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ORIGINAL ARTICLE

Exact travelling wave solutions for some nonlinear partial differential equations

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Abstract In this work, we construct explicit by the travelling wave solutions involving parameters of the Boussinesq and Benjamin–Ono equations by using a new approach, namely the $\left(\frac{G'}{G}\right)$-expansion method. The travelling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions.

1. Introduction

During the past four decades or so, some efficient and powerful methods have been developed by a diverse group of scientists to find the exact analytic solutions of physically important nonlinear evolution equations. For example, Hirota's bilinear method (Hirota, 2004), inverse scattering method (Ablowitz and Segur, 1981), the tanh method (Fan, 2000; Malfliet, 1992; Parkes and Duffy, 1996; Wang and Li, 2005; Chow, 1995), Backlund transformation (Miura, 1973), symmetry method (Bluman and Kumei, 1989), the sinecosine function method (Yan, 1996), the exp-function method (He and Wu, 2006; Zi and Aslan, 2008) and so on. All the methods mentioned above have some limitations in their applications and a majority of the well-known methods involve tedious computation if it is performed by hand.

The objective of this paper is to use a new method which is called the (G'/G)-expansion method (Bekir, 2008; Wang et al., 2008; Zhang et al., 2008). The main idea of this method is that the travelling wave solutions of non-linear equations can be expressed by a polynomial in (G'/G) where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G'' + \lambda G' + \mu G = 0$, where $\xi = x - vt$. The rest of the Letter is organized as follows. In Section 2, we describe briefly the (G'/G)-expansion method. In Sections 3 and 4, we apply the method to Boussinesq and Benjamin–Ono Equations. In section 5 some conclusions are given.

2. Description of The $\frac{G'}{G}$-expansion method

Suppose that a nonlinear equation, say in two independent variables x and t, is given by

$$P(u, u_x, u_t, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0, \quad (1)$$

where $u = u(x, t)$ is an unknown function, P is a polynomial in $u = u(x, t)$ and its various partial derivatives, in which the

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highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the $\frac{G'}{G}$-expansion method.

Step 1: Combining the independent variables x and t into one variable $\xi = x - vt$, we suppose that

$$u(x,t) = u(\xi), \quad \xi = x - vt. \tag{2}$$

The travelling wave variable (2) permits us to reduce Eq. (1) to an ODE for $u = u(\xi)$, namely

$$P(u, -vu', u', v^2u'', -vu'', u'', \dots) = 0. \tag{3}$$

Step 2: Suppose that the solution of ODE (3) can be expressed by a polynomial in $\frac{G'}{G}$ as follows

$$u(\xi) = \alpha_m \left(\frac{G'}{G}\right)^m + \dots, \tag{4}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{5}$$

$\alpha_m, \dots, \lambda$ and μ are constants to be determined later, $\alpha_m \neq 0$, the unwritten part in (4) is also a polynomial in $\frac{G'}{G}$, but the degree of which is generally equal to or less than $m-1$, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in ODE (3).

Step 3: By substituting (4) into Eq. (3) and using the second order linear ODE (5), collecting all terms with the same order of $\frac{G'}{G}$ together, the left-hand side of Eq. (3) is converted into another polynomial in $\frac{G'}{G}$. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $\alpha_m, \dots, \lambda$ and μ.

Step 4: Assuming that the constants $\alpha_m, \dots, \lambda$ and μ can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order LODE (5) have been well known for us, then substituting $\alpha_m, \dots, v$ and the general solutions of Eq. (5) into (4) we have more travelling wave solutions of the nonlinear evolution Eq. (1).

3. Boussinesq equation

We now consider the Boussinesq equation in the form

$$u_{tt} - u_{xx} - u_{xxxx} - 3(u^2)_{xx} = 0. \tag{6}$$

In what follows, we study the travelling wave solutions to Eq. (6). Substituting $u = u(\xi), \xi = x - vt$ into Eq. (6) and integrating twice, we have

$$(v^2 - 1)u - u'' - 3u^2 + c = 0, \tag{7}$$

where c is the integration constant, and the first integrating constant is taken to zero. Suppose that the solutions of the O.D.E. (7) can be expressed by a polynomial in $\frac{G'}{G}$ as follows:

$$u(\xi) = \alpha_m \left(\frac{G'}{G}\right)^m + \cdots, \tag{8}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0. \tag{9}$$

By using (8) and (9) and considering the homogeneous balance between u'' and u^2 in Eq. (7) we required that $2m = m + 2$ then $m = 2$. So we can write (8) as

$$u(\xi) = \alpha_2 \left(\frac{G'}{G}\right) + \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0. \tag{10}$$

So by using (9) and (10) it is derived that

$$\begin{aligned} u'' &= 6\alpha_2 \left(\frac{G'}{G}\right)^4 + (2\alpha_1 + 10\alpha_2\lambda)\left(\frac{G'}{G}\right)^3 \\ &\quad + (8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)\left(\frac{G'}{G}\right)^2 \\ &\quad + (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)\left(\frac{G'}{G}\right) + 2\alpha_2\mu^2 + \alpha_1\lambda\mu. \end{aligned} \tag{11}$$

On substituting (10)-(11) into (7), collecting all terms with the same powers of $\frac{G'}{G}$ and setting each coefficient to zero, we obtain the following system of algebraic equations:

$$\begin{aligned} &c + (v^2-1)\alpha_0 - 3\alpha_0^2 - (2\alpha_2\mu^2 + \alpha_1\lambda\mu) = 0 \\ &(v^2-1)\alpha_1 - 6\alpha_0\alpha_1 - (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2) = 0 \\ &(v^2-1)\alpha_2 - 3(\alpha_1^2 + 2\alpha_2\alpha_0) - (8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2) = 0 \\ &-6\alpha_1\alpha_2 - 2\alpha_1 - 10\alpha_2\lambda = 0 \\ &-3\alpha_2^2 - 6\alpha_2 = 0. \end{aligned} \tag{12}$$

On solving the algebraic equations above yield

$$\begin{aligned} &\alpha_2 = -2, \quad \alpha_1 = 2\lambda \\ &v = \pm\sqrt{1 - 5\lambda^2 + 8\mu + 6\alpha_0} \\ &c = -3\alpha_0^2 + 2\lambda^2\mu - 4\mu^2 - 8\alpha_0\mu + 5\alpha_0\lambda^2, \end{aligned} \tag{13}$$

λ, μ and α_0 are arbitrary constants.

By using (13), expression (10) can be written as

$$u(\xi) = -2\left(\frac{G'}{G}\right)^2 + 2\lambda\left(\frac{G'}{G}\right) + \alpha_0, \tag{14}$$

where $\xi = x \mp \sqrt{1 - 5\lambda^2 + 8\mu + 6\alpha_0}t$.

On solving Eq. (9), we deduce after some reduction that

$$\frac{G'}{G} = \frac{1}{2}\sqrt{\lambda^2 - 4\mu} \times \left(\frac{C_1 sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right) - \frac{\lambda}{2},$$

where C_1 and C_2 are arbitrary constants. Substituting the general solutions of Eq. (9) into (10) we have three types of travelling wave solutions of the Boussinesq equation (6) as follows:

Case 1: When $\lambda^2 - 4\mu > 0$

$$u(\xi) = -2(\lambda^2 - 4\mu) \times \left(\frac{C_1 sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2 + \frac{3\lambda}{2} + \alpha_0,$$

where $\xi = x \mp \sqrt{1 - 5\lambda^2 + 8\mu + 6\alpha_0}t$, C_1 and C_2 are arbitrary constants.If C_1 and C_2 are taken as special values, the various known results in the literature can be rediscovered, for instance, if $C_1 > 0$, $C_1^2 > C_2^2$, then $u = u(\xi)$ can be written as

$$u(\xi) = -12\epsilon^2 s^2(\lambda^2 - 4\mu) \times sech^2\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + \xi_0\right) + \frac{3\lambda}{2} + \alpha_0.$$

Case 2: When $\lambda^2 - 4\mu < 0$

$$u(\xi) = -2(4\mu - \lambda^2) \times \left(\frac{-C_1 sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right)^2 + \frac{3\lambda}{2} + \alpha_0.$$

Case 3: When $\lambda^2 - 4\mu = 0$

$$u(\xi) = \frac{-2C_2^2}{(C_1 + C_2\xi)^2} + \frac{3\lambda}{2} + \alpha_0 \tag{15}$$

4. Benjamin–Ono equation

In this section we consider the Benjamin–Ono Equation in the form

$$u_t + hu_{xx} + uu_x = 0. \tag{16}$$

And look for the travelling wave solution of Eq. (16) in the form

$$u(x,t) = u(\xi), \quad \xi = x - vt, \tag{17}$$

where the speed v of the travelling waves is to be determined later.

By using the travelling wave variable (17), Eq. (16) is converted into an O.D.E. for $u = u(\xi)$

$$-vu' + hu'' + \frac{1}{2}(u^2)' = 0.$$

Integrating it with respect to ξ once yields

$$c - vu + hu' + \frac{1}{2}u^2 = 0, \tag{18}$$

where c is the integration constant. Suppose that the solutions of the O.D.E. (18) can be expressed by a polynomial in $\frac{G'}{G}$ as follows:

$$u(\xi) = \alpha_m\left(\frac{G'}{G}\right)^m + \ldots, \tag{19}$$

where $G = G(\xi)$ satisfies the second order LODE in the form.

$$G'' + \lambda G' + \mu G = 0. \tag{20}$$

By using (19) and (20) and considering the homogeneous balance between u'' and u^2 in Eq. (18) we required that $2m = m + 1$ then $m = 1$. So we can write (19) as

$$u(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0. \tag{21}$$

On substituting (20)–(21) into (18), collecting all terms with the same powers of $\frac{G'}{G}$ and setting each coefficient to zero, we obtain the system of algebraic equations. And by solving this algebraic equations we obtain

$$\alpha_1 = 2h \tag{22}$$

$$v = h\lambda - \alpha_0, c = 2h^2\mu - h\lambda\alpha_0 - \frac{3}{2}\alpha_0^2. \tag{23}$$

By using (22), expression (21) can be written as

$$u(\xi) = 2h\left(\frac{G'}{G}\right) + \alpha_0, \tag{24}$$

where $\xi = x - (h\lambda - \alpha_0)t$. Eq. (24) is the formula of a solution of Eq. (18), provided that the integration constant c in Eq. (18) is taken as that in (23). Substituting the general solutions of Eq. (9) into (21) we have three types of travelling wave solutions of the Benjamin–Ono Equation (16) as follows:

$$u(\xi) = 2h\sqrt{\lambda^2 - 4\mu} \times \left(\frac{C_1 sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right) - \frac{\lambda}{2} + \alpha_0,$$

where $x - (h\lambda - \alpha_0)t$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$.

$$u(\xi) = 2h\sqrt{4\mu - \lambda^2} \times \left(\frac{-C_1 sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right) - \frac{\lambda}{2} + \alpha_0.$$

When $\lambda^2 - 4\mu = 0$

$$u(\xi) = \frac{2hC_2}{C_1 + C_2\xi},$$

where C_1 and C_2 are arbitrary constants.

5. Conclusions

In this paper, we have seen the three types of travelling wave solutions in terms of hyperbolic, trigonometric and rational functions for Boussinesq and Benjamin–Ono Equations. These equations are very difficult to be solved by traditional methods. The performance of this method is reliable, simple and gives many new exact solutions. We have noted that the $\frac{G'}{G}$-expansion method changes the given difficult problems into simple problems which can be solved easily.

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