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Approximations for higher order boundary value problems using non-polynomial quadratic spline based on off-step points

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ABSTRACT

Non-polynomial quadratic spline method based on off-step points is used to develop a numerical algorithm for obtaining an approximate solution of higher even order boundary value problems. For the employment of the method, decomposition procedure is used. Higher order boundary value problems are reduced into the corresponding system of second order boundary value problems. Convergence analysis of the method is also discussed. Seven numerical examples are given to illustrate the applicability and efficiency of new method. It is also shown that the new method gives approximations, which are better than those produced by other existing fourth order methods except higher degree splines.

1. Introduction

The higher order boundary value problems have been studied as their mathematical applications are in diversified applied sciences. Conditions for the existence and uniqueness of solutions of the higher-order boundary-value problems was discussed in Agarwal (1986, pp. 89–93). Several numerical algorithms have been developed to determine the approximate solution of higher-order boundary-value problems. In this paper, we consider the higher order boundary value problems of the form

$$y^{(2N)} = f(x, y, y^{(1)}, y^{(2)}, y^{(3)} \dots, y^{(2N-1)}), a \leqslant x \leqslant b, where\ N = 2, 3, 4 \tag{1}$$

subject to the boundary conditions

$$\begin{aligned} &y(a) = \lambda_1, y^{(2)}(a) = \lambda_2, \dots, y^{(2N-2)}(a) = \lambda_N, \\ &y(b) = \delta_1, y^{(2)}(b) = \delta_2, \dots, y^{(2N-2)}(b) = \delta_N. \end{aligned} \tag{2}$$

We rewrite the Eq. (1) and (2) as system of second order boundary value problems:

$$\begin{aligned} &y_i^{(2)}(x) = F_i\Big(x, y_1, y_2, \dots, y_i, \dots, y_N, y_1^{(1)}, y_2^{(1)}, \dots, y_i^{(1)}, \dots, y_N^{(1)}\Big), \\ &i = 1, 2, \dots, N \end{aligned} \tag{3}$$

subject to the modified boundary conditions;

$$y_i(a) = \lambda_i, y_i(b) = \delta_i, i = 1, 2, \dots, N. \tag{4}$$

We assume that, for $a < x < b$, for $m = 1, 2, \dots, N, i = 1, 2, \dots, N$ and $-\infty < y_m < y_m^{(1)} < \infty$:

(i) F_i is continous,
(ii) $\frac{\partial F_i}{\partial y_m}$ and $\frac{\partial F_i}{\partial y_m^{(1)}}$ exist and are continous,
(iii) $\frac{\partial F_i}{\partial y_m} > 0$ and $|\frac{\partial F_i}{\partial y_m}| \leqslant C$, for some constant C.

These conditions from Keller (1968, (pp. 6–8 and 15–16)) ensure the existence and uniqueness of the solution of the above boundary value problem (3) and (4).

Here, our aim is to solve the boundary value problems of fourth, sixth and eighth orders of the form (1) and (2) with $N = 2, 3$ and 4. There are various methods available in literature for solving these boundary value problems. For example, non-polynomial spline technique was used by Siddiqi and Akram (2007) to determine the solution of eighth-order and Akram

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and Siddiqi (2006) to determine the solution of sixth order boundary value problems. Meŝtroviĉ (2007) used modified decomposition method for the solution of eighth-order boundary value problems. Islam et al. (2008) used non-polynomial splines approach to the solution of sixth-order non-linear boundary value problems of the form (1) and (2). Solution of special sixth order boundary-value problems was given by Boutayeb and Twizell (1992). Khan and Khandelwal (2012) used parametric septic splines and Jalilian and Rashidinia (2010) used nonic-spline for the solution of non-linear sixth-order two point boundary value problems.

The above system involves the second order boundary value problems. Polynomial and non-polynomial spline approach (Ramadan et al., 2007), quintic non-polynomial spline method (Srivastava et al., 2011), cubic spline method (Al-Said, 2001), non-polynomial spline (Jha and Mohanty, 2011; Mohanty et al., 2017), spline in tension (Mohanty et al., 2005), spline in compression (Mohanty et al., 2004) were used in previous papers to solve second order boundary value problems. Here, we use lower degree non-polynomial quadratic spline for solving system of second order boundary value problems. This non-polynomial quadratic spline is based on off-step points.

When linear BVPs are implemented over the method, we get a linear system of equations which are solved by using LU decomposition method and when non-linear BVPs are implemented we get a non-linear system which is solved by Block Newton Raphson method. The outline of this paper is organised into five sections. In Section 2, consistency relation and in Section 2.1 truncation error and end equations are discussed. In Section 3, application of the method to solve the sixth order BVPs is given. Section 4 gives the convergence analysis of the method. In Section 5, seven examples are considered to illustrate the accuracy and performance of the method presented in the paper.

2. Non-polynomial quadratic spline function

To develop the new method based on off-step points, we divide the interval $[a, b]$ into $n+1$ subintervals, s.t.

$$a = x_0 < x_{1/2} < x_{3/2} < \cdots < x_{n-1/2} < x_n = b.$$

We introduce a finite set of grid points x_i as

$$x_i = a + (i - 1/2)h, \quad i = 0, 1, \ldots, n \quad \textit{and} \quad h = (b-a)/n.$$

Let

$$P_i(x) = a_i \cos k(x - x_i) + b_i e^{k(x-x_i)} + c_i \tag{5}$$

be a non-polynomial quadratic spline P_i is defined on [a, b] which interpolates the uniform mesh points x_i depends on a parameter k, reduces to an ordinary quadratic spline in [a,b] as $k \longrightarrow 0$ and $k > 0$.

To determine the coefficients a_i, b_i and c_i, we define the following interpolatory conditions as

$$P_i(x_{i+1/2}) = y_{i+1/2}, P_i^{(1)}(x_i) = Q_i, P_i^{(2)}(x_{i+1/2}) = R_{i+1/2}, \ i = 0, 1, \ldots, n \tag{6}$$

By using the conditions (6) we calculated the coefficients as

$$a_i = -\frac{1}{k^2} R_{i+1/2} \sec\left(\frac{\theta}{2}\right) + \frac{1}{k} Q_i e^{\frac{\theta}{2}} \sec\left(\frac{\theta}{2}\right)$$
$$b_i = \frac{1}{k} Q_i$$
$$c_i = y_{i+1/2} + \frac{1}{k^2} R_{i+1/2} - \frac{2}{k} Q_i e^{\frac{\theta}{2}}$$

where, $\theta = kh$.

Using the continuity of first derivative, $P_{i-1}^{(m)}(x_i) = P_i^{(m)}(x_i)$, $m = 0, 1$ the following consistency relation is derived

$$\begin{aligned} &\alpha_1 y_{i-3/2} + \beta_1 y_{i-1/2} + \gamma_1 y_{i+1/2} \\ &= h^2(\alpha_2 R_{i-3/2} + \beta_2 R_{i-1/2} + \gamma_2 R_{i+1/2}), \quad i = 2, 3, \ldots, n-1 \end{aligned} \tag{7}$$

where,

$$\begin{aligned}
\alpha_2 =& \frac{1}{\theta^2}\left[e^{2\theta} + \left[(e^{\frac{\theta}{2}} - e^{\theta}) \sin\theta \sec\frac{\theta}{2} + e^{\frac{3\theta}{2}}(1 + \cos\theta \csc\theta) \right.\right.\\
&\left.\left. - e^{\theta} \cos\theta \sec\frac{\theta}{2}\right] e^{\frac{\theta}{2}} \sin\theta \sec\frac{\theta}{2}\right]\\
\beta_2 =& \frac{1}{\theta^2}\left[-1 + e^{\theta} + e^{2\theta} + \left[e^{\theta} \sec\frac{\theta}{2} - e^{\frac{3\theta}{2}} \csc\theta - e^{\frac{\theta}{2}} \sin\theta \sec\frac{\theta}{2} + \cos\theta \sec\frac{\theta}{2}\right.\right.\\
&\left.\left. + \sin\theta \sec\frac{\theta}{2} - e^{\frac{\theta}{2}} - e^{\frac{\theta}{2}} \cos\theta \csc\theta\right] e^{\frac{\theta}{2}} \sin\theta \sec\frac{\theta}{2}\right]\\
\gamma_2 =& \frac{1}{\theta^2}\left[-e^{\theta} - e^{\frac{\theta}{2}} \sin\theta \sec^2\frac{\theta}{2} + e^{\frac{\theta}{2}} \sin\theta \sec\frac{\theta}{2} + e^{\theta} \sec\frac{\theta}{2}\right]\\
\gamma_1 =& -e^{\theta} + e^{\frac{\theta}{2}} \sin\theta \sec\frac{\theta}{2}\\
\alpha_1 =& [\gamma_1]^2\\
\beta_1 =& \gamma_1 - [\gamma_1]^2.
\end{aligned}$$

Remark 1. Our method reduces to Ramadan et al. (2007) based on quadratic spline when

$$(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2,) = (1, -2, 1, \frac{1}{8}, \frac{6}{8}, \frac{1}{8}). \tag{8}$$

2.1. Truncation error

Expanding the scheme (7) by using Taylor series, we obtained the following truncation error

$$\begin{aligned}
t_i =& [\alpha_1 + \beta_1 + \gamma_1] y_i + \left[\frac{-3\alpha_1 - \beta_1 + \gamma_1}{2}\right] h y_i^{(1)}\\
&+ h^2 \left[\frac{9\alpha_1 + \beta_1 + \gamma_1}{2^2 2!} - (\alpha_2 + \beta_2 + \gamma_2)\right] y_i^{(2)}\\
&+ h^3 \left[\frac{-27\alpha_1 - \beta_1 + \gamma_1}{2^3 3!} - \left(\frac{-3\alpha_2 - \beta_2 + \gamma_2}{2}\right)\right] y_i^{(3)}\\
&+ h^4 \left[\frac{81\alpha_1 + \beta_1 + \gamma_1}{2^4 4!} - \left(\frac{9\alpha_2 + \beta_2 + \gamma_2}{2^2 2!}\right)\right] y_i^{(4)}\\
&+ h^5 \left[\frac{-243\alpha_1 - \beta_1 + \gamma_1}{2^5 5!} - \left(\frac{-27\alpha_2 - \beta_2 + \gamma_2}{2^3 3!}\right)\right] y_i^{(5)}\\
&+ h^6 \left[\frac{729\alpha_1 + \beta_1 + \gamma_1}{2^6 6!} - \left(\frac{81\alpha_2 + \beta_2 + \gamma_2}{2^4 4!}\right)\right] y_i^{(6)}\\
&+ O(h^7), i = 2, 3, \ldots, n-1.
\end{aligned} \tag{9}$$

For different values of parameters, we get the second as well as fourth order method. For $\alpha_2 + \beta_2 + \gamma_2 = 1$ and $\alpha_2 = \gamma_2$ we get the second order method. For the choice of parameters $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2) = (1, -2, 1, 1/12, 10/12, 1/12)$ we get the fourth order method.

Eq. (7) forms a system of $(n-2)$ linear equations in n unknowns $y_{i-1/2}, i = 1, 2, \ldots, n$. Thus, we need two more equations, one at each end of range of integration. These boundary conditions are obtained by using method of undetermined coefficients.

The equations for second order method are given by Islam et al. (2006) as

$$2y_0 - 3y_{1/2} + y_{3/2} = \frac{h^2}{24}[15R_{1/2} + 3R_{3/2}] + O(h^4), i = 1 \tag{10}$$

$$2y_n - 3y_{n-1/2} + y_{n-3/2} = \frac{h^2}{24}[15R_{n-1/2} + 3R_{n-3/2}] + O(h^4), i = n \tag{11}$$

The truncation error for second order method for $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2) = (1, -2, 1, 1/8, 6/8, 1/8)$ is given as follows:

$$t_i = \begin{cases} -\frac{1}{64}h^4 y_0^{(4)} + O(h^5), & \text{i} = 1 \\ -\frac{1}{24}h^4 y_i^{(4)} + O(h^5), & \text{i} = 2, 3, \ldots, \text{n} - 1 \\ -\frac{1}{64}h^4 y_n^{(4)} + O(h^5), & \text{i} = \text{n} \end{cases}$$

and for fourth order method are obtained as

$$2y_0 - 3y_{1/2} + y_{3/2} = \frac{h^2}{384}[233R_{1/2} + 63R_{3/2} - 9R_{5/2} + R_{7/2}] + O(h^6), i = 1 \tag{12}$$

$$2y_n - 3y_{n-1/2} + y_{n-3/2} = \frac{h^2}{384}[233R_{n-1/2} + 63R_{n-3/2} - 9R_{n-5/2} + R_{n-7/2}] + O(h^6), i = n. \tag{13}$$

The truncation error for the fourth order method is given as follows:

$$t_i = \begin{cases} \frac{11}{7680}h^6 y_0^{(6)} + O(h^7), & \text{i} = 1 \\ -\frac{1}{240}h^6 y_i^{(6)} + O(h^7), & \text{i} = 2, 3, \ldots, \text{n} - 1 \\ \frac{11}{7680}h^6 y_n^{(6)} + O(h^7), & \text{i} = \text{n}. \end{cases}$$

3. Application of the algorithm to the higher order boundary value problems for $N = 3$

We consider a sixth order boundary value problem i.e N = 3 in (1–2) of the form

$$y^{(6)}(x) = a(x)y^{(5)}(x) + b(x)y^{(4)}(x) + c(x)y^{(3)}(x) + d(x)y^{(2)}(x) + e(x)y^{(1)}(x) + f(x)y(x) + g(x) \tag{14}$$

subject to boundary conditions

$$y(a) = \lambda_1, y^{(2)}(a) = \lambda_2, y^{(4)}(a) = \lambda_3$$
$$y(b) = \delta_1, y^{(2)}(b) = \delta_2, y^{(4)}(b) = \delta_3$$

where λ_i and $\delta_i (i = 1, 2, 3)$ are finite real constants and the functions $a(x), b(x), c(x), d(x), e(x), f(x)$ and $g(x)$ are continuous on $[a, b]$. We decompose the above problem into the system of second order boundary value problems as follows:

$$y^{(2)}(x) = u(x), \tag{15}$$

$$u^{(2)}(x) = v(x), \tag{16}$$

$$v^{(2)}(x) = a(x)v^{(1)}(x) + b(x)v(x) + c(x)u^{(1)}(x) + d(x)u(x) + e(x)y^{(1)}(x) + f(x)y(x) + g(x). \equiv F(x, y, u, v, y^{(1)}, u^{(1)}, v^{(1)}) \tag{17}$$

subject to modified boundary conditions

$$y(a) = \lambda_1, y(b) = \delta_1, \tag{18}$$

$$u(a) = \lambda_2, u(b) = \delta_2, \tag{19}$$

$$v(a) = \lambda_3, v(b) = \delta_3. \tag{20}$$

Therefore by implementing the scheme (7) on the boundary value problems (15)–(17), we get the following system of equations

$$\alpha_1 y_{i-3/2} + \beta_1 y_{i-1/2} + \gamma_1 y_{i+1/2} = h^2(\alpha_2 u_{i-3/2} + \beta_2 u_{i-1/2} + \gamma_2 u_{i+1/2}), i = 2, 3, \ldots, n-1 \tag{21}$$

$$\alpha_1 u_{i-3/2} + \beta_1 u_{i-1/2} + \gamma_1 u_{i+1/2} = h^2(\alpha_2 v_{i-3/2} + \beta_2 v_{i-1/2} + \gamma_2 v_{i+1/2}), i = 2, 3, \ldots, n-1 \tag{22}$$

$$\alpha_1 v_{i-3/2} + \beta_1 v_{i-1/2} + \gamma_1 v_{i+1/2} = h^2(\alpha_2 F_{i-3/2} + \beta_2 F_{i-1/2} + \gamma_2 F_{i+1/2}), i = 2, 3, \ldots, n-1 \tag{23}$$

The fourth order method is obtained by using the parameters $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{12}, \frac{10}{12}, \frac{1}{12})$ in the scheme (21)–(23) and replacing (23) by the following as:

$$\alpha_1 v_{i-3/2} + \beta_1 v_{i-1/2} + \gamma_1 v_{i+1/2} = h^2\left(\frac{1}{12}F_{i-3/2} + \frac{10}{12}\widetilde{F}_{i-1/2} + \frac{1}{12}F_{i+1/2}\right),$$
$i = 2, 3, \ldots, n-1$

where,

$$F_{i-3/2} = F\left(x, y_{i-3/2}, u_{i-3/2}, v_{i-3/2}, y'_{i-3/2}, u'_{i-3/2}, v'_{i-3/2}\right),$$
$$F_{i-1/2} = F\left(x, y_{i-1/2}, u_{i-1/2}, v_{i-1/2}, y'_{i-1/2}, u'_{i-1/2}, v'_{i-1/2}\right),$$
$$F_{i+1/2} = F\left(x, y_{i+1/2}, u_{i+1/2}, v_{i+1/2}, y'_{i+1/2}, u'_{i+1/2}, v'_{i+1/2}\right),$$
$$\widetilde{F}_{i-1/2} = F\left(x, y_{i-1/2}, u_{i-1/2}, v_{i-1/2}, \tilde{y}'_{i-1/2}, \tilde{u}'_{i-1/2}, \tilde{v}'_{i-1/2}\right),$$

and the finite difference approximations to derivatives are

$$y'_{i-1/2} = \frac{y_{i+1/2} - y_{i-3/2}}{2h},$$
$$y'_{i-3/2} = \frac{-3y_{i-3/2} + 4y_{i-1/2} - y_{i+1/2}}{2h},$$
$$y'_{i+1/2} = \frac{y_{i-3/2} - 4y_{i-1/2} + 3y_{i+1/2}}{2h},$$
$$\tilde{y}'_{i-1/2} = \frac{y_{i+1/2} - y_{i-3/2}}{2h} - \frac{h}{20}(R_{i+1/2} - R_{i-3/2}).$$

The fourth order approximations of derivatives involved in the end Eqs. (12) and (13) are as follows:

$$hy^{(1)}_{1/2} = -\frac{32}{35}y_0 + \frac{1}{6}y_{1/2} + y_{3/2} - \frac{3}{10}y_{5/2} + \frac{1}{21}y_{7/2}$$
$$hy^{(1)}_{3/2} = \frac{32}{105}y_0 - y_{1/2} + \frac{1}{6}y_{3/2} + \frac{3}{5}y_{5/2} - \frac{1}{14}y_{7/2}$$
$$hy^{(1)}_{5/2} = -\frac{32}{105}y_0 + \frac{5}{6}y_{1/2} - \frac{5}{3}y_{3/2} + \frac{9}{10}y_{5/2} + \frac{5}{21}y_{7/2}$$
$$hy^{(1)}_{7/2} = \frac{32}{35}y_0 - \frac{7}{3}y_{1/2} + \frac{7}{2}y_{3/2} - \frac{21}{5}y_{5/2} + \frac{89}{42}y_{7/2}.$$

Here, we derive only the **second order method**. On combining (21)–(23), we obtain the vector difference equation for the boundary value problem (15)–(17) as

$$A_i W_{i-3/2} + B_i W_{i-1/2} + C_i W_{i+1/2} = H_i \tag{24}$$

which are as follows:

$$\begin{bmatrix} ai_{11} & ai_{12} & ai_{13} \\ ai_{21} & ai_{22} & ai_{23} \\ ai_{31} & ai_{32} & ai_{33} \end{bmatrix} \begin{bmatrix} y_{i-3/2} \\ u_{i-3/2} \\ v_{i-3/2} \end{bmatrix} + \begin{bmatrix} bi_{11} & bi_{12} & bi_{13} \\ bi_{21} & bi_{22} & bi_{23} \\ bi_{31} & bi_{32} & bi_{33} \end{bmatrix} \begin{bmatrix} y_{i-1/2} \\ u_{i-1/2} \\ v_{i-1/2} \end{bmatrix} + \begin{bmatrix} ci_{11} & ci_{12} & ci_{13} \\ ci_{21} & ci_{22} & ci_{23} \\ ci_{31} & ci_{32} & ci_{33} \end{bmatrix} \begin{bmatrix} y_{i+1/2} \\ u_{i+1/2} \\ v_{i+1/2} \end{bmatrix} = \begin{bmatrix} h_{i1} \\ h_{i2} \\ h_{i3} \end{bmatrix}, i = 2, 3, \ldots, n-1$$

where,

$ai_{11} = -\alpha_1, ai_{12} = h^2\alpha_2, ai_{13} = 0, ai_{21} = 0,$
$ai_{22} = -\alpha_1, ai_{23} = h^2\alpha_2,$
$ai_{31} = -\frac{3h}{2}\alpha_2 e_{i-3/2} + h^2\alpha_2 f_{i-3/2} - \frac{h}{2}\beta_2 e_{i-1/2} + \frac{h}{2}\gamma_2 e_{i+1/2},$
$ai_{32} = -\frac{3h}{2}\alpha_2 c_{i-3/2} + h^2\alpha_2 d_{i-3/2} - \frac{h}{2}\beta_2 c_{i-1/2} + \frac{h}{2}\gamma_2 c_{i+1/2},$
$ai_{33} = -\alpha_1 - \frac{3h}{2}\alpha_2 a_{i-3/2} + h^2\alpha_2 b_{i-3/2} - \frac{h}{2}\beta_2 a_{i-1/2} + \frac{h}{2}\gamma_2 a_{i+1/2},$
$bi_{11} = -\beta_1, bi_{12} = h^2\beta_2, bi_{13} = 0, bi_{21} = 0,$
$bi_{22} = -\beta_1, bi_{23} = h^2\beta_2,$
$bi_{31} = 2h\alpha_2 e_{i-3/2} + h^2\beta_2 f_{i-1/2} - 2h\gamma_2 e_{i+1/2},$
$bi_{32} = 2h\alpha_2 c_{i-3/2} + h^2\beta_2 d_{i-1/2} - 2h\gamma_2 c_{i+1/2},$
$bi_{33} = -\beta_1 + 2h\alpha_2 a_{i-3/2} + h^2\beta_2 b_{i-1/2} - 2h\gamma_2 a_{i+1/2},$
$ci_{11} = -\gamma_1, ci_{12} = h^2\gamma_2, ci_{13} = 0, ci_{21} = 0,$
$ci_{22} = -\gamma_1, ci_{23} = h^2\gamma_2,$
$ci_{31} = -\frac{1}{2}h\alpha_2 e_{i-3/2} + \frac{1}{2}h\beta_2 e_{i-1/2} + \frac{3}{2}h\gamma_2 e_{i+1/2} + h^2\gamma_2 f_{i+1/2},$
$ci_{32} = -\frac{1}{2}h\alpha_2 c_{i-3/2} + \frac{1}{2}h\beta_2 c_{i-1/2} + \frac{3}{2}h\gamma_2 c_{i+1/2} + h^2\gamma_2 d_{i+1/2},$
$ci_{33} = -\gamma_1 - \frac{1}{2}h\alpha_2 a_{i-3/2} + \frac{1}{2}h\beta_2 a_{i-1/2} + \frac{3}{2}h\gamma_2 a_{i+1/2} + h^2\gamma_2 b_{i+1/2},$
$h_{i1} = 0,$
$h_{i2} = 0,$
$h_{i3} = -h^2(\alpha_2 g_{i-3/2} + \beta_2 g_{i-1/2} + \gamma_2 g_{i+1/2}),$
$i = 2, 3, \ldots, n-1.$

Now for i = 1, we have

$$B_1 W_{1/2} + C_1 W_{3/2} = H_1 \tag{25}$$

which can be written as

$$\begin{bmatrix} b1_{11} & b1_{12} & b1_{13} \\ b1_{21} & b1_{22} & b1_{23} \\ b1_{31} & b1_{32} & b1_{33} \end{bmatrix}\begin{bmatrix} y_{1/2} \\ u_{1/2} \\ v_{1/2} \end{bmatrix} + \begin{bmatrix} c1_{11} & c1_{12} & c1_{13} \\ c1_{21} & c1_{22} & c1_{23} \\ c1_{31} & c1_{32} & c1_{33} \end{bmatrix}\begin{bmatrix} y_{3/2} \\ u_{3/2} \\ v_{3/2} \end{bmatrix} = \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \end{bmatrix}$$

where,

$b1_{11} = 3, b1_{12} = \frac{15h^2}{24}, b1_{13} = 0, b1_{21} = 0, b1_{22} = 3, b1_{23} = \frac{15h^2}{24},$
$b1_{31} = -\frac{15}{24}he_{1/2} + \frac{15}{24}h^2 f_{1/2} - \frac{3}{24}he_{3/2},$
$b1_{32} = -\frac{15}{24}hc_{1/2} + \frac{15}{24}h^2 d_{1/2} - \frac{3}{24}hc_{3/2},$
$b1_{33} = 3 - \frac{15}{24}ha_{1/2} + \frac{15}{24}h^2 b_{1/2} - \frac{3}{24}ha_{3/2},$
$c1_{11} = -1, c1_{12} = \frac{3h^2}{24}, c1_{13} = 0, c1_{21} = 0, c1_{22} = -1, c1_{23} = \frac{3h^2}{24},$

$c1_{31} = \frac{15}{24}he_{1/2} + \frac{3}{24}he_{3/2} + \frac{3}{24}h^2 f_{3/2},$
$c1_{32} = \frac{15}{24}hc_{1/2} + \frac{3}{24}hc_{3/2} + \frac{3}{24}h^2 d_{3/2},$
$c1_{33} = -1 + \frac{15}{24}ha_{1/2} + \frac{3}{24}ha_{3/2} + \frac{3}{24}h^2 b_{3/2},$
$h_{11} = 2y_0, h_{12} = 2u_0, h_{13} = 2v_0 - \frac{h^2}{24}(15g_{1/2} + 3g_{3/2}).$

Now for i = n, we have

$$A_n W_{n-3/2} + B_n W_{n-1/2} = H_n \tag{26}$$

which can be written as

$$\begin{bmatrix} an_{11} & an_{12} & an_{13} \\ an_{21} & an_{22} & an_{23} \\ an_{31} & an_{32} & an_{33} \end{bmatrix}\begin{bmatrix} y_{n-3/2} \\ u_{n-3/2} \\ v_{n-3/2} \end{bmatrix} + \begin{bmatrix} bn_{11} & bn_{12} & bn_{13} \\ bn_{21} & bn_{22} & bn_{23} \\ bn_{31} & bn_{32} & bn_{33} \end{bmatrix}\begin{bmatrix} y_{n-1/2} \\ u_{n-1/2} \\ v_{n-1/2} \end{bmatrix} = \begin{bmatrix} h_{n1} \\ h_{n2} \\ h_{n3} \end{bmatrix}$$

where,

$an_{11} = 3, an_{12} = \frac{15h^2}{24},$
$an_{13} = 0, an_{21} = 0,$
$an_{22} = 3, an_{23} = \frac{15h^2}{24},$
$an_{31} = -\frac{15}{24}he_{n-1/2} + \frac{15}{24}h^2 f_{n-1/2} - \frac{3}{24}he_{n-3/2},$
$an_{32} = -\frac{15}{24}hc_{n-1/2} + \frac{15}{24}h^2 d_{n-1/2} - \frac{3}{24}hc_{n-3/2},$
$an_{33} = 3 - \frac{15}{24}ha_{n-1/2} + \frac{15}{24}h^2 b_{n-1/2} - \frac{3}{24}ha_{n-3/2}$
$bn_{11} = -1, bn_{12} = \frac{3h^2}{24},$
$bn_{13} = 0, bn_{21} = 0,$
$bn_{22} = -1, bn_{23} = \frac{3h^2}{24},$
$bn_{31} = \frac{15}{24}he_{n-1/2} + \frac{3}{24}he_{n-3/2} + \frac{3}{24}h^2 f_{n-3/2},$
$bn_{32} = \frac{15}{24}hc_{n-1/2} + \frac{3}{24}hc_{n-3/2} + \frac{3}{24}h^2 d_{n-3/2},$
$bn_{33} = -1 + \frac{15}{24}ha_{n-1/2} + \frac{3}{24}ha_{n-3/2} + \frac{3}{24}h^2 b_{n-3/2},$
$h_{n1} = 2y_n,$
$h_{n2} = 2u_n,$
$h_{n3} = 2v_n - \frac{h^2}{24}(15g_{n-1/2} + 3g_{n-3/2}).$

4. Convergence analysis

In this section, we study the convergence analysis of the second order method developed in Section 2 where $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2) = (1, -2, 1, 1/8, 6/8, 1/8)$. The method has the following form

$$AW = M \tag{27}$$

where,

$$A = \begin{bmatrix} B_1 & C_1 & & & & & \\ A_2 & B_2 & C_2 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & \ddots \\ & & & & A_{n-1} & B_{n-1} & C_{n-1} \\ & & & & & A_n & B_n \end{bmatrix} \tag{28}$$

where, A is a triblockdiagonal matrix in which $A_i (i = 2, 3, \ldots, n)$, $B_i (i = 1, 2, \ldots, n)$ and $C_i (i = 2, 3, \ldots, n-1)$ are matrices of order 2×2, $W = [w_{1/2}, w_{3/2}, \ldots, w_{n-1/2}]^T$ where, $w_{i-1/2} = [y_{i-1/2}, u_{i-1/2}, v_{i-1/2}]^T, i = 1, 2, \ldots, n$ and the right hand side vector $M = [m_1, m_2, \ldots, m_n]^T$, where $m_i = [m_{i1}, m_{i2}]^T, i = 1, 2, \ldots, n$.

We also have,

$$A\widetilde{W} = M + T \tag{29}$$

where $\widetilde{W} = [\tilde{w}_{1/2}, \tilde{w}_{3/2}, \ldots, \tilde{w}_{n-1/2}]^T$ where, $\tilde{w}_{i-1/2} = [\tilde{y}_{i-1/2}, \tilde{u}_{i-1/2}, \tilde{v}_{i-1/2}]^T, i = 1, 2, \ldots, n$ be the exact solution and $T = [t_1, t_2, \ldots, t_n]^T$ where, $t_i = [\tilde{y}_{i-1/2} - y_{i-1/2}, \tilde{u}_{i-1/2} - u_{i-1/2}, \tilde{v}_{i-1/2} - v_{i-1/2}]^T, i = 1, 2, \ldots, n$ be the local truncation error. From (27) and (29) we have,

$A(\widetilde{W} - W) = T,$
$AE = T,$
$E = \widetilde{W} - W = [\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{n-1}]^T$

Let $0 < R \in Z^+$ is the minimum of $|a_i|, |b_i|, |c_i|, |d_i|, |e_i|$ and $|f_i|$.

Then

$$\|C_i\| \leqslant \max_{1\leqslant i\leqslant n-1}\begin{cases} 1+\gamma_1+h^2(\gamma_2+\frac{3}{24}), & j=1\\ 1+\gamma_1+h^2(\gamma_2+\frac{3}{24}), & j=2\\ 1+\gamma_1+\frac{1}{2}h(\alpha_2+\beta_2+3\gamma_2)R+h^2\gamma_2R+\frac{18}{24}hR+\frac{3}{24}h^2R, & j=3\end{cases}$$

$$\|A_i\| \leqslant \max_{2\leqslant i\leqslant n}\begin{cases} 3+\alpha_1+h^2(\alpha_2+\frac{15}{24}), & j=1\\ 3+\alpha_1+h^2(\alpha_2+\frac{15}{24}), & j=2\\ 3+\alpha_1+\frac{1}{2}h(3\alpha_2+\beta_2+\gamma_2)R+h^2\alpha_2R+\frac{18}{24}hR+\frac{15}{24}h^2R, & j=3\end{cases}$$

Further,

$$\|C_i\|_\infty \leqslant \max_{1\leqslant i\leqslant n-1}\begin{cases} 2+\frac{1}{4}h^2, & j=1\\ 2+\frac{1}{4}h^2, & j=2\\ 2+\frac{11}{8}hR+\frac{1}{4}h^2R, & j=3\end{cases}$$

$$\|A_i\|_\infty \leqslant \max_{2\leqslant i\leqslant n}\begin{cases} 4+\frac{1}{2}h^2, & j=1\\ 4+\frac{1}{2}h^2, & j=2\\ 4+\frac{11}{8}hR+\frac{1}{2}h^2R, & j=3\end{cases}$$

This shows that $\|C_i\|_\infty \le 2$ and $\|A_i\|_\infty \le 4$ for sufficiently small h. Hence, A is irreducible. Now we have to show that A is monotone.

To show matrix A is monotone first we calculate the sum of each row of the matrix A.

$$S_{1j}=\begin{cases}\sum_{k=1}^{3}a1_{1k}+b1_{1k}, & j=1\\ \sum_{k=1}^{3}a1_{2k}+b1_{2k}, & j=2\\ \sum_{k=1}^{3}a1_{3k}+b1_{3k}, & j=3\end{cases}$$

$$S_{ij}=\begin{cases}\sum_{k=1}^{3}ai_{1k}+bi_{1k}+ci_{1k}, & j=1, i=1,4,7,\dots,n-3\\ \sum_{k=1}^{3}ai_{2k}+bi_{2k}+ci_{2k}, & j=2, i=2,5,8,\dots,n-2\\ \sum_{k=1}^{3}ai_{3k}+bi_{3k}+ci_{3k}, & j=3, i=3,6,9,\dots,n-1\end{cases}$$

$$S_{n\mathbf{j}}=\begin{cases}\sum_{k=1}^{3}an_{1k}+bn_{1k}, & j=1\\ \sum_{k=1}^{3}an_{2k}+bn_{2k}, & j=2\\ \sum_{k=1}^{3}an_{3k}+bn_{3k}, & j=3\end{cases}$$

We have

$$S_{1j}=\begin{cases}2+\frac{18}{24}h^2, & j=1\\ 2+\frac{18}{24}h^2, & j=2\\ 2+\frac{15}{24}h^2(f_{1/2}+d_{1/2}+b_{1/2})-\frac{3}{24}h^2(f_{3/2}+d_{3/2}+b_{3/2}), & j=3\end{cases}$$

$$S_{ij}=\begin{cases}-(\alpha_1+\beta_1+\gamma_1)+2h^2(\alpha_2+\beta_2+\gamma_2), & i=1,4,7,\dots,n-3, \quad j=1\\ -(\alpha_1+\beta_1+\gamma_1)+2h^2(\alpha_2+\beta_2+\gamma_2), & i=2,5,8,\dots,n-2, \quad j=2\\ -(\alpha_1+\beta_1+\gamma_1)+h^2\alpha_2(f_{i-3/2}+d_{i-3/2}+b_{i-3/2}) & \\ \quad +h^2\beta_2(f_{i-1/2}+d_{i-1/2}+b_{i-1/2}) & \\ \quad +h^2\gamma_2(f_{i+1/2}+d_{i+1/2}+b_{i+1/2}), & i=3,6,9,\dots,n-1, \quad j=3\end{cases}$$

$$=\begin{cases}2h^2,\ i=1,4,7,\dots,n-3, & j=1\\ 2h^2,\ i=2,5,8,\dots,n-2, & j=2\\ \frac{1}{8}h^2(f_{i-3/2}+d_{i-3/2}+b_{i-3/2})+\frac{6}{8}h^2(f_{i-1/2} & \\ +d_{i-1/2}+b_{i-1/2})+\frac{1}{8}h^2(f_{i+1/2}+d_{i+1/2}+b_{i+1/2}), \quad i=3,6,9,\dots,n-1, & j=3\end{cases}$$

$$S_{n\mathbf{j}}=\begin{cases}2+\frac{18}{24}h^2, & j=1\\ 2+\frac{18}{24}h^2, & j=2\\ 2+\frac{15}{24}h^2(f_{n-1/2}+d_{n-1/2}+b_{n-1/2})-\frac{3}{24}h^2(f_{n-3/2}+d_{n-3/2}+b_{n-3/2}), & j=3\end{cases}$$

For sufficiently small h, we can easily show that the matrix A is irreducible and monotone. Therefore, A^{-1} exist and $A^{-1}\ge 0$.

Hence,

$$\|E\| = \|A^{-1}\|\|T\|.$$

Now for sufficiently small h, we have

$$S_{1j}\geqslant\begin{cases}\frac{18}{24}h^2, & j=1\\ \frac{18}{24}h^2, & j=2\\ \frac{18}{24}h^2R, & j=3\end{cases}$$

$$S_{ij}\geqslant\begin{cases}h^2, i=1,4,\dots,n-3, & j=1\\ h^2, i=2,5,\dots,n-2, & j=2\\ h^2R,\ i=3,6,\dots,n-1, & j=3\end{cases}$$

$$S_{n\mathbf{j}}\geqslant\begin{cases}\frac{18}{24}h^2, & j=1\\ \frac{18}{24}h^2, & j=2\\ \frac{18}{24}h^2R, & j=3\end{cases}$$

$$S_1 \geqslant max[\frac{18}{24}h^2,\frac{18}{24}h^2,\frac{18}{24}Rh^2]=\frac{18}{24}Rh^2,\ i=1$$

$$S_i \geqslant max[h^2,h^2,Rh^2]=Rh^2, i=2,\dots,n-1$$

$$S_n \geqslant max[\frac{18}{24}h^2,\frac{18}{24}h^2,\frac{18}{24}Rh^2]=\frac{18}{24}Rh^2,\ i=n$$

Therefore, we get

$$\frac{1}{S_1}\leqslant\left\{\frac{24}{18Rh^2},\right. \quad i=1$$

$$\frac{1}{S_i}\leqslant\left\{\frac{1}{Rh^2},\right. \quad i=2,3,\dots,n-1$$

$$\frac{1}{S_n}\leqslant\left\{\frac{24}{18Rh^2},\right. \quad i=n$$

Further,

$$\frac{1}{S_i}\leqslant\begin{cases}\frac{24}{18h^2R}, & i=1\\ \frac{1}{h^2R}, & i=2,3,\dots,n-1\\ \frac{24}{18h^2R}, & i=n\end{cases}$$

Let $A^{-1}=(a^*_{i,j})$, then by theory of matrices (Varga, 1962), we get $\sum_{i=1}^{n}a^*_{i,j}S_i=1, j=1,\dots,n.$

Therefore,

$$a^*_{i,j} \leqslant \frac{1}{S_i}$$

$$\|A^{-1}\| = \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |a^*_{i,j}| \leqslant \sum_{i=1}^{n} \frac{1}{S_i} = \frac{1}{h^2 R}\left(\frac{24}{9}+1\right),$$

$i = 1, \ldots, n$ and

$$\|T_i\| = \max_{1\leqslant i\leqslant n} \sum_{i=1}^{n} |T_i|$$

The error is given by

$$\|E\| = \|A^{-1}\|\|T\| \leqslant \frac{1}{h^2 R}\left(\frac{33}{9}\right)\|T\|.$$

Therefore, using (9) we get $\|T\| = O(h^4)$ for second order method.

$$\|E\| \leqslant \frac{1}{h^2 R}\left(\frac{33}{9}\right)O(h^4) = O(h^2).$$

Hence, our method is second order convergent. By repeating the same procedure we can find the bound for error of fourth order method which is as follows:

$$\|E\| = \|A^{-1}\|\|T\| \leqslant \frac{1}{h^2 R}\left(\frac{33}{9}\right)\|T\|.$$

Therefore, using (9) we get $\|T\| = O(h^6)$ for fourth order method.

$$\|E\| \leqslant \frac{1}{h^2 R}\left(\frac{33}{9}\right)O(h^6) = O(h^4).$$

Hence, our method is also fourth order convergent.

Theorem. *The method given by Eq. (7) for solving the boundary value problem (1), (2) for sufficiently small h has a second as well as fourth order convergence depending upon the parameters.*

5. Numerical Illustrations

We now consider seven numerical examples involving higher order non-linear and linear boundary value problems along with two singular problems involving first and third derivatives illustrating the efficiency of the presented method. We compared the results with the existing methods of fourth order. We have also provided the numerical rate of convergence (ρ^n) for the non-linear singular fourth order BVPs. The numerical rate of convergence is computed using

$$\rho^n = \ln_2(E^n/E^{2n})$$

The maximum absolute errors for h = 1/8, 1/16 and 1/32 are tabulated in the Tables 1–7.

Table 1
Maximum absolute errors for Example 5.1.

Our Method	$h = 1/8$	$h = 1/16$	$h = 1/32$
Fourth order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{12}, \frac{10}{12}, \frac{1}{12})$	1.8041×10^{-8}	1.2112×10^{-9}	7.7281×10^{-11}
Second order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{8}, \frac{6}{8}, \frac{1}{8})$	7.4684×10^{-6}	1.9008×10^{-6}	4.7705×10^{-7}
Wazwaz (2002)	9.14×10^{-5}	–	–

Table 2
Maximum absolute errors for Example 5.2.

Our Method	$h = 1/8$	$h = 1/16$	$h = 1/32$
Fourth order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{12}, \frac{10}{12}, \frac{1}{12})$	2.6956×10^{-6}	2.3833×10^{-7}	1.6610×10^{-8}
Khan and Khandelwal (2012)	3.06×10^{-6}	4.34×10^{-7}	3.53×10^{-8}
Second order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{8}, \frac{6}{8}, \frac{1}{8})$	5.8363×10^{-5}	1.5815×10^{-5}	4.0262×10^{-6}
Islam et al. (2008)	2.3×10^{-3}	3.4×10^{-4}	1.54×10^{-4}

Table 3
Maximum absolute errors for Example 5.3.

Our Method	$h = 1/8$	$h = 1/16$	$h = 1/32$
Fourth order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{12}, \frac{10}{12}, \frac{1}{12})$	1.8014×10^{-7}	1.1957×10^{-8}	7.5853×10^{-10}
Khan and Khandelwal (2012)	7.02×10^{-6}	4.35×10^{-6}	7.87×10^{-7}
Second order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{8}, \frac{6}{8}, \frac{1}{8})$	1.1216×10^{-4}	3.1010×10^{-5}	7.7930×10^{-6}
Khan and Khandelwal (2012)	2.19×10^{-4}	3.88×10^{-5}	–

Table 4
Maximum absolute errors for Example 5.4.

Our Method	$h = 1/8$	$h = 1/16$	$h = 1/32$
Fourth order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{12}, \frac{10}{12}, \frac{1}{12})$	1.8034×10^{-7}	1.1971×10^{-8}	7.5941×10^{-10}
Second order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{8}, \frac{6}{8}, \frac{1}{8})$	1.2179×10^{-4}	3.1046×10^{-5}	7.8021×10^{-6}
Meŝtroviĉ (2007)	1.43×10^{-4}	–	–

Table 5
Maximum absolute errors for Example 5.5.

Our Method	$h = 1/8$	$h = 1/16$	$h = 1/32$
Fourth order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{12}, \frac{10}{12}, \frac{1}{12})$	3.0468×10^{-5}	2.7373×10^{-6}	1.8321×10^{-7}
Islam et al. (2008)	6.97×10^{-4}	3.60×10^{-5}	7.44×10^{-7}
Second order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{8}, \frac{6}{8}, \frac{1}{8})$	3.30×10^{-3}	9.2908×10^{-4}	2.3651×10^{-4}
Islam et al. (2008)	1.23×10^{-2}	2.80×10^{-3}	1.60×10^{-3}

Table 6
Maximum absolute errors for Example 5.6.

Our Method	$h = 1/8$	$h = 1/16$	$h = 1/32$
Fourth order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{12}, \frac{10}{12}, \frac{1}{12})$	8.7479×10^{-6}	5.2045×10^{-7}	3.0265×10^{-8}
ρ^n	4.0711	4.1040	–
Mohanty et al. (2017)	3.0756×10^{-5}	1.8795×10^{-6}	1.1320×10^{-7}
Second order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{8}, \frac{6}{8}, \frac{1}{8})$	4.1197×10^{-3}	9.1777×10^{-4}	2.1645×10^{-4}

Table 7
Maximum absolute errors for Example 5.7.

Our Method	$h = 1/8$	$h = 1/16$	$h = 1/32$
Fourth order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{12}, \frac{10}{12}, \frac{1}{12})$	2.3953×10^{-6}	1.4326×10^{-7}	8.4006×10^{-9}
ρ^n	4.0635	4.0920	–
Second order method for $(\alpha_2, \beta_2, \gamma_2) = (\frac{1}{8}, \frac{6}{8}, \frac{1}{8})$	1.005×10^{-3}	2.2503×10^{-4}	5.3021×10^{-5}

Example 5.1. Consider the following fourth order non-linear boundary value problem for $0 \le x \le 4 - e$ as

$$y^{(4)}(x) = -6\exp(-4y(x)), \tag{30}$$

where,

$$y(0) = 1, y(4 - e) = \ln(4),$$
$$y^{(2)}(0) = -\frac{1}{e^2}, y^{(2)}(4 - e) = -\frac{1}{16}.$$

The exact solution of the problem is $y(x) = \ln(e + x)$. The maximum absolute errors of the problem (30) are given in Table 1 and results are compared with (Wazwaz, 2002). Graph between the exact and the approximate solutions of Example 5.1 for N = 16 is shown in Fig. 1.

Example 5.2. Consider the following sixth order non-linear boundary value problem as

$$y^{(6)}(x) = 20\exp[-36y(x)] - 40(1 + x)^{-6}, x \in [0, 1] \tag{31}$$

where,

$$y(0) = 0, y(1) = \frac{1}{6}\log 2,$$
$$y^{(2)}(0) = -\frac{1}{6}, y^{(2)}(1) = -\frac{1}{24},$$
$$y^{(4)}(0) = -1, y^{(4)}(1) = -\frac{1}{16}.$$

The exact solution of the problem is $y(x) = \frac{1}{6}\log(1 + x)$. The maximum absolute errors of the problem (31) are given in Table 2 and results are compared with (Khan and Khandelwal, 2012; Islam et al., 2008). Graph between the exact and the approximate solutions of Example 5.2 for N = 32 is shown in Fig. 2.

Example 5.3. For $0 \le x \le 1$, the sixth order non-linear boundary value problem is considered as,

$$y^{(6)}(x) = \exp[-x]y^2(x), \tag{32}$$

where,

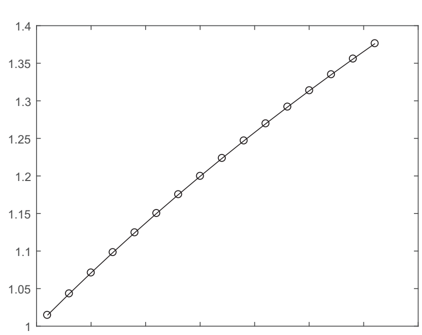

Fig. 1. Graph of the exact solution versus the approximate solution for N = 16 for Example 5.1.

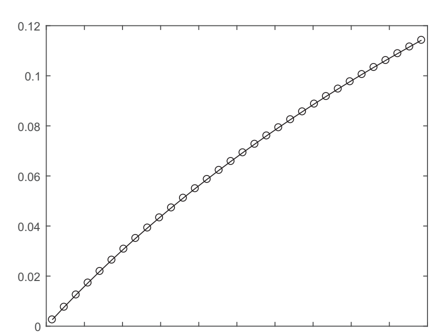

Fig. 2. Graph of the exact solution versus the approximate solution for N = 32 for Example 5.2.

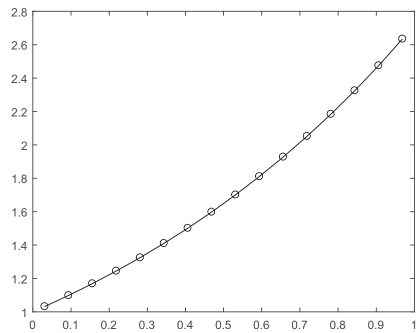

Fig. 3. Graph of the exact solution versus the approximate solution for N = 16 for Example 5.3.

$$y(0) = 1, y(1) = \exp(1),$$
$$y^{(2)}(0) = 1, y^{(2)}(1) = \exp(1),$$
$$y^{(4)}(0) = 1, y^{(4)}(1) = \exp(1).$$

The analytical solution of the above problem is $y(x) = \exp(x)$. The maximum absolute errors of the problem (32) are given in Table 3 and results are compared with (Khan and Khandelwal, 2012). Graph between the exact and the approximate solutions of Example 5.3 for N = 16 is shown in Fig. 3.

Example 5.4. For $0 \le x \le 1$, the eighth order non-linear boundary value problem is considered as,

$$y^{(8)}(x) = \exp[-x]y^2(x), \tag{33}$$

where,

$$y(0) = 1, y(1) = \exp(1),$$
$$y^{(2)}(0) = 1, y^{(2)}(1) = \exp(1),$$
$$y^{(4)}(0) = 1, y^{(4)}(1) = \exp(1),$$
$$y^{(6)}(0) = 1, y^{(6)}(1) = \exp(1).$$

The analytical solution of the above problem is $y(x) = \exp(x)$. The maximum absolute errors of the problem (33) are given in Table 4 and results are compared with (Meŝtroviĉ, 2007). Graph between the exact and the approximate solutions of Example 5.4 for N = 32 is shown in Fig. 4.

Example 5.5. Consider the following sixth order linear boundary value problem as

$$y^{(6)}(x) + y(x) = 6[2x\cos(x) + 5\sin(x)], x \in [-1, 1] \tag{34}$$

where,

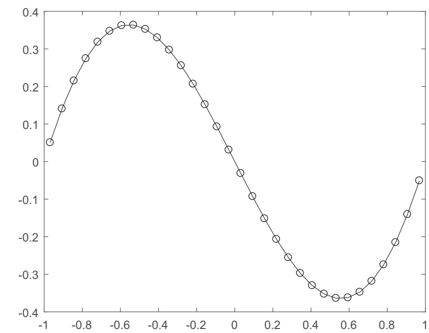

Fig. 5. Graph of the exact solution versus the approximate solution for N = 32 for Example 5.5.

$$y(-1) = 0, y(1) = 0,$$
$$y^{(2)}(-1) = -4\cos(-1) + 2\sin(-1),$$
$$y^{(2)}(1) = 4\cos(1) + 2\sin(1),$$
$$y^{(4)}(-1) = 8\cos(-1) - 12\sin(-1),$$
$$y^{(4)}(1) = -8\cos(1) - 12\sin(1).$$

The exact solution of the problem is $y(x) = (x^2 - 1)\sin(x)$. The maximum absolute errors of the problem (34) are given in Table 5 and results are compared with (Islam et al., 2008). Graph between the exact and the approximate solutions of Example 5.5 for N = 32 is shown in Fig. 5.

Example 5.6. Consider the following non-linear singular fourth order boundary value problem as

$$y^{(4)}(x) + \frac{4}{x}y^{(3)}(x) = y^3 + \frac{\cos(x)}{x}\left[x\frac{\sin(2x)}{2} - 4\right], x \in [0, 1] \tag{35}$$

where,

$$y(0) = 0, y(1) = \sin(1),$$
$$y^{(2)}(0) = 0, y^{(2)}(1) = -\sin(1).$$

The exact solution of the problem is $y(x) = \sin(x)$. The results of the problem (35) are given in Table 6 and compared with (Mohanty et al., 2017). Graph between the exact and the approximate solutions of Example 5.6 for N = 32 is shown in Fig. 6.

Example 5.7. Consider the following non-linear singular fourth order boundary value problem as

$$y^{(4)}(x) + \frac{4}{x}y^{(3)}(x) + \frac{5}{x}y^{(1)}(x) = y^3 + \frac{\cos(x)}{x} + \sin x\cos^2(x), x \in [0, 1] \tag{36}$$

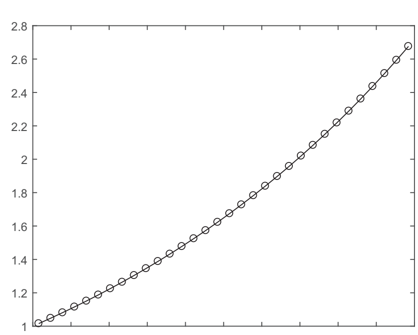

Fig. 4. Graph of the exact solution versus the approximate solution for N = 32 for Example 5.4.

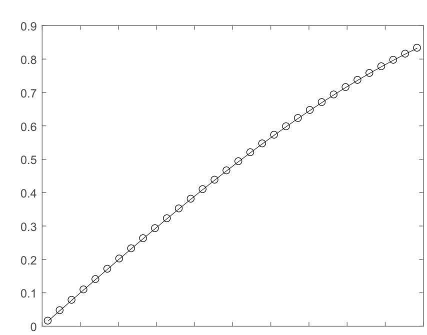

Fig. 6. Graph of the exact solution versus the approximate solution for N = 32 for Example 5.6.

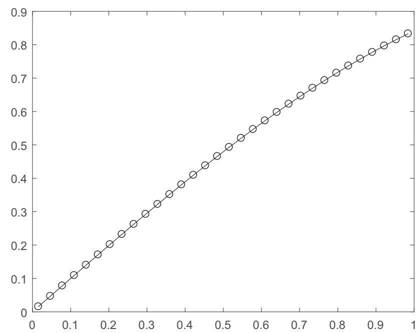

Fig. 7. Graph of the exact solution versus the approximate solution for N = 32 for Example 5.7.

where,

$$y(0) = 0, y(1) = \sin(1),$$
$$y^{(2)}(0) = 0, y^{(2)}(1) = -\sin(1).$$

The exact solution of the problem is $y(x) = \sin(x)$. The results of the problem (36) are given in Table 7. Graph between the exact and approximate solutions of Example 5.7 for N = 32 is shown in Fig. 7.

6. Conclusion

In this paper, we developed a non-polynomial quadratic spline method based on off-step points for solving higher even order boundary value problems. Advantage of the off-step points is to solve the higher order singular boundary value problems. We reduced the given problem into system of second order boundary value problems. The developed scheme (7) is second as well as fourth order accurate depending upon the parameters. Comparison of our method with the existing methods are shown in Tables 1–6 and graphs between exact and approximate solutions of the Examples 5.1–5.7 are shown in Figs. 1–7 respectively. These results shows that our fourth order method is far better than the existing fourth order methods except higher degree splines.

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