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ORIGINAL ARTICLE

On the Hadamard's type inequalities for co-ordinated convex functions via fractional integrals

Abdullah Akkurt [a,*], Mehmet Zeki Sarıkaya [b], Hüseyin Budak [b], Hüseyin Yıldırım [a]

[a] *Department of Mathematics, Faculty of Science and Arts, University of Kahramanmaraş Sütçü İmam, 46100 Kahramanmaraş, Turkey*
[b] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey*

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Abstract In this paper, we establish two identities for functions of two variables and apply them to give new Hermite–Hadamard type fractional integral inequalities for double fractional integrals involving functions whose derivatives are bounded or co-ordinates convex function on $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b,\ c < d$.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

is known in the literature as Hermite–Hadamard inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave.

It is well known that the Hermite–Hadamard's inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of studies on Hermite–Hadamard's inequality reporting its role in nonlinear analysis (Alomari et al., 2009; Azpeitia, 1994; Bakula and Pečarić, 2004; Dragomir and Pearce, 2000), later, this classical inequality has been improved (Kırmacı and Dikici, 2013; Set et al., 2011; Latif and Dragomir, 2012; Ozdemir et al., 2010) and is generalized in a number of ways (Hussain et al., 2009; Sarikaya and Aktan, 2011; Sarikaya et al., 2014a).

Let us now consider a bidemensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leqslant tf(x, y) + (1-t)f(z, w) \tag{2}$$

holds, for all $(x, y), (z, w) \in \Delta$ and $t \in [0, 1]$. A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R},\ f_y(u) = f(u, y)$ and

* Corresponding author.
E-mail address: abdullahmat@gmail.com (A. Akkurt).
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$f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b]$ and $y \in [c, d]$ (Dragomir and Pearce, 2000).

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. A function $f : \Delta \to \mathbb{R}$ will be called co-ordinated convex on Δ, for all $t, s \in [0, 1]$ and $(x, y), (u, w) \in \Delta$, if the following inequality holds:

$$f(tx + (1-t)y, su + (1-s)w) \leqslant tsf(x, u) + s(1-t)f(y, u) + t(1-s)f(x, w) + (1-t)(1-s)f(y, w). \tag{3}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex (Dragomir, 2001). Several recent studies have expressed concerns on Hermite–Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$ (Sarikaya and Yaldiz, 2013; Ozdemir et al., 2011; Sarikaya et al., 2012; Sarikaya et al. (2014c)). More details, one can consult Sarikaya (2014), Sarikaya et al. (2014b) and Sarikaya (2015).

Earlier, Dragomir (2001) establish the following inequality of Hermite–Hadamard type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$. Later, another proof of a special version of the following theorem, using the definition of the co-ordinated convex function was reported (Sarikaya and Yaldiz, 2013).

Theorem 2. *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leqslant \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ &\leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ &\leqslant \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] \\ &\leqslant \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \end{aligned} \tag{4}$$

The above inequalities are sharp.

In the following section, some relevant definitions and mathematical preliminaries of fractional calculus theory are presented. For more details, one can consult Gorenflo and Mainardi (1997), Kilbas et al. (2006), Samko et al. (1993), Miller and Ross (1993).

Definition 3. Let $f \in L_1[a, b]$. The Riemann–Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geqslant 0$ are defined by

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a, \tag{5}$$

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b \tag{6}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function.

It is remarkable that Sarikaya et al. (2012) first give the following interesting integral inequalities of Hermite–Hadamard type involving Riemann–Liouville fractional integrals.

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leqslant a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)\right] \leqslant \frac{f(a)+f(b)}{2} \tag{7}$$

with $\alpha > 0$.

Meanwhile, Sarikaya et al. (2012) presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite–Hadamard type inequalities for convexity functions via Riemann–Liouville fractional integrals of the order $\alpha > 0$.

Lemma 5. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L_1[a, b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)\right] \tag{8}$$

$$= \frac{b-a}{2}\int_0^1 [(1-t)^{\alpha} - t^{\alpha}]f'(ta + (1-t)b)dt. \tag{9}$$

Definition 6. Let $f \in L_1([a, b] \times [c, d])$. The Riemann–Liouville integrals $J^{\alpha,\beta}_{a^+,c^+}$, $J^{\alpha,\beta}_{a^+,d^-}$, $J^{\alpha,\beta}_{b^-,c^+}$ and $J^{\alpha,\beta}_{b^-,d^-}$ of order $\alpha, \beta > 0$ with $a, c \geqslant 0$ are defined by

$$J^{\alpha,\beta}_{a^+,c^+}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}f(t,s)dsdt, \tag{10}$$

$$J^{\alpha,\beta}_{a^+,d^-}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt, \tag{11}$$

$$J^{\alpha,\beta}_{b^-,c^+}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1}f(t,s)dsdt, \tag{12}$$

and

$$J^{\alpha,\beta}_{b^-,d^-}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt, \tag{13}$$

respectively. Similar to Definitions 3 and 6 we introduce the following fractional integrals:

$$J^{\alpha}_{a^+}f\left(x, \frac{c+d}{2}\right) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f\left(t, \frac{c+d}{2}\right)dt, \tag{14}$$

$$J^{\alpha}_{b^-}f\left(x, \frac{c+d}{2}\right) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f\left(t, \frac{c+d}{2}\right)dt, \tag{15}$$

$$J^{\beta}_{c^+}f\left(\frac{a+b}{2}, y\right) = \frac{1}{\Gamma(\beta)}\int_c^y (y-s)^{\beta-1}f\left(\frac{a+b}{2}, s\right)ds, \tag{16}$$

$$J_{d^-}^{\beta}f\left(\frac{a+b}{2},y\right)=\frac{1}{\Gamma(\beta)}\int_y^d(s-y)^{\beta-1}f\left(\frac{a+b}{2},s\right)ds. \tag{17}$$

Objective of the present study is to state and prove the Hermite–Hadamard type inequality for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$. In order to achieve our goal, we first give two important identities and then by using these identities we prove some integral inequalities. We have obtained some results which are a simpler proof of the results presented by Sarikaya (2012).

2. Main results

To establish our main results, we need the following first identity:

Lemma 7. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$ and $f_{\sigma\tau}\in L(\Delta)$. Then the following equality holds:*

$$\begin{aligned}&\frac{4\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}\left[J_{a^+,c^+}^{\alpha,\beta}f(b,d)+J_{a^+,d^-}^{\alpha,\beta}f(b,c)+J_{b^-,c^+}^{\alpha,\beta}f(a,d)\right.\\&\left.+J_{b^-,d^-}^{\alpha,\beta}f(a,c)\right]-\frac{2\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b,c)+J_{a^+}^{\alpha}f(b,d)\right.\\&\left.+J_{b^-}^{\alpha}f(a,c)+J_{b^-}^{\alpha}f(a,d)\right]-\frac{2\Gamma(\beta+1)}{(d-c)^{\beta}}\left[J_{c^+}^{\beta}f(a,d)+J_{c^+}^{\beta}f(b,d)\right.\\&\left.+J_{d^-}^{\beta}f(a,c)+J_{d^-}^{\beta}f(b,c)\right]+F\\&=\frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}}\left\{\int_a^b\int_c^d\left[(b-x)^{\alpha-1}(d-y)^{\beta-1}\right.\right.\\&+(b-x)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}\\&\left.\left.+(x-a)^{\alpha-1}(d-y)^{\beta-1}\right]\times I(x,y)dydx\right\}\end{aligned} \tag{18}$$

where

$$\begin{aligned}I(x,y)=&\int_a^x\int_c^y f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma+\int_a^x\int_d^y f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma\\&+\int_b^x\int_c^y f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma+\int_b^x\int_d^y f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma,\end{aligned} \tag{19}$$

and

$$F=f(a,c)+f(a,d)+f(b,c)+f(b,d). \tag{20}$$

Proof. For any $x,t\in[a,b]$ and $s,y\in[c,d], x\neq t,\ s\neq y$, we have

$$\begin{aligned}\int_t^x\int_s^y f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma&=\int_t^x[f_{\sigma}(\sigma,y)-f_{\sigma}(\sigma,s)]d\sigma\\&=[f(\sigma,y)-f(\sigma,s)]|_t^x\\&=f(x,y)-f(x,s)-f(t,y)+f(t,s).\end{aligned} \tag{21}$$

Choose $t=a,\ s=c;\ t=a,\ s=d;\ t=b,\ s=c;\ t=b,\ s=d$ in (21), respectively, we get

$$I_1=\int_a^x\int_c^y f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma=f(x,y)-f(x,c)-f(a,y)+f(a,c), \tag{22}$$

$$I_2=\int_a^x\int_d^y f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma=f(x,y)-f(x,d)-f(a,y)+f(a,d), \tag{23}$$

$$I_3=\int_b^x\int_c^y f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma=f(x,y)-f(x,c)-f(b,y)+f(b,c), \tag{24}$$

and

$$I_4=\int_b^x\int_d^y f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma=f(x,y)-f(x,d)-f(b,y)+f(b,d). \tag{25}$$

Adding these four integrals side by side, we obtain

$$\begin{aligned}I(x,y)&=I_1+I_2+I_3+I_4\\&=4f(x,y)-2[f(x,c)+f(x,d)]-2[f(a,y)+f(b,y)]\\&\quad+f(a,c)+f(a,d)+f(b,c)+f(b,d).\end{aligned} \tag{26}$$

Multiplying (26) by $\frac{(b-x)^{\alpha-1}(d-y)^{\beta-1}}{4\Gamma(\alpha)\Gamma(\beta)}$ and integrating the resulting equality with respect to (x,y) on $[a,b]\times[c,d]$, we have

$$\begin{aligned}&\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1}I(x,y)dydx\\&=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1}f(x,y)dydx\\&\quad-\frac{1}{2\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1}[f(x,c)+f(x,d)]dx\\&\quad-\frac{1}{2\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1}[f(a,y)+f(b,y)]dydx\\&\quad+\frac{F}{4\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1}dydx.\end{aligned} \tag{27}$$

Thus, in (27) by means of simple calculations, we have

$$\begin{aligned}&J_{a^+,c^+}^{\alpha,\beta}(b,d)-\frac{(d-c)^{\beta}}{2\Gamma(\beta+1)}\left[J_{a^+}^{\alpha}f(b,c)+J_{a^+}^{\alpha}f(b,d)\right]\\&-\frac{(b-a)^{\alpha}}{2\Gamma(\alpha+1)}\left[J_{c^+}^{\beta}f(a,d)+J_{c^+}^{\beta}f(b,d)\right]+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}F\\&=\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1}I(x,y)dydx.\end{aligned} \tag{28}$$

Multiplying (26) by $\frac{(b-x)^{\alpha-1}(y-c)^{\beta-1}}{4\Gamma(\alpha)\Gamma(\beta)}$ and integrating the resulting equality with respect to (x,y) on $[a,b]\times[c,d]$, and by similar calculations, we have

$$\begin{aligned}&J_{a^+,d^-}^{\alpha,\beta}f(b,c)-\frac{(d-c)^{\beta}}{2\Gamma(\beta+1)}\left[J_{a^+}^{\alpha}f(b,c)+J_{a^+}^{\alpha}f(b,d)\right]\\&-\frac{(b-a)^{\alpha}}{2\Gamma(\alpha+1)}\left[J_{d^-}^{\beta}f(a,c)+J_{d^-}^{\beta}f(b,c)\right]+\frac{(b-a)^{\alpha}(d-c)^{\beta}}{4\Gamma(\alpha+1)\Gamma(\beta+1)}F\\&=\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-x)^{\alpha-1}(y-c)^{\beta-1}I(x,y)dydx.\end{aligned} \tag{29}$$

Multiplying (26) by $\frac{(x-a)^{\alpha-1}(y-c)^{\beta-1}}{4\Gamma(\alpha)\Gamma(\beta)}$ and integrating the resulting equality with respect to (x,y) on $[a,b]\times[c,d]$, we have

$$J^{\alpha,\beta}_{b^-,c^+}f(a,d)-\frac{(d-c)^\beta}{2\Gamma(\beta+1)}\left[J^\alpha_{b^-}f(a,c)+J^\alpha_{b^-}f(a,d)\right]-\frac{(b-a)^\alpha}{2\Gamma(\alpha+1)}\left[J^\beta_{c^+}f(a,d)+J^\beta_{c^+}f(b,d)\right]+\frac{(b-a)^\alpha(d-c)^\beta}{4\Gamma(\alpha+1)\Gamma(\beta+1)}F$$
$$=\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(x-a)^{\alpha-1}(y-c)^{\beta-1}I(x,y)dydx. \quad (30)$$

Multiplying (26) by $\frac{(x-a)^{\alpha-1}(d-y)^{\beta-1}}{4\Gamma(\alpha)\Gamma(\beta)}$ and integrating the resulting equality with respect to (x,y) on $[a,b]\times[c,d]$, we have

$$J^{\alpha,\beta}_{b^-,d^-}f(a,c)-\frac{(d-c)^\beta}{2\Gamma(\beta+1)}\left[J^\alpha_{b^-}f(a,c)+J^\alpha_{b^-}f(a,d)\right]-\frac{(b-a)^\alpha}{2\Gamma(\alpha+1)}\left[J^\beta_{d^-}f(a,c)+J^\beta_{d^-}f(b,c)\right]+\frac{(b-a)^\alpha(d-c)^\beta}{4\Gamma(\alpha+1)\Gamma(\beta+1)}F$$
$$=\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(x-a)^{\alpha-1}(d-y)^{\beta-1}I(x,y)dydx. \quad (31)$$

Adding these (28)–(31) side by side, which completes the proof.

Corollary 8. *If we take $\alpha=\beta=1$ in Lemma 7, we get*

$$\frac{4}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-\frac{2}{(b-a)}\int_a^b[f(x,c)+f(x,d)]dx-\frac{2}{(d-c)}\int_c^d[f(a,y)+f(b,y)]dy+F=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d I(x,y)dydx. \quad (32)$$

Theorem 9. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$ and $f_{\sigma\tau}\in L(\Delta)$. If $f_{\sigma\tau}\in L_\infty(\Delta)$, i.e $\|f_{\sigma\tau}\|_\infty=\sup_{(\sigma,\tau)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(\sigma,\tau)}{\partial\sigma\partial\tau}\right|<\infty$, then one has the inequality:*

$$\Bigg|\frac{4\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]-\frac{2\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{a^+}f(b,c)+J^\alpha_{a^+}f(b,d)+J^\alpha_{b^-}f(a,c)+J^\alpha_{b^-}f(a,d)\right]-\frac{2\Gamma(\beta+1)}{(d-c)^\beta}\left[J^\beta_{c^+}f(a,d)+J^\beta_{c^+}f(b,d)+J^\beta_{d^-}f(a,c)+J^\beta_{d^-}f(b,c)\right]+F\Bigg|\leqslant 4\|f_{\sigma\tau}\|_\infty(b-a)(d-c). \quad (33)$$

Proof. From Lemma 7, taking the modulus, it follows that

$$|J|=\Bigg|\frac{4\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]-\frac{2\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{a^+}f(b,c)+J^\alpha_{a^+}f(b,d)+J^\alpha_{b^-}f(a,c)+J^\alpha_{b^-}f(a,d)\right]-\frac{2\Gamma(\beta+1)}{(d-c)^\beta}\left[J^\beta_{c^+}f(a,d)+J^\beta_{c^+}f(b,d)+J^\beta_{d^-}f(a,c)+J^\beta_{d^-}f(b,c)\right]+F\Bigg| \quad (34)$$

$$\leqslant\frac{\alpha\beta}{(b-a)^\alpha(d-c)^\beta}\left\{\int_a^b\int_c^d\left[(b-x)^{\alpha-1}(d-y)^{\beta-1}+(b-x)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(d-y)^{\beta-1}\right]\times\left[\int_a^x\int_c^y|f_{\sigma\tau}(\sigma,\tau)|d\tau d\sigma+\int_a^x\int_y^d|f_{\sigma\tau}(\sigma,\tau)|d\tau d\sigma+\int_x^b\int_c^y|f_{\sigma\tau}(\sigma,\tau)|d\tau d\sigma+\int_x^b\int_y^d|f_{\sigma\tau}(\sigma,\tau)|d\tau d\sigma\right]dydx\right\}. \quad (35)$$

Since $f_{\sigma\tau}\in L_\infty(\Delta)$, we get

$$|J|\leqslant\frac{\alpha\beta\|f_{\sigma\tau}\|_\infty}{(b-a)^\alpha(d-c)^\beta}\left\{\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1}\left[\int_a^b\int_c^d d\tau d\sigma\right]dydx+\int_a^b\int_c^d(b-x)^{\alpha-1}(y-c)^{\beta-1}\left[\int_a^b\int_c^d d\tau d\sigma\right]dydx \quad (36)\right.$$

$$\left.+\int_a^b\int_c^d(x-a)^{\alpha-1}(y-c)^{\beta-1}\left[\int_a^b\int_c^d d\tau d\sigma\right]dydx+\int_a^b\int_c^d(x-a)^{\alpha-1}(d-y)^{\beta-1}\left[\int_a^b\int_c^d d\tau d\sigma\right]dydx\right\}$$
$$=\frac{\alpha\beta\|f_{\sigma\tau}\|_\infty}{(b-a)^\alpha(d-c)^\beta}\frac{4(b-a)^{\alpha+1}}{\alpha}\frac{(d-c)^{\beta+1}}{\beta}$$
$$=4\|f_{\sigma\tau}\|_\infty(b-a)(d-c). \quad (37)$$

This completes the proof.

Corollary 10. *If we take $\alpha=\beta=1$ in Theorem 9, we get*

$$\left|\frac{4}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-\frac{2}{(b-a)}\int_a^b[f(x,c)+f(x,d)]dx-\frac{2}{(d-c)}\int_c^d[f(a,y)+f(b,y)]dy+F\right|\leqslant 4\|f_{\sigma\tau}\|_\infty(b-a)(d-c). \quad (38)$$

Theorem 11. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$ and $f_{\sigma\tau}\in L(\Delta)$. If $|f_{\sigma\tau}|$ is a convex function on the co-ordinates on Δ, then the following inequality holds:*

$$\Bigg|\frac{4\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]-\frac{2\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{a^+}f(b,c)+J^\alpha_{a^+}f(b,d)+J^\alpha_{b^-}f(a,c)+J^\alpha_{b^-}f(a,d)\right]-\frac{2\Gamma(\beta+1)}{(d-c)^\beta}\left[J^\beta_{c^+}f(a,d)+J^\beta_{c^+}f(b,d)+J^\beta_{d^-}f(a,c)+J^\beta_{d^-}f(b,c)\right]+F\Bigg|\leqslant(b-a)(d-c)\times[|f_{\sigma\tau}(a,c)+||f_{\sigma\tau}(a,d)|+|f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|] \quad (39)$$

Proof. Since $|f_{\sigma\tau}(\sigma,\tau)|$ is co-ordinates on Δ, we know that $x\in[a,b],\ y\in[c,d]$

$$|f_{\sigma\tau}(\sigma,\tau)| = \left|f_{\sigma\tau}\left(\frac{b-\sigma}{b-a}a+\frac{\sigma-a}{b-a}b, \frac{d-\tau}{d-c}c+\frac{\tau-c}{d-c}d\right)\right| \leqslant \frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(a,c)|+\frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(a,d)| + \frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(b,c)|+\frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(b,d)|. \tag{40}$$

From Lemma 7, we have

$$|J| \leqslant \frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}}\left\{\int_a^b\int_c^d\left[(b-x)^{\alpha-1}(d-y)^{\beta-1} + (b-x)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1} + (x-a)^{\alpha-1}(d-y)^{\beta-1}\right]\times\left[\int_a^x\int_c^y|f_{\sigma\tau}(\sigma,\tau)|d\tau d\sigma + \int_a^x\int_y^d|f_{\sigma\tau}(\sigma,\tau)|d\tau d\sigma+\int_x^b\int_c^y|f_{\sigma\tau}(\sigma,\tau)|d\tau d\sigma + \int_x^b\int_y^d|f_{\sigma\tau}(\sigma,\tau)|d\tau d\sigma\right]dydx\right\} \tag{41}$$

By using co-ordinated convexity of $|f_{\sigma\tau}|$, we get

$$|J| \leqslant \frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}}\left\{\int_a^b\int_c^d\left[(b-x)^{\alpha-1}(d-y)^{\beta-1} + (b-x)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1} + (x-a)^{\alpha-1}(d-y)^{\beta-1}\right]\times\left(\int_a^x\int_c^y\left[\frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(a,c)| + \frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(a,d)|+\frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(b,c)| + \frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(b,d)|\right]d\tau d\sigma+\int_a^x\int_y^d\left[\frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(a,c)| + \frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(a,d)|+\frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(b,c)| + \frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(b,d)|\right]d\tau d\sigma\right. \tag{42}$$

$$+\int_x^b\int_c^y\left[\frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(a,c)|+\frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(a,d)| + \frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(b,c)|+\frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(b,d)|\right]d\tau d\sigma + \int_x^b\int_y^d\left[\frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(a,c)|+\frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(a,d)| + \frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(b,c)|+\frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(b,d)|\right]d\tau d\sigma\left.\vphantom{\int}\right)dydx\Bigg\} \tag{43}$$

$$=\frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}}\left\{\int_a^b\int_c^d\left[(b-x)^{\alpha-1}(d-y)^{\beta-1} + (b-x)^{\alpha-1}(y-c)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1} + (x-a)^{\alpha-1}(d-y)^{\beta-1}\right]\times\left(\int_a^b\int_c^d\left[\frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(a,c)| + \frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(a,d)|+\frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(b,c)| + \frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(b,d)|\right]d\tau d\sigma\right\}dydx = A_1+A_2+A_3+A_4. \tag{44}$$

With a simple calculation, we have

$$A_1=\frac{\alpha\beta}{(b-a)^{\alpha+1}(d-c)^{\beta+1}}\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1} \times\left\{\int_a^b\int_c^d[(b-\sigma)(d-\tau)|f_{\sigma\tau}(a,c)|+(b-\sigma)(\tau-c)|f_{\sigma\tau}(a,d)| + (\sigma-a)(d-\tau)|f_{\sigma\tau}(b,c)|+(\sigma-a)(\tau-c)|f_{\sigma\tau}(b,d)|]d\tau d\sigma\right\}dydx \tag{45}$$

$$=\frac{\alpha\beta}{(b-a)^{\alpha+1}(d-c)^{\beta+1}} \times\left\{\frac{(b-a)^{\alpha+2}}{2\alpha}\frac{(d-c)^{\beta+2}}{2\beta}[|f_{\sigma\tau}(a,c)+||f_{\sigma\tau}(a,d)| + |f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|]\right. = \frac{(b-a)(d-c)}{4}[|f_{\sigma\tau}(a,c)+||f_{\sigma\tau}(a,d)| + |f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|]. \tag{46}$$

Similarly, we also have the following equalities

$$A_2=\frac{\alpha\beta}{(b-a)^{\alpha+1}(d-c)^{\beta+1}}\int_a^b\int_c^d(b-x)^{\alpha-1}(y-c)^{\beta-1} \times\left\{\int_a^b\int_c^d[(b-\sigma)(d-\tau)|f_{\sigma\tau}(a,c)|+(b-\sigma)(\tau-c)|f_{\sigma\tau}(a,d)| + (\sigma-a)(d-\tau)|f_{\sigma\tau}(b,c)|+(\sigma-a)(\tau-c)|f_{\sigma\tau}(b,d)|]d\tau d\sigma\right\}dydx =\frac{(b-a)(d-c)}{4}[|f_{\sigma\tau}(a,c)+||f_{\sigma\tau}(a,d)|+|f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|], \tag{47}$$

$$A_3=\frac{\alpha\beta}{(b-a)^{\alpha+1}(d-c)^{\beta+1}}\int_a^b\int_c^d(x-a)^{\alpha-1}(y-c)^{\beta-1} \times\left\{\int_a^b\int_c^d[(b-\sigma)(d-\tau)|f_{\sigma\tau}(a,c)|+(b-\sigma)(\tau-c)|f_{\sigma\tau}(a,d)| + (\sigma-a)(d-\tau)|f_{\sigma\tau}(b,c)|+(\sigma-a)(\tau-c)|f_{\sigma\tau}(b,d)|]d\tau d\sigma\right\}dydx =\frac{(b-a)(d-c)}{4}[|f_{\sigma\tau}(a,c)+||f_{\sigma\tau}(a,d)|+|f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|] \tag{48}$$

and

$$A_4=\frac{\alpha\beta}{(b-a)^{\alpha+1}(d-c)^{\beta+1}}\int_a^b\int_c^d(x-a)^{\alpha-1}(d-y)^{\beta-1} \times\left\{\int_a^b\int_c^d[(b-\sigma)(d-\tau)|f_{\sigma\tau}(a,c)|+(b-\sigma)(\tau-c)|f_{\sigma\tau}(a,d)| + (\sigma-a)(d-\tau)|f_{\sigma\tau}(b,c)|+(\sigma-a)(\tau-c)|f_{\sigma\tau}(b,d)|]d\tau d\sigma\right\}dydx =\frac{(b-a)(d-c)}{4}[|f_{\sigma\tau}(a,c)+||f_{\sigma\tau}(a,d)|+|f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|]. \tag{49}$$

Adding these (46)–(49) side by side, if we put in (44), we obtain (39). This completes the proof of the theorem.

Corollary 12. *If we take $\alpha = \beta = 1$ in Theorem 11, we get*

$$\left|\frac{4}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx - \frac{2}{(b-a)}\int_a^b[f(x,c)+f(x,d)]dx - \frac{2}{(d-c)}\int_c^d[f(a,y)+f(b,y)]dy + F\right| \leqslant (b-a)(d-c)[|f_{\sigma\tau}(a,c) + ||f_{\sigma\tau}(a,d)|+|f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|]. \tag{50}$$

Lemma 13. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b, c<d$ and $f_{\sigma\tau}\in L(\Delta)$. Then the following equality holds:*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{\Gamma(\beta+1)}{2(d-c)^\beta}\left\{J^\beta_{c^+}f\left(\frac{a+b}{2},d\right)+J^\beta_{d^-}f\left(\frac{a+b}{2},c\right)\right\}$$
$$-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left\{J^\alpha_{b^-}f\left(a,\frac{c+d}{2}\right)+J^\alpha_{a^+}f\left(b,\frac{c+d}{2}\right)\right\}$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)\right.$$
$$\left.+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]=\frac{\alpha\beta}{4(b-a)^\alpha(d-c)^\beta}\int_a^b\int_c^d\left\{\left[(b-t)^{\alpha-1}\right.\right.$$
$$\left.+(t-a)^{\alpha-1}\right]\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]$$
$$\times\left(\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma\right)\Bigg\}dsdt. \tag{51}$$

Proof. Choose $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (21), we have

$$\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma=f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},s\right)-f\left(t,\frac{c+d}{2}\right)+f(t,s). \tag{52}$$

Multiplying (52) by $\frac{(b-t)^{\alpha-1}(d-s)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$ and integrating the resulting equality with respect to (s,t) on $[a,b]\times[c,d]$, we get

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}\left\{\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma\right\}dsdt$$
$$=\frac{f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}dsdt$$
$$-\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}f\left(\frac{a+b}{2},s\right)dsdt$$
$$-\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}f\left(t,\frac{c+d}{2}\right)dsdt$$
$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}f(t,s)dsdt. \tag{53}$$

By simple calculations, we have

$$\frac{(b-a)^\alpha(d-c)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}J^\beta_{c^+}f\left(\frac{a+b}{2},d\right)$$
$$-\frac{(d-c)^\beta}{\Gamma(\beta+1)}J^\alpha_{a^+}f\left(b,\frac{c+d}{2}\right)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)$$
$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}$$
$$\times\left\{\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma\right\}dsdt. \tag{54}$$

Multiplying (52) by $\frac{(b-t)^{\alpha-1}(s-c)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$, integrating the resulting equality with respect to (s,t) on $[a,b]\times[c,d]$, and by similar methods above we have

$$\frac{(b-a)^\alpha(d-c)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}J^\beta_{d^-}f\left(\frac{a+b}{2},c\right)$$
$$-\frac{(d-c)^\beta}{\Gamma(\beta+1)}J^\alpha_{a^+}f\left(b,\frac{c+d}{2}\right)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)$$
$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}$$
$$\times\left\{\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma\right\}dsdt. \tag{55}$$

Multiplying (52) by $\frac{(t-a)^{\alpha-1}(d-s)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$ integrating the resulting equality with respect to (s,t) on $[a,b]\times[c,d]$, and by similar methods above we have

$$\frac{(b-a)^\alpha(d-c)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}J^\beta_{c^+}f\left(\frac{a+b}{2},d\right)$$
$$-\frac{(d-c)^\beta}{\Gamma(\beta+1)}J^\alpha_{b^-}f\left(a,\frac{c+d}{2}\right)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)$$
$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}\left\{\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma\right\}dsdt. \tag{56}$$

Multiplying (52) by $\frac{(t-a)^{\alpha-1}(s-c)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$ integrating the resulting equality with respect to (s,t) on $[a,b]\times[c,d]$, and by similar methods above we have

$$\frac{(b-a)^\alpha(d-c)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}J^\beta_{d^-}f\left(\frac{a+b}{2},c\right)$$
$$-\frac{(d-c)^\beta}{\Gamma(\beta+1)}J^\alpha_{b^-}f\left(a,\frac{c+d}{2}\right)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)$$
$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d(b-t)^{\alpha-1}(d-s)^{\beta-1}\left\{\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma\right\}dsdt. \tag{57}$$

Adding these (54)–(57) side by side and multiplying both sides by $\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}$, we get the desired equality (51).

Corollary 14. *If we take $\alpha=\beta=1$ in Lemma 13, we get*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx$$
$$-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx$$
$$=\frac{1}{16(b-a)(d-c)}\int_a^b\int_c^d\left\{\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma\right\}dsdt. \tag{58}$$

Theorem 15. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $f_{\sigma\tau}\in L_\infty(\Delta)$, then the following equality holds:*

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{\Gamma(\beta+1)}{2(d-c)^\beta}\left\{J^\beta_{c^+}f\left(\frac{a+b}{2},d\right)+J^\beta_{d^-}f\left(\frac{a+b}{2},c\right)\right\}\right.$$
$$-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left\{J^\alpha_{b^-}f\left(a,\frac{c+d}{2}\right)+J^\alpha_{a^+}f\left(b,\frac{c+d}{2}\right)\right\}$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)\right.$$
$$\left.\left.+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\right]\right|\leqslant\frac{\|f_{\sigma\tau}\|_\infty(b-a)(d-c)}{4}$$
$$\times\left[\frac{(2^{1-\alpha}+(\alpha-1))}{\alpha+1}\frac{(2^{1-\beta}+(\beta-1))}{\beta+1}\right]. \tag{59}$$

Proof. In Lemma 13, taking the modulus, it follows that

$$\Bigg|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\left\{J^{\beta}_{c^+}f\left(\frac{a+b}{2},d\right)+J^{\beta}_{d^-}f\left(\frac{a+b}{2},c\right)\right\}$$
$$-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left\{J^{\alpha}_{b^-}f\left(a,\frac{c+d}{2}\right)+J^{\alpha}_{a^+}f\left(b,\frac{c+d}{2}\right)\right\}$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\Big[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)$$
$$+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\Big]\Bigg| \tag{60}$$

$$\leqslant\frac{\alpha\beta\|f_{\sigma\tau}\|_{\infty}}{4(b-a)^{\alpha}(d-c)^{\beta}}\int_a^b\int_c^d\left\{\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\right.$$
$$\left.\times\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\times\left|\frac{a+b}{2}-t\right|\left|\frac{c+d}{2}-s\right|\right\}dsdt$$
$$=\frac{\|f_{\sigma\tau}\|_{\infty}(b-a)(d-c)}{4}\left[\frac{(2^{1-\alpha}+(\alpha-1))}{\alpha+1}\frac{(2^{1-\beta}+(\beta-1))}{\beta+1}\right] \tag{61}$$

for $f_{\sigma\tau}\in L_{\infty}(\Delta)$.

Remark 16. If we take $\alpha=\beta=1$ in Theorem 15, we get

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right|\leqslant\frac{\|f_{\sigma\tau}\|_{\infty}}{16}(b-a)(d-c). \tag{62}$$

which is proved by Sarikaya in Sarikaya (2012).

Theorem 17. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $|f_{\sigma\tau}|$ is a convex function on the co-ordinates on Δ, then the following equality holds:*

$$\Bigg|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\left\{J^{\beta}_{c^+}f\left(\frac{a+b}{2},d\right)+J^{\beta}_{d^-}f\left(\frac{a+b}{2},c\right)\right\}$$
$$-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left\{J^{\alpha}_{b^-}f\left(a,\frac{c+d}{2}\right)+J^{\alpha}_{a^+}f\left(b,\frac{c+d}{2}\right)\right\}$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\Big[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)$$
$$+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\Big]\Bigg| \tag{63}$$

$$\leqslant(b-a)(d-c)\frac{\alpha2^{\alpha}-(\alpha+1)2^{\alpha-1}+1}{2^{\alpha}(\alpha+1)}\frac{\beta2^{\beta}-(\beta+1)2^{\beta-1}+1}{2^{\beta}(\beta+1)}$$
$$\times\frac{|f_{\sigma\tau}(a,c)|+|f_{\sigma\tau}(a,d)|+|f_{\sigma\tau}(b,c)|+|f_{\sigma\tau}(b,d)|}{4}. \tag{64}$$

Proof. Since $|f_{\sigma\tau}(\sigma,\tau)|$ is co-ordinates on Δ, we know that $t\in[a,b],\ s\in[c,d]$

$$|f_{\sigma\tau}(\sigma,\tau)|=\left|f_{\sigma\tau}\left(\frac{b-\sigma}{b-a}a+\frac{\sigma-a}{b-a}b,\frac{d-\tau}{d-c}c+\frac{\tau-c}{d-c}d\right)\right|$$
$$\leqslant\frac{b-\sigma}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(a,c)|+\frac{b-\sigma}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(a,d)|$$
$$+\frac{\sigma-a}{b-a}\frac{d-\tau}{d-c}|f_{\sigma\tau}(b,c)|+\frac{\sigma-a}{b-a}\frac{\tau-c}{d-c}|f_{\sigma\tau}(b,d)|. \tag{65}$$

From Lemma 13, using co-ordinated convexity of $|f_{\sigma\tau}|$, we have

$$\Bigg|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\left\{J^{\beta}_{c^+}f\left(\frac{a+b}{2},d\right)+J^{\beta}_{d^-}f\left(\frac{a+b}{2},c\right)\right\}$$
$$-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left\{J^{\alpha}_{b^-}f\left(a,\frac{c+d}{2}\right)+J^{\alpha}_{a^+}f\left(b,\frac{c+d}{2}\right)\right\}$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\Big[J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)$$
$$+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)\Big]\Bigg| \tag{66}$$

$$\leqslant\frac{\alpha\beta}{4(b-a)^{\alpha}(d-c)^{\beta}}\int_a^b\int_c^d\left\{\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\right.$$
$$\left.\times\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\times\left|\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}|f_{\sigma\tau}(\sigma,\tau)|d\tau d\sigma\right|\right\}dsdt \tag{67}$$

$$\leqslant\frac{\alpha\beta}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}}\int_a^b\int_c^d\left\{\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\right.$$
$$\times\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\times\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}[(b-\sigma)(d-\tau)|f_{\sigma\tau}(a,c)|$$
$$+(b-\sigma)(\tau-c)|f_{\sigma\tau}(a,d)|+(\sigma-a)(d-\tau)|f_{\sigma\tau}(b,c)|$$
$$\left.+(\sigma-a)(\tau-c)|f_{\sigma\tau}(b,d)|\Big]d\tau d\sigma\right\}dsdt=K_1+K_2+K_3+K_4. \tag{68}$$

With a simple calculation, we have

$$K_1=\frac{\alpha\beta}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}}\int_a^b\int_c^d\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]$$
$$\times\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\times|f_{\sigma\tau}(a,c)|$$
$$\times\left|\int_t^{\frac{a+b}{2}}\int_s^{\frac{c+d}{2}}(b-\sigma)(d-\tau)d\tau d\sigma\right|dsdt=\frac{\alpha\beta|f_{\sigma\tau}(a,c)|}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}}$$
$$\times\left[\int_a^b\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\left|\int_t^{\frac{a+b}{2}}(b-\sigma)d\sigma\right|dt\right]$$
$$\times\left[\int_c^d\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\left|\int_s^{\frac{c+d}{2}}(d-\tau)d\tau\right|ds\right] \tag{69}$$

$$=\frac{\alpha\beta|f_{\sigma\tau}(a,c)|}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}}\left[\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\int_t^{\frac{a+b}{2}}(b-\sigma)d\sigma dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^b\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\int_{\frac{a+b}{2}}^t(b-\sigma)d\sigma dt\right]$$
$$\times\left[\int_c^{\frac{c+d}{2}}\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\int_s^{\frac{c+d}{2}}(d-\tau)d\tau ds\right.$$
$$\left.+\int_{\frac{c+d}{2}}^d\left[(d-s)^{\beta-1}+(s-c)^{\beta-1}\right]\int_{\frac{c+d}{2}}^s(d-\tau)d\tau ds\right]$$
$$=\frac{|f_{\sigma\tau}(a,c)|}{4}\frac{\alpha2^{\alpha}-(\alpha+1)2^{\alpha-1}+1}{2^{\alpha}(\alpha+1)}\frac{\beta2^{\beta}-(\beta+1)2^{\beta-1}+1}{2^{\beta}(\beta+1)}(b-a)(d-c). \tag{70}$$

Similarly, we also have the following equalities

$$K_2 = \frac{\alpha\beta}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}} \int_a^b \int_c^d \left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right] \times \left[(d-s)^{\beta-1} + (s-c)^{\beta-1}\right] \times |f_{\sigma\tau}(a,d)| \left| \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} (b-\sigma)(\tau-c)d\tau d\sigma \right| dsdt = \frac{|f_{\sigma\tau}(a,d)|}{4} \frac{\alpha 2^\alpha - (\alpha+1)2^{\alpha-1}+1}{2^\alpha(\alpha+1)} \frac{\beta 2^\beta - (\beta+1)2^{\beta-1}+1}{2^\beta(\beta+1)}(b-a)(d-c), \quad (71)$$

$$K_3 = \frac{\alpha\beta}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}} \int_a^b \int_c^d \left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right] \left[(d-s)^{\beta-1} + (s-c)^{\beta-1}\right] \times |f_{\sigma\tau}(b,c)| \left| \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} (\sigma-a)(d-\tau)d\tau d\sigma \right| dsdt = \frac{|f_{\sigma\tau}(b,c)|}{4} \frac{\alpha 2^\alpha - (\alpha+1)2^{\alpha-1}+1}{2^\alpha(\alpha+1)} \frac{\beta 2^\beta - (\beta+1)2^{\beta-1}+1}{2^\beta(\beta+1)}(b-a)(d-c) \quad (72)$$

and

$$K_4 = \frac{\alpha\beta}{4(b-a)^{\alpha+1}(d-c)^{\beta+1}} \int_a^b \int_c^d \left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right] \times \left[(d-s)^{\beta-1} + (s-c)^{\beta-1}\right] \times |f_{\sigma\tau}(b,d)| \left| \int_t^{\frac{a+b}{2}} \int_s^{\frac{c+d}{2}} (\sigma-a)(\tau-c)d\tau d\sigma \right| dsdt = \frac{|f_{\sigma\tau}(b,d)|}{4} \frac{\alpha 2^\alpha - (\alpha+1)2^{\alpha-1}+1}{2^\alpha(\alpha+1)} \frac{\beta 2^\beta - (\beta+1)2^{\beta-1}+1}{2^\beta(\beta+1)}(b-a)(d-c). \quad (73)$$

Thus, if we put the last four equalities in (68), we obtain (64). This completes the proof of the theorem.

Corollary 18. *If we take $\alpha = \beta = 1$ in Theorem 17, we get*

$$\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) dx - \frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right) dy + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx \right| \leqslant \frac{(b-a)(d-c)}{16} \times \left(\frac{|f_{\sigma\tau}(a,c)| + |f_{\sigma\tau}(a,d)| + |f_{\sigma\tau}(b,c)| + |f_{\sigma\tau}(b,d)|}{4} \right). \quad (74)$$

3. Conclusion

In this work we give two identities for functions of two variables and apply them to give new Hermite–Hadamard type Fractional integral inequalities for double Fractional integrals involving functions whose derivatives are bounded or co-ordinates convex function on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b,\ c < d$.

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