\ L \propto\ & (n_u+n_a)ln\ \alpha -\ (n_u+n_a)ln\ a + n_a ln\ \beta - (n_u\ ln\ k_1 + n_a\ ln\ k_2)b \\ & -(b+\alpha+1)[\sum_{i=1}^{n_u} ln\ t_i\ + \sum_{i=1}^{n_a} ln\ (\beta(t_i - \tau)+\tau)] \\ & -(1/a)\ [k_1^{-\,b}\sum_{i=1}^{n_u} t_i^{-\ (b+\alpha)}\ + k_2^{-\,b}\sum_{i=1}^{n_a}\ (\beta(t_i - \tau)+\tau)^{-\ (b+\alpha)}] \\ & +(1/a)\ k_2^{-b}\ n_c(\beta(\eta - \tau)+\tau)^{-\ (b+\alpha)} \end{aligned} \tag{10}$$

The MLEs of ***a***, ***b***, α and β can be obtained by solving the following likelihood equations:

$$\begin{aligned} \frac{\partial \ln L}{\partial a} = & -\frac{(n_u+n_a)}{a} +\ (1/a^2)\ \left[k_1^{-\,b}\sum_{i=1}^{n_u} t_i^{-(b+\alpha)}\ + k_2^{-\,b}\sum_{i=1}^{n_a}\ \psi_i^{-(b+\alpha)}\right. \\ & \left. -\ n_c\ k_2^{-\,b}\psi_\eta^{-(b+\alpha)}\right] = 0, \end{aligned} \tag{11}$$

Where $\psi_i = \beta(t_i - \tau) + \tau$ and $\psi_\eta = \beta(\eta - \tau) + \tau$.

$$\begin{aligned} \frac{\partial \ln L}{\partial b} = &\ (n_u\ ln\ k_1 + n_a\ ln\ k_2)\ -\ [\sum_{i=1}^{n_u} ln\ t_i + \sum_{i=1}^{n_a}\ ln\ \psi_i] \\ & +(1/a)\{\ (k_1^{-\,b}(ln\ k_1)\sum_{i=1}^{n_u} t_i^{-(b+\alpha)}\ + k_2^{-\,b}(ln\ k_2)\sum_{i=1}^{n_a}\ \psi_i^{-(b+\alpha)})\ + n_c k_2^{-\,b}\psi_\eta^{-(b+\alpha)}\ ln\ k_2\} = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{\partial \ln L}{\partial \alpha} = &\ (n_u+n_a)/\alpha - [\sum_{i=1}^{n_u} ln\ t_i \\ & + \sum_{i=1}^{n_a}\ ln\ \psi_i] + (1/a)[k_1^{-\,b}\sum_{i=1}^{n_u} t_i^{-\ (b+\alpha)} ln\ t_i \\ & + k_2^{-\,b}\sum_{i=1}^{n_a}\ \psi_i^{-\ (b+\alpha)} ln\ \psi_i]\ -(1/a)n_c k_2^{-\,b}\psi_\eta^{-\ (b+\alpha)} ln\ \psi_\eta = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{\partial \ln L}{\partial \beta} = &\ \frac{n_a}{\beta} - (b+\alpha+1)\sum_{i=1}^{n_a}\ \frac{(t_i - \tau)}{\psi_i} \\ & + (1/\mathrm{a})\ (b+\alpha)[k_2^{-\,b}\sum_{i=1}^{n_a}(t_i - \tau)\psi_i^{-(b+\alpha)-1} \\ & + n_c k_2^{-\,b}(\eta - \tau)\psi_\eta^{-(b+\alpha)-1}] = 0. \end{aligned} \tag{14}$$

From (11), the MLE of ***a*** can be obtained as.

$$\hat{a} = \frac{k_1^{-\hat{b}} \sum_{i=1}^{n_u} t_i^{-(\hat{b}+\hat{\alpha})} + k_2^{-\hat{b}} \sum_{i=1}^{n_a} \psi_i^{-(\hat{b}+\hat{\alpha})} - n_c\, k_2^{-\hat{b}} \psi_\eta^{-(\hat{b}+\hat{\alpha})}}{n_u + n_a} \quad (15)$$

When α is known, one can eliminate (13) and can put the known values of $\boldsymbol{a}$ and α in (12) and (14). Then, solve for $\boldsymbol{b}$ and β.

4. Properties of estimators when $\alpha = 1$

The ML estimators of the model parameters for $\alpha = 1$ have the following properties.

4.1. $\hat{b}$ and K_i, i = 1, 2 are related by test data

From the maximum likelihood equations, we have.

$$(n_u + n_a) + \left(\sum_{i=1}^{n_u} \ln t_i + \sum_{i=1}^{n_a} \ln \psi_i\right) - \frac{\left(\sum_{i=1}^{n_u} t_i^{(\hat{b}+1)} + \sum_{i=1}^{n_a} \psi_i^{(\hat{b}+1)}\right)}{(n_u + n_a)\left[\sum_{i=1}^{n_u} t_i^{(\hat{b}+1)} \ln t_i + \sum_{i=1}^{n_a} \psi_i^{(\hat{b}+1)} \ln \psi_i\right]} = 0 \quad (16)$$

where $\psi_i = \beta(t_i - \tau) + \tau$

$\hat{b}$ can be found without K_i, *i* = *1, 2*. That is, $\hat{b}$ is indirectly related to K_i, *i* = *1, 2* by test data.

4.2. Uniqueness of $\hat{b}$

Let.

$$f \equiv \sum_{i=1}^{n_u} \ln t_i + \sum_{i=1}^{n_a} \ln \psi_i \quad (17)$$

and

$$g(x) \equiv \frac{\left(\sum_{i=1}^{n_u} t_i^x + \sum_{i=1}^{n_a} \psi_i^x\right)}{(n_u + n_a)\left[\sum_{i=1}^{n_u} t_i^x \ln t_i + \sum_{i=1}^{n_a} \psi_i^x \ln \psi_i\right]} - \frac{(n_u + n_a)}{x} \quad (18)$$

then,

$$\frac{dg(x)}{dx} = \frac{\left[\sum_{i=1}^{n_u} t_i^x \ln t_i + \sum_{i=1}^{n_a} \psi_i^x \ln \psi_i\right] (n_u+n_a)\left[\sum_{i=1}^{n_u} t_i^x \ln t_i + \sum_{i=1}^{n_a} \psi_i^x \ln \psi_i)\right]}{(n_u+n_a)^2 \left[\sum_{i=1}^{n_u} t_i^x \ln t_i + \sum_{i=1}^{n_a} \psi_i^x \ln \psi_i)\right]^2}$$
$$- \frac{\left(\sum_{i=1}^{n_u} t_i^x + \sum_{i=1}^{n_a} \psi_i^x\right)(n_u+n_a)\left[\sum_{i=1}^{n_u} t_i^x (\ln t_i)^2 + \sum_{i=1}^{n_a} \psi_i^x (\ln \psi_i)^2\right]}{(n_u+n_a)^2 \left[\left[\sum_{i=1}^{n_u} t_i^x \ln t_i + \sum_{i=1}^{n_a} \psi_i^x \ln \psi_i)\right]^2} + \frac{(n_u+n_a)}{x^2}$$
$$= \frac{\left[\sum_{i=1}^{n_u} t_i^x \ln t_i + \sum_{i=1}^{n_a} \psi_i^x \ln \psi_i\right]}{(n_u+n_a)\left[\sum_{i=1}^{n_u} t_i^x \ln t_i + \sum_{i=1}^{n_a} \psi_i^x \ln \psi_i)\right]}$$
$$- \frac{\left(\sum_{i=1}^{n_u} t_i^x + \sum_{i=1}^{n_a} \psi_i^x\right)\left[\sum_{i=1}^{n_u} t_i^x (\ln t_i)^2 + \sum_{i=1}^{n_a} \psi_i^x (\ln \psi_i)^2\right]}{(n_u+n_a)\left[\sum_{i=1}^{n_u} t_i^x \ln t_i + \sum_{i=1}^{n_a} \psi_i^x \ln \psi_i)\right]^2} + \frac{(n_u+n_a)}{x^2}$$

Since $t_i > 0$ $(i = 1, 2, \ldots, n_u + n_a)$, $\frac{dg(x)}{dx} > 0$.

That is, $g(x)$ is a strictly monotone increasing function. In addition, it is noted that $\hat{b} > 0$ and if $\hat{b}$ is the solution of (16), then $g(\hat{b} + 1) = f$. Therefore, with the strict monotonicity and continuity of $g(x)$ in $[1,\infty)$, we conclude that $\hat{b}$ determined by (16) is unique. Similarly, we can conclude that $\hat{\beta}$ is also unique.

4.3. Invariance of $\hat{b}$

Let $\omega_i = \frac{t_i}{\zeta}$, where ζ is an arbitrary constant. Then, substituting $\zeta\omega_i$ for t_i into (16), we can obtain.

$$(n_u + n_a) + \left(\sum_{i=1}^{n_u} \ln \omega_i + \sum_{i=1}^{n_a} \ln \phi_i\right) + (n_u + n_a)\ln \zeta$$
$$- \frac{\left(\sum_{i=1}^{n_u} \omega_i^{(\hat{b}+1)} \zeta^{(\hat{b}+1)} + \sum_{i=1}^{n_a} \phi_i^{(\hat{b}+1)} \zeta^{(\hat{b}+1)}\right)}{(n_u+n_a)\left[\sum_{i=1}^{n_u} \omega_i^{(\hat{b}+1)} \zeta^{(\hat{b}+1)} (\ln \omega_i + \ln \zeta) + \sum_{i=1}^{n_a} \phi_i^{(\hat{b}+1)} \zeta^{(\hat{b}+1)} (\ln \phi_i + \ln \zeta)\right]} = 0,$$

where $\phi_i = \beta(\omega_i - \tau) + \tau$.

That is,

$$(n_u + n_a) + \left(\sum_{i=1}^{n_u} \ln \omega_i + \sum_{i=1}^{n_a} \ln \phi_i\right) - \frac{\left(\sum_{i=1}^{n_u} \omega_i^{(\hat{b}+1)} + \sum_{i=1}^{n_a} \phi_i^{(\hat{b}+1)}\right)}{(n_u + n_a)\left[\sum_{i=1}^{n_u} \omega_i^{(\hat{b}+1)} \ln \omega_i + \sum_{i=1}^{n_a} \phi_i^{(\hat{b}+1)} \ln \phi_i\right]} = 0 \quad (19)$$

It is clear that Eqs. (16) and (19) have the same form. Eq. (19) does not include ζ. This means that if the test data are divided (or multiplied) by an arbitrary constant, $\hat{b}$ does not change.

4.4. Existence $\hat{b}$

By definition of $g(x)$.

$$g(1) - f = \frac{\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a} \psi_i}{(n_u + n_a)\left[\sum_{i=1}^{n_u} t_i \ln t_i + \sum_{i=1}^{n_a} \psi_i \ln \psi_i\right]} - (n_u + n_a) - \sum_{j=1}^{n_u} \ln t_j - \sum_{j=1}^{n_a} \ln \psi_j$$
$$= \{[\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a} \psi_i]/\{(n_u + n_a)[\sum_{i=1}^{n_u}\sum_{j=1}^{n_u} (\ln t_i - \ln t_j).\ t_i + \sum_{i=1}^{n_a}\sum_{j=1}^{n_a} (\ln \psi_i - \ln \psi_j).\ \psi_i]\}\} - (n_u + n_a)$$
$$= \{[\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a} \psi_i]/\{[\sum_{i=1}^{n_u}\sum_{j=1}^{n_u} (\ln t_i - \ln t_j).\ [t_i - t_j] + \sum_{i=1}^{n_a}\sum_{j=1}^{n_a} (\ln \psi_i - \ln \psi_j).\ [\psi_i - \psi_j]]\}\} - (n_u + n_a)\ ,\ i < j$$

If

$$\{[\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a} \psi_i]/\{[\sum_{i=1}^{n_u}\sum_{j=1}^{n_u} (\ln t_i - \ln t_j).\ [t_i - t_j] + \sum_{i=1}^{n_a} \times \sum_{j=1}^{n_a} (\ln \psi_i - \ln \psi_j).\ [\psi_i - \psi_j]]\}\} \geqslant (n_u + n_a),\ i < j \quad (20)$$

then

$g(1) \geqslant f$.

When $g(1) \geqslant f$, it is certain that $g(x) > f$, (x<1) according to the strict monotonically and continuity of $g(x)$ in [1, +∞), ie, Eq. (16) can't provide solution $\hat{b}$ < 0 when Eq. (20) is true. Inversely, if.

$$\{[\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a} \psi_i]/\{[\sum_{i=1}^{n_u}\sum_{j=1}^{n_u} (\ln t_i - \ln t_j).\ [t_i - t_j] + \sum_{i=1}^{n_a}\sum_{j=1}^{n_a} (\ln \psi_i - \ln \psi_j).\ [\psi_i - \psi_j]]\}\} < (n_u + n_a),\ i < j$$

then

$g(1) < f$

According to the continuity of $g(x)$ on [1+ξ, γ), where ξ is a small constant and γ is a large constant such that $g(1 + \xi) > f$ and $g(\gamma) < f$, there exists an $x^* \in$ [1+ξ, γ], which satisfies $g(x^*) = f$ and

$\hat{b}$ = $x^* - 1$ is the solution of (16).

Thus the necessary and sufficient condition for (19) providing solution $\hat{b}$ < 0 is:

$\{[\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a} \psi_i]/\{[\sum_{i=1}^{n_u}\sum_{j=1}^{n_u} (\ln t_i - \ln t_j).\ [t_i - t_j] + \sum_{i=1}^{n_a}\sum_{j=1}^{n_a} (\ln \psi_i - \ln \psi_j).\ [\psi_i - \psi_j]]\}\} < (n_u + n_a),\ i < j.$

5. Numerical results

In this section simulation studies are considered to assess the performance of the MLEs in terms of their mean squared errors (MSEs) for different choices of α, β, a, b,τ, K_1, K_2 and η values.

The simulation study is implemented according to the following algorithm:

(1) Specify the values of n,τ, K_1, K_2 and η.
(2) Specify the values of the parametersα, β, a and b.
(3) Generate a random sample of size n from the random variable Y given by (7) and sort it.
(4) Use the model given by (8) to generate Type-I censored data for a given n,τ, K_1, K_2, η, α, β, a and b.
(5) Use the Type-I censored data to compute the MLEs of the model parameters. Newton-Raphson method is applied for solving the nonlinear equations given by (12) - (14) to obtain the MLEs of the parameters b, α and β. Then one can easily obtain the value of a using (15). Accordingly, λ is determined from the relation $\lambda = aS^b$ to obtain the estimated value of the parameter θ, where $\theta = 1/\lambda$.
(6) Replicate the steps 3–5 10,000 times.
(7) Compute the average values of MSEs associated with the MLEs of the parameters.
(8) Steps 1–7 are done with different values of n,τ, K_1, K_2, η, α, β, a and b.
(9) Conducting the above algorithm, the average values of MSEs are obtained using 10,000 replications to avoid randomness. The results are reported in Tables 1 and 2 based on different values of n,τ, K_1, K_2, η, α, β, a and b to investigate the performance of the MLEs of the model parameters.

From Tables 1 and 2 the following notes can be detected.

1) For fixed τ and η, the MSEs decrease as n increases.
2) For fixed τ and n, the MSEs decrease as η increases.
3) For fixed η and n, the MSEs are to be larger when τ is getting to be large and $\alpha > 1$.
4) For fixed η and n, the MSEs are to be slightly larger when τ is getting to be large and $\alpha < 1$.
5) It is also observed that the MLEs of the model parameters are very close to the true values as n increases.

Accordingly, under the proposed model developed in this paper, we have obtained good estimates of the model parameters.

Table 1
Average values of the MLEs and MSEs, when a, b, α and β set at 1.5, 1.2, 1.7 and 2.4, respectively, with $K_1 = 2$ and $K_2 = 5$.

n Parameters / (τ, η)	(7, 9)	(6, 9)	(7, 12)
30 a	1.3716 0.2718	1.3972 0.2311	1.4051 0.2149
b	0.9013 0.2411	0.9782 0.2162	0.9274 0.1942
α	1.1327 0.2130	1.1756 0.1822	1.1547 0.1527
β	2.1711 0.2619	2.2189 0.2216	2.1135 0.1677
50 a	1.4389 0.2513	1.4416 0.2063	1.4521 0.1756
b	1.1245 0.2177	1.1378 0.1709	0.9733 0.1591
α	1.4944 0.1658	1.5346 0.1281	1.2378 0.1105
β	2.2382 0.2475	2.2755 0.1944	2.2385 0.1224
75 a	1.4766 0.1492	1.4805 0.1243	1.4865 0.1033
b	1.1791 0.1083	1.1861 0.0854	1.1508 0.0743
α	1.6727 0.0790	1.6829 0.0521	1.6452 0.0456
β	2.2855 0.1304	2.3420 0.1265	2.3454 0.0688
100 a	1.4968 0.0561	1.5043 0.0375	1.4988 0.0267
b	1.2013 0.0473	1.1991 0.0351	1.1896 0.0312
α	1.7124 0.0418	1.7083 0.0294	1.6985 0.0216
β	2.3879 0.0527	2.3976 0.0411	2.4105 0.0307

Table 2
Average values of the MLEs and MSEs, when a, b, α and β set at 1.5, 1.2, 0.5 and 2.4, respectively, with $K_1 = 2$ and $K_2 = 5$.

n Parameters / (τ, η)	(7, 9)	(6, 9)	(7, 12)
30 a	1.3562 0.2654	1.3852 0.2270	1.4183 0.2106
b	0.9347 0.2316	0.9722 0.2192	1.1158 0.1962
α	0.3722 0.1985	0.4365 0.1803	0.4413 0.1578
β	2.1853 0.2514	2.1053 0.2261	2.1955 0.2065
50 a	1.4133 0.2279	1.4106 0.1967	1.4460 0.1689
b	1.1510 0.2034	1.1741 0.1851	1.1868 0.1521
α	0.4381 0.1843	0.4582 0.1654	0.4672 0.1253
β	2.2792 0.2182	2.1846 0.1935	2.2813 0.1676
75 a	1.4691 0.1292	1.4490 0.0820	1.4732 0.0755
b	1.1835 0.0974	1.1934 0.0811	1.1975 0.0685
α	0.4805 0.0720	0.4855 0.0694	0.4956 0.0531
β	2.3469 0.1146	2.2768 0.0817	2.3694 0.0743
100 a	1.4876 0.0764	1.4985 0.0638	1.5003 0.0522
b	1.2106 0.0655	1.2052 0.0562	1.2014 0.0472
α	0.5041 0.0485	0.5011 0.0381	0.5007 0.0324
β	2.3954 0.0685	2.3956 0.0584	0.0513

6. Conclusion

In this article we have considered PALTs under PS when the PS is directly proportional to time. That is, the stress is a linearly increasing function with time. PSPALTs can substantially shorten the duration of the test without affecting the accuracy of lifetime distribution estimates. That is, The PS test pattern is more effective in time and money compared with constant- or step-stress. For highly reliable products, PSPALTs have been proposed to obtain timely information of the products whose lifetime follows inverse Weibull distribution. We have established some statistical properties of the MLEs of the model parameters such as existence, uniqueness and invariance under PSPALTs. For further studies, the progressive stress testing wherein the stress on every unit is increased continuously in a non-linear pattern with time will be considered.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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