Journal of King Saud University – Science 31 (2019) 1070–1076

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Haar wavelet series solution for solving neutral delay differential equations

Akmal Raza *, Arshad Khan *

Department of Mathematics, Jamia Millia Islamia, New Delhi 110025, India

ARTICLE INFO

Article history:
Received 13 July 2018
Accepted 17 September 2018
Available online 19 September 2018

2000 Mathematics Subject Classification:
65M99
65N35
65N55
65L10

Keywords:
Haar wavelet
Neutral delay differential equations (NDDE)
Collocation points

ABSTRACT

We solved linear and nonlinear neutral delay differential equations using Haar wavelet series. We apply Haar wavelet and obtain its integration for neutral delay differential equations with respect to the collocation points to obtain the numerical solution of the problems. Six problems are solved and results are compared with the exact solution and existing methods from literature to show the accuracy and applicability of the Haar wavelet series. On increasing the level of resolutions maximum absolute error decreases uniformly.

1. Introduction

Let us assume the neutral delay differential equation (NDDE) of the type

$$y'(x) = f(x,\ y(x),\ y(x-\mu(x,\ y(x))),\ y'(x-\nu(x,\ y(x)))),\quad x_1 \leqslant x \leqslant x_f,$$

with

$$y(x) = \phi(x) \qquad x \leqslant x_1. \tag{1.1}$$

where $f : [x_1,\ x_f] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function, $\mu(x,\ y(x))$ and $\nu(x,\ y(x))$ are continuous functions on $[x_1,\ x_f] \times \mathbb{R}$ such that $x - \mu(x,\ y(x)) < x_f$ and $x - \nu(x,\ y(x)) < x_f$. Also $\phi(x)$ represents the initial function (Karimi and Aminataei, 2008).

This problem can be considered as a particular class of delay differential equation which arises frequently in mathematical modelling of biological, physiological, chemical process, electronic, transportation system (control of ships and aircrafts), neural networks and economic growth. Neutral delay differential equation systems are of immense interest by many researchers (Driver, 1977; El'sgol'ts and Norkin, 1973; Gopalsamy, 1992; Halanay, 1966; Jamshidi and Wang, 1984; Kuang, 1993; Kolmanovskii and Myshkis, 1992; Lelarsmee et al., 1982). Some researchers replaces the delay differential equation by a system of ordinary differential equation to find the solution, which inherit partial differences among delay differential equation and system of ordinary differential equation (Kuang, 1993). So it is good to solve delay differential equation independently. Many researchers try to solve delay differential equation using Runge–Kutta method (Bellen and Zennaro, 2004), Radau, waveform relaxation (Lelarsmee et al., 1982) and Bellman's method (Bellen and Zennaro, 2004). Wang et al. used Runge–Kutta-type methods (Wang et al., 2009a), one-leg Θ-method (Wang et al., 2009b), variational iteration method (Wang and Chen, 2010) and Xueqin Lv and Yue Gao used reproducing kernel Hilbert space method (RKHSM) (Lv and Gao, 2012) to solve NDDE. In this paper we solved NDDE using Haar wavelet which transform the delay differential equation into algebraic equations. We describe the basic Haar wavelet and their integration in Section 2, solution of neutral delay differential equation using Haar wavelet is described in Section 3. We have shown six benchmark problems and report the maximum absolute error of each problem and compared with exact solution in Section 4.

* Corresponding authors.
E-mail addresses: akmalrazataqvi@gmail.com (A. Raza), akhan2@jmi.ac.in (A. Khan).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2018.09.013
1018-3647/

2. Haar wavelet

Haar wavelet has excellent features, compact support, symmetry, orthogonality and better approximations of square integrable functions (Chui, 1992; Debnath and Shah, 2015; Yves, 1999). Thats why Haar wavelet method is becoming more attractive among researchers. Solution of lumped and distributed-parameter system was given by Chen and Hsiao using Haar wavelet (Chen and Hsiao, 1997), after that Lepik (2007), Lepik and Hein (2014) and Islam et al. (2010) solved the ordinary differential equation by Haar wavelet. Also many researchers are using wavelets to solve the ordinary and partial differential equations, which can be seen in Oruç (2017a, 2018, 2017b), Pandit and Kumar (2012, 2014).

Table 1
Comparison between analytical solution and Haar wavelet series method for problem 1 with J = 2.

x($=\frac{1}{16}$)	Exact	HWSM	Error
1	1.0645	1.0643	2.000E−004
3	1.2062	1.2159	9.600E−002
5	1.3668	1.3856	1.880E−002
7	1.5488	1.5516	2.700E−002
9	1.7551	1.7450	1.01E−002
11	1.9887	1.9893	6.000E−003
13	2.2535	2.2575	4.000E−003
15	2.5536	2.5525	1.100E−003

Table 2
Maximum absolute errors with different resolutions for problem 1.

Level of Resolution J	Max. abs. Error
4	1.8877 E−004
5	3.1031 E−005
6	3.4746 E−006
7	1.5236 E−006
8	2.2379 E−007
9	3.7991 E−008
10	5.8536 E−009

The Haar wavelet family for $x \in [0,1]$ is given as follows (Gu and Jiang, 1996; Oruç, 2017a).

$$h_i(x) = \begin{cases} 1, & \zeta_1(i) \leqslant x < \zeta_2(i) \\ -1, & \zeta_2(i) \leqslant x < \zeta_3(i) \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where i indicates the wavelet number and
$\zeta_1(i) = \frac{k}{m}, \quad \zeta_2(i) = \frac{k+0.5}{m}, \quad \zeta_3(i) = \frac{k+1}{m}$,
$m = 2^j,\ j = 0, 1, 2 \ldots, J$, and integer $k = 0, 1 \ldots, m-1$.

J indicates the level of resolution, k represents the translations parameter and index i is calculated as $i = m + k + 1$ which is true for $i \geqslant 2$. For $i = 1$ the Haar wavelet is given as

$$h_1(x) = \begin{cases} 1, & 0 \leqslant x < 1 \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

The Integration of Haar wavelet can be obtained by following (Gu and Jiang, 1996; Lepik and Hein, 2014):

$$I_1 h_i(x) = \begin{cases} x - \zeta_1(i), & \zeta_1(i) \leqslant x < \zeta_2(i) \\ \zeta_3(i) - x, & \zeta_2(i) \leqslant x < \zeta_3(i) \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

Any square integrable function $f(x) \in L^2[0,1]$ can be approximated by the dilation and translation of Haar wavelet as (Lepik and Hein, 2014)

$$f(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{2.4}$$

The Haar wavelet series coefficients a_i are calculated as:

$$a_i = < y(x), h_i(x) > = \int_0^1 y(x).\overline{h_i(x)}dx. \tag{2.5}$$

Collocation points are given as

$$X(u) = \frac{2u-1}{2M}, \tag{2.6}$$

$u = 1, 2, \ldots, 2M$ and $M = 2^J$.

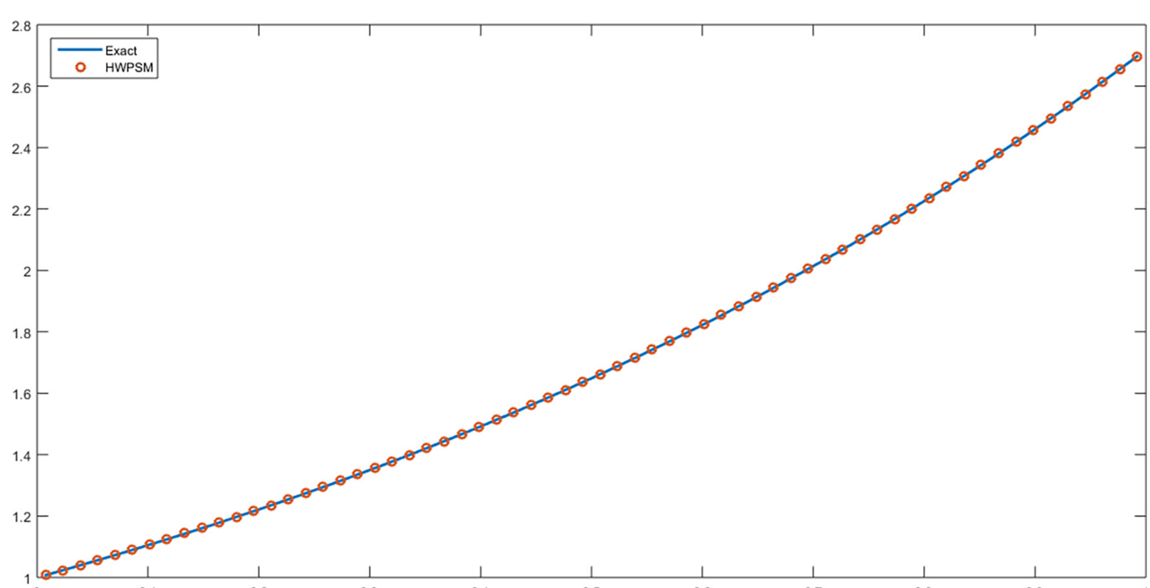

Fig. 1. Graph of problem 1 for J = 5.

3. Haar wavelet series solution of neutral delay differential equations

Let us assume the following neutral delay differential equation

$$y'(x) = f(x,\ y(x),\ y(x-\mu(x,\ y(x))),\ y'(x-\nu(x,\ y(x)))),\quad x_1 \leqslant x \leqslant x_f, \tag{3.1}$$

with

$y(x) = \phi(x) \qquad x \leqslant x_1.$

Let the haar wavelet series of first derivative is

$$y'(x) = \sum_{i=1}^{2M} a_i h_i(x). \tag{3.2}$$

Now integrating from 0 to x w.r.t to x we get

$$y(x) = \sum_{i=1}^{2M} a_i I_1 h_i(x) + y(0). \tag{3.3}$$

Also

$$y'(x-\nu(x,y(x))) = \sum_{i=1}^{2M} a_i h_i(x-\nu(x,y(x))). \tag{3.4}$$

$$y(x-\mu(x,y(x))) = \sum_{i=1}^{2M} a_i I_1 h_i(x-\mu(x,y(x))) + y(0). \tag{3.5}$$

Using Eqs. (3.2)–(3.5) in Eq. (3.1), we get the following system of algebraic equations

Table 3
Comparison between analytical solution and Haar wavelet series method for problem 2 with J = 2.

$x(=\frac{1}{16})$	Exact	HWM	Error
1	0.0625	0.0629	4.0000E−003
3	0.1864	0.1886	2.200E−003
5	0.3074	0.3099	2.500E−003
7	0.4237	0.4243	6.000E−003
9	0.5333	0.5325	8.000E−003
11	0.6346	0.6386	6.000E−003
13	0.7260	0.7112	1.480E−002
15	0.8061	0.8830	2.310E−002

Table 5
Comparison between analytical solution and Haar wavelet method for problem 3 with J = 2.

$x(=\frac{1}{16})$	Exact	HWM	Error
1	0.0625	0.0629	4.000E−004
3	0.1864	0.1886	2.200E−003
5	0.3074	0.3099	2.500E−003
7	0.4237	0.4243	6.000E−003
9	0.5333	0.5325	8.000E−003
11	0.6346	0.6286	6.000E−002
13	0.7260	0.7112	1.480E−002
15	0.7806	0.7830	2.310E−003

Table 4
Maximum absolute errors with different resolutions for problem 2.

Level of Resolution J	Max. abs. Error
3	1.5000 E−003
4	8.2354 E−004
5	3.8198 E−004
6	2.0353 E−004
7	1.0458 E−004
8	5.3230E−005
9	2.6683E−005
10	1.3362E−006

Table 6
Maximum absolute errors with different resolutions for problem 3.

Level of Resolution J	Max. abs. Error
3	9.3374E−004
4	5.0517E−004
5	1.5678E−004
6	6.9632E−005
7	3.7205E−005
8	9.6130E−006
9	6.8820E−006
10	1.9670E−006

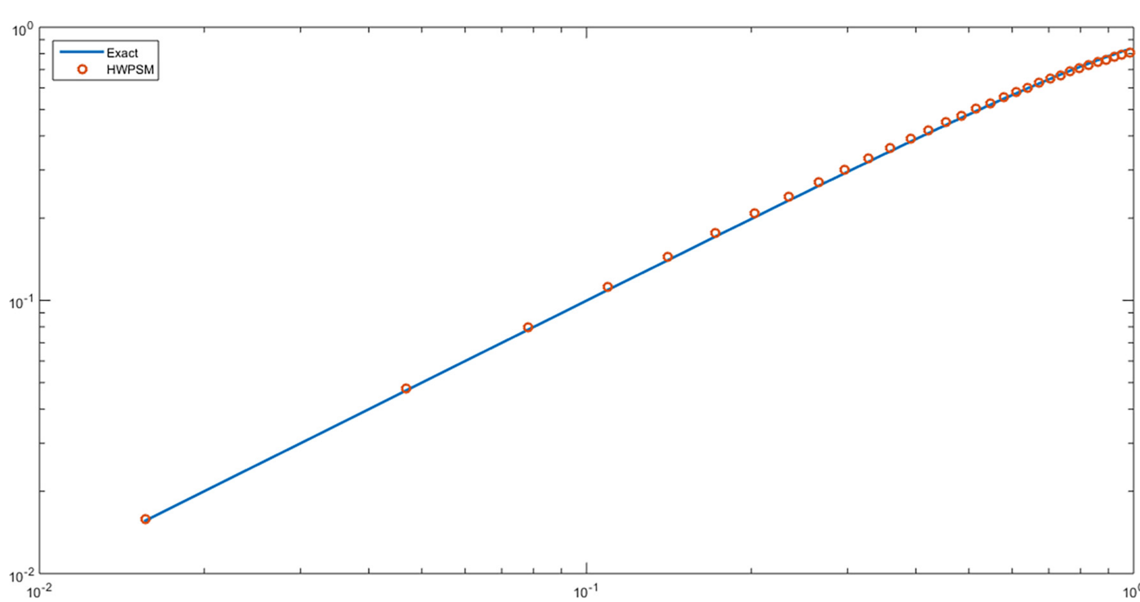

Fig. 2. loglog plot of problem 2 for J = 4.

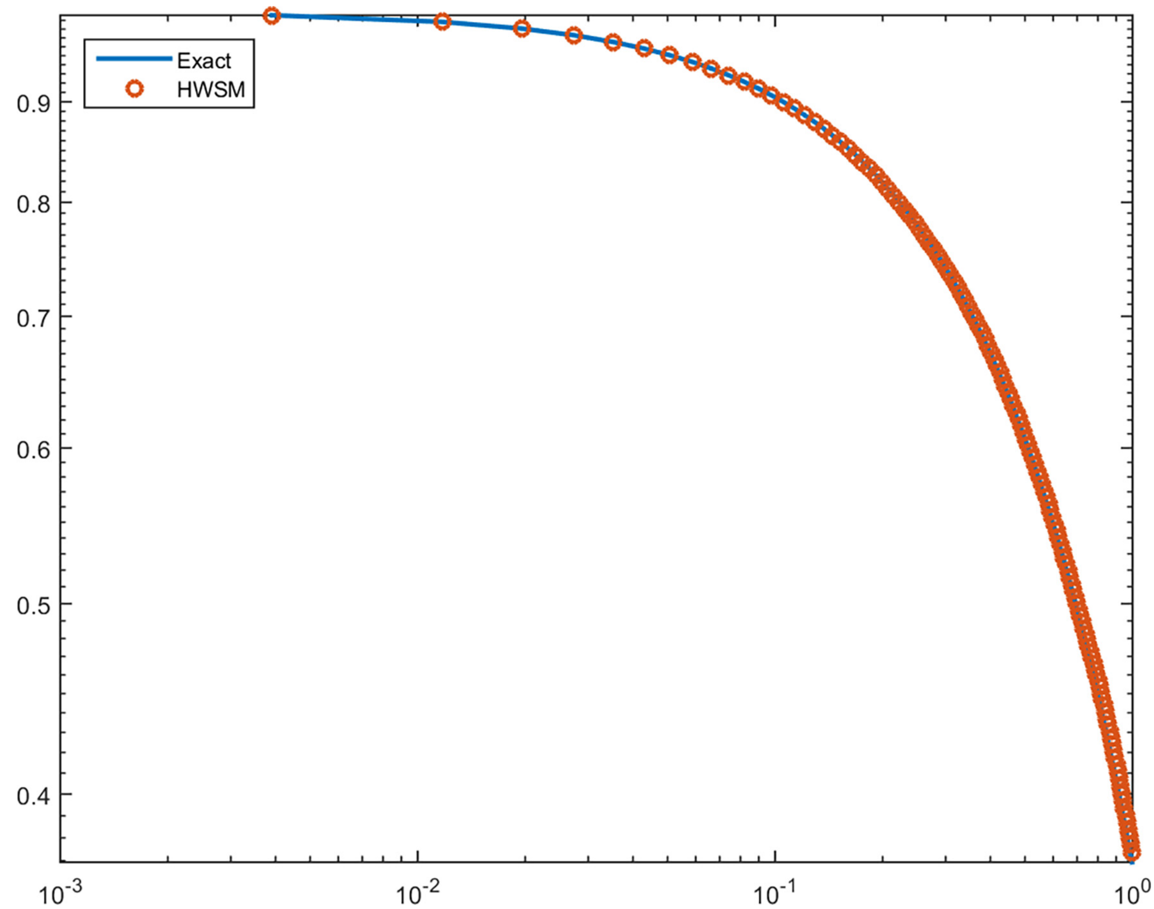

Fig. 3. loglog plot of problem 3 for J = 6.

Table 7
Maximum absolute errors of problem 4 with J = 3.

Our Method	Wang et al. (2009a)	Wang et al. (2009b)	Wang and Chen (2010)	Lv and Gao (2012)
3.6461E−004	2.31E−003	5.47E−002	5.96E−003	9.45E−004

$$\sum_{i=1}^{2M} a_i h_i(x) = f\left(x,\ \sum_{i=1}^{2M} a_i I_1 h_i(x) + y(0),\ \sum_{i=1}^{2M} a_i I_1 h_i(x-\mu(x,\ y(x))) + y(0),\ \sum_{i=1}^{2M} a_i h_i(x - \nu(x,\ y(x)))\right) \tag{3.6}$$

Solving this system of algebraic equation we can find the unknown Haar wavelet series coefficients a_i's and then put in Eq. (3.3) to get the approximate Haar wavelet series solution of the NDDE.

In case of nonlinear NDDE we applied the same method and obtained the system of nonlinear equations which are solved by Newton's method. We obtain the unknown Haar wavelet series coefficients a_i's and then put in Eq. (3.3) to get the approximate Haar wavelet series solution of the nonlinear neutral delay differential equations.

3.1. Convergence analysis of the Haar wavelet series method

Lemma. *Let $y(x)$ be a square integrable function with bounded first derivative on $(0,1)$ and $y(x_j)$ be Haar wavelet approximation of $y(x)$ then the error norm at J^{th} level satisfies the inequality.*

$$\|e_j(x)\| \leqslant \sqrt{\frac{K}{7}} C \times 2^{-\frac{3}{2}M}. \tag{3.7}$$

where K is a positive constant and C is given by

$$C = \int_0^1 |x h_2(x)| dx \tag{3.8}$$

Proof. For the proof see Islam et al. (2010), Pandit and Kumar (2014). □

Table 8
Maximum absolute errors with different resolutions for problem 4.

Level of Resolution J	Max. abs. Error
4	7.0605E−005
5	1.4873E−005
6	3.6656E−006
7	8.7681E−007
8	2.1541E−007

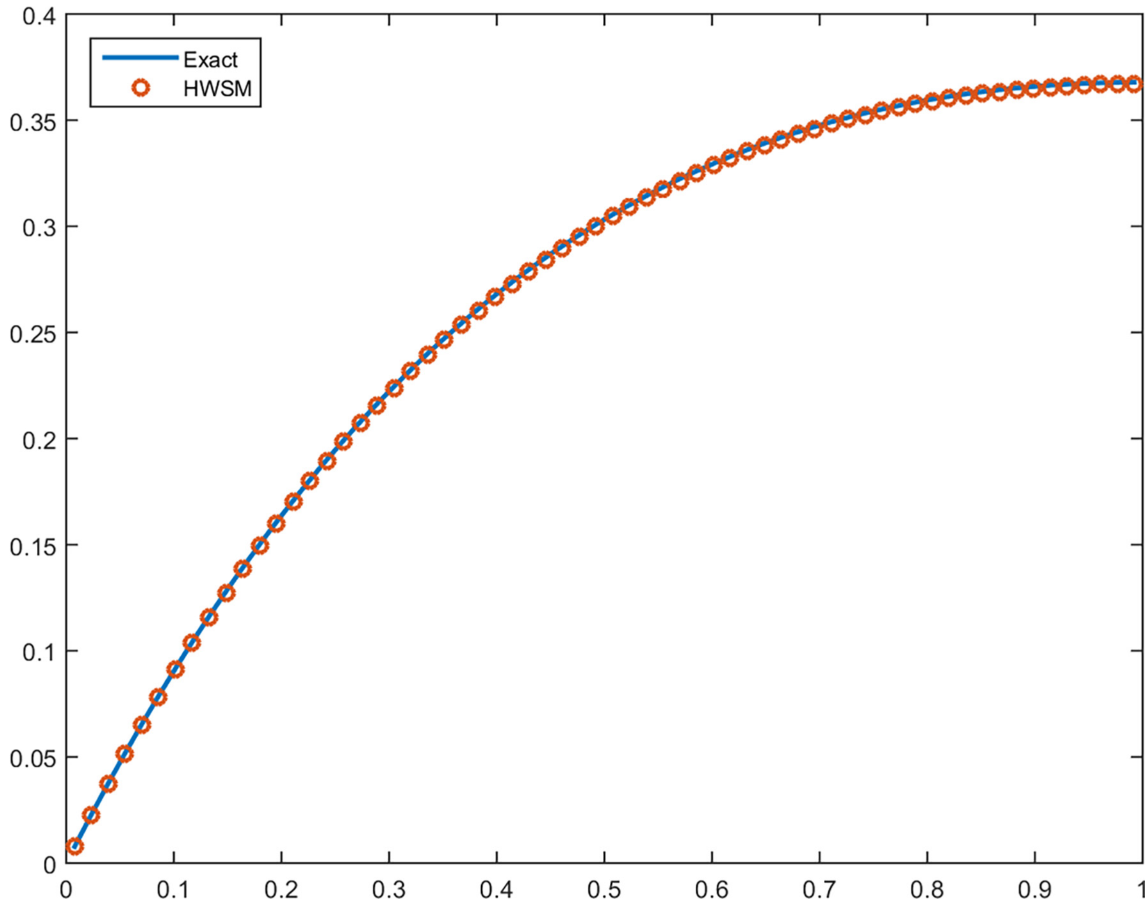

Fig. 4. Plot of problem 4 for J = 5.

Table 9
Maximum absolute error for problem 5 with J = 3.

Our Method	Wang et al. (2009a)	Wang et al. (2009b)	Wang and Chen (2010)	Lv and Gao (2012)
8.9532E−005	1.85E−003	7.66E−002	4.94E−003	4.01E−004

4. Numerical illustrations

Here we present six problems to illustrate the developed method and obtain maximum absolute errors (M.A.E.). The results are compared with the exact solution of the problems and other methods such as Wang et al. (2009a,b), Wang and Chen (2010), Lv and Gao (2012).

Problem 1. Let us take the following NDDE

$$y'(x) + \sqrt{cosx}y'(\sqrt{x}) + (sin(\sqrt{x}) + e^x)y(sinx) = e^x + \sqrt{cosx}e^{\sqrt{x}} + (sin(\sqrt{x}) + e^x)e^{(sinx)}. \qquad x \in [0,1] \tag{4.1}$$

with initial condition

$$y(x) = e^x, \qquad x \leqslant 0 \tag{4.2}$$

Analytical solution is

$$y(x) = e^x. \tag{4.3}$$

Obtained maximum absolute errors with different resolutions are given in the Tables 1, 2 and graph is given in the Fig. 1.

Problem 2. Let us take the following NDDE

$$y'(x) + \sqrt{x}(y'(e^{-\frac{x}{2}}) - y(\sqrt{x}e^{-x}) + y(x)) = cosx + \sqrt{x}(cos(e^{-\frac{x}{2}}) - sin(\sqrt{x}e^{-x}) + sinx). \qquad x \in [0,1] \tag{4.4}$$

with the initial condition

$$y(x) = sinx, \qquad x \leqslant 0. \tag{4.5}$$

The analytical solution is

$$y(x) = sinx. \tag{4.6}$$

Table 10
Maximum absolute errors with different resolutions for problem 5.

Level of Resolution J	Max. abs. Error
4	1.3456E−005
5	1.9630E−006
6	2.8093E−007
7	3.9528E−008
8	5.4910E−009

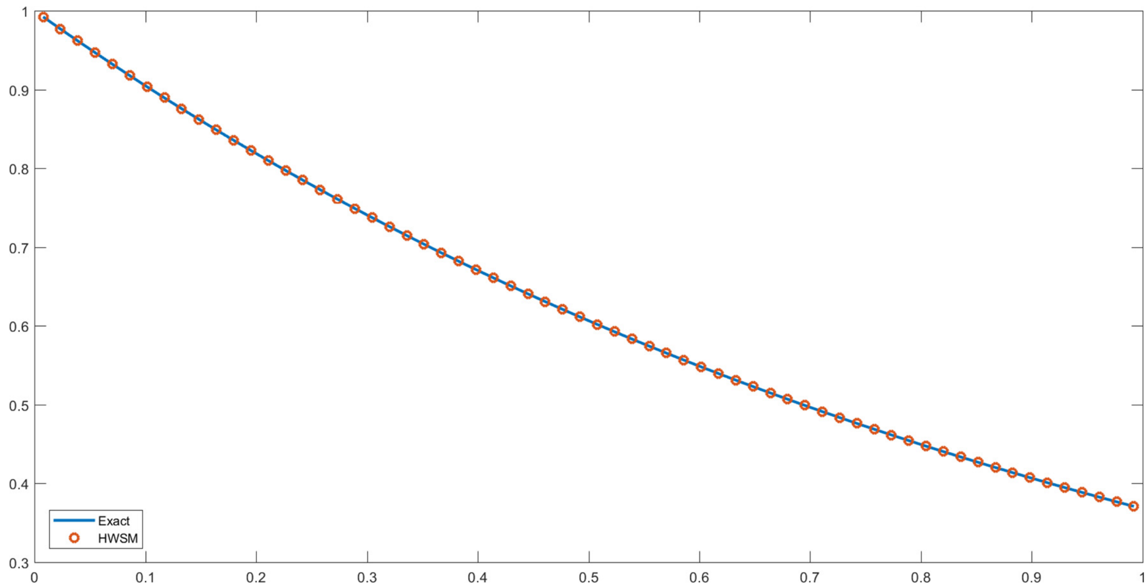

Fig. 5. Plot of problem 5 for J = 5.

Obtained maximum absolute errors with different resolutions are given in the Tables 3, 4 and graph is given in Fig. 2.

Problem 3. Let us take the following NDDE

$$y'(x) + e^{x}y'\left(x - \sin\left(x^2\right)\right) + \cos(x)y(x - \sin x) = -e^{-x} - e^{\sin x^2} + \cos(x)e^{\sin(x)-x}, \qquad x \in [0,1] \tag{4.7}$$

with initial condition

$$y(x) = e^{-x}, \qquad x \leqslant 0 \tag{4.8}$$

The analytical solution is

$$y(x) = e^{-x}. \tag{4.9}$$

Obtained maximum absolute errors with different resolutions are given in the Tables 5, 6 and graph is given in Fig. 3.

Problem 4. Let us take the following NDDE

$$y'(x) = 0.5y'(0.80x) + 0.10y(0.80x) - y(x) + (0.80x - 0.50)e^{-0.80x} + e^{-x}, \quad x \in [0,1]$$

with the initial condition

$$y(0) = 0, \qquad x \leqslant 0 \tag{4.10}$$

The analytical solution is

$$y(x) = xe^{-x}. \tag{4.11}$$

Obtained maximum absolute errors along with comparison to existing methods are given in the Tables 7, 8 and graph is given in Fig. 4.

Problem 5. Let us take the following NDDE

$$y'(x) = 0.50y'(0.50x) + 0.50y(0.50x) - y(x), \quad x \in [0,1]$$

with initial condition

$$y(0) = 1, \qquad x \leqslant 0 \tag{4.12}$$

The analytical solution is

$$y(x) = e^{-x}. \tag{4.13}$$

Obtained maximum absolute errors along with the comparison to existing methods are given in the Tables 9, 10 and graph is given in Fig. 5.

Problem 6. Let us take the following nonlinear NDDE.

$$y(x)y'(x) + \sqrt{\cos x}\,y'\left(\sqrt{x}\right) + \left(\sin\left(\sqrt{x}\right) + e^{x}\right)y(\sin x) = e^{2x} + \sqrt{\cos x}\,e^{\sqrt{x}} + \left(\sin\left(\sqrt{x}\right) + e^{x}\right)e^{(\sin x)}. \qquad x \in [0,1] \tag{4.14}$$

with the initial condition

$$y(x) = e^{x}, \qquad x \leqslant 0 \tag{4.15}$$

The analytical solution is

$$y(x) = e^{x}. \tag{4.16}$$

Obtained maximum absolute errors with different resolutions are given in the Tables 11, 12 and graph is given in Fig. 6.

Table 11
Comparison between analytical solution and Haar wavelet series method for problem 6 with J = 2.

$x(=\frac{1}{16})$	Exact	HWSM	Error
1	1.0645	1.0608	3.700E−003
3	1.2062	1.2006	5.700E−003
5	1.3668	1.3664	4.000E−004
7	1.5488	1.5463	2.500E−003
9	1.7551	1.7468	8.300E−003
11	1.9887	1.9819	6.900E−003
13	2.2535	2.2485	5.000E−003
15	2.5536	2.5488	4.800E−003

Table 12
Maximum absolute errors with different resolutions for problem 6.

Level of Resolution J	Max. abs. Error
2	8.300E−003
3	2.5862E−04
4	4.5075E−05
5	1.3195E−05
6	1.9151E−06

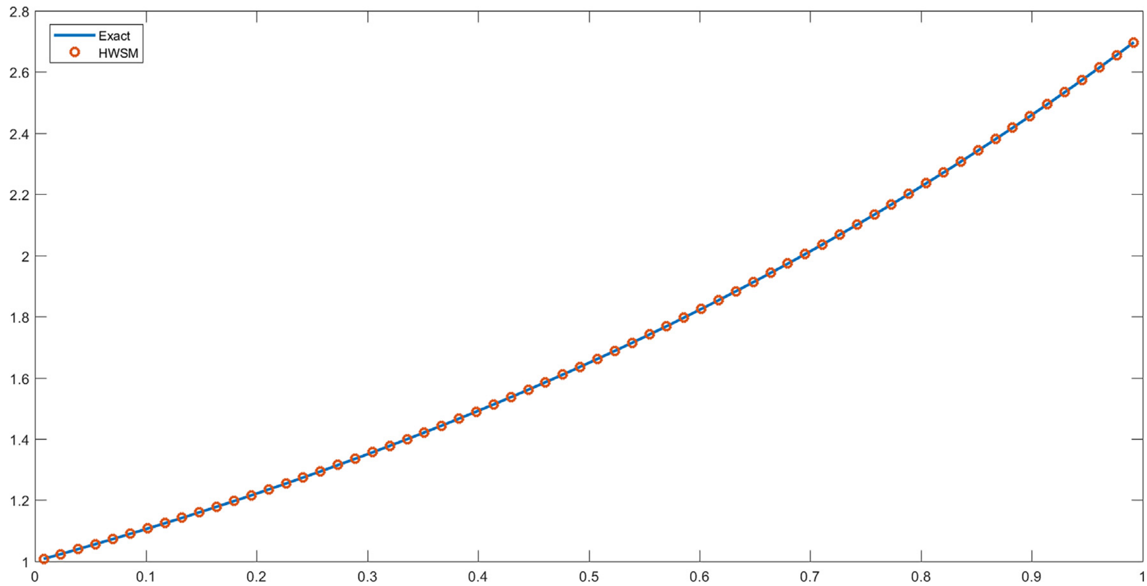

Fig. 6. Graph of problem 6 for J = 5.

5. Conclusion

Haar wavelet series method has been applied successfully to solve the linear and nonlinear neutral delay differential equations, which is easy to apply directly. As we have observed that this method has less calculation and takes less time to solve the problems and yield accurate results on increasing the resolution level.

Acknowledgements

We would like to express our sincere gratitude to Professor Khalil Ahmad and the anonymous reviewers for their helpful and valuable comments which greatly improved the quality of the paper.

References

Bellen, A., Zennaro, M., 2004. Adaptive integration of delay differential equations. Advances in time-delay systems. Lect. Notes Comput. Sci. Eng., 38. Springer, Berlin, pp. 155–165.
Chen, C.F., Hsiao, C.H., 1997. Haar wavelet method for solving lumped and distributed-parameter system. IEEE Proc.-Control Theory Appl. 144, 87–94.
Chui, C.K., 1992. An Introduction to Wavelets. Academic Press, San Diego, pp. 49–74.
Debnath, L., Shah, F.A., 2015. Wavelet Transform and Their Application. Birkhauser, Springer New york, pp. 337–440.
Driver, R.D., 1977. Ordinary and delay differential equations. Applied Mathematical Sciences, 20. Springer-Verlag, New York-Heidelberg, pp. 225–331.
El'sgol'ts, L.E., Norkin, S.B., 1973. Introduction to the theory and applications of differential equations with deviating arguments. Mathematics in Science and Engineering, 105. Academic Press [A Subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, pp. 165–182.
Gopalsamy, K., 1992. Stability and oscillations in delay differential equations of population dynamics. Mathematics and its Applications, 74. Kluwer Academic Publishers Group, Dordrecht, pp. 394–461.
Gu, J.S., Jiang, W.S., 1996. The Haar wavelets operational matrix of integration. Int. J. Syst. Sci., 623–628
Halanay, A., 1966. Differential Equations: Stability, Oscillations, Time Lags. Academic Press, New York-London, pp. 394–510.
Islam, S., Aziz, I., Sarler, B., 2010. The numerical solution of second order boundary value problems by collocation method with Haar wavelets. Math. Comput. Model. 50, 1577–1590.
Jamshidi, M., Wang, C.M., 1984. A computational algorithm for large-scale nonlinear time delays systems. IEEE Trans. Syst. Man. Cybern. SMC-14, 2–9.
Karimi, S. Vanani, Aminataei, A., 2008. On the numerical solution of neutral delay differential equations using multiquadric approximation scheme. Bull. Korean Math. Soc. 45, 663–670.
Kolmanovskii, V.B., Myshkis, A., 1992. Applied theory of functional-differential equations. Mathematics and its Applications (Soviet Series), 85. Kluwer Academic Publishers Group, Dordrecht, pp. 48–54.
Kuang, Y., 1993. Delay differential equations with applications in population dynamics. Mathematics in Science and Engineering, 191. Academic Press Inc., Boston, MA, pp. 3–116.
Lelarsmee, E., Ruehli, A., Sangiovanni-Vincentelli, A., 1982. The waveform relaxation method for time domain analysis of large scale integrated circuits. IEEE Trans. CAD 1 (3), 131–145.
Lepik, Ulo, 2007. Application of the Haar wavelet transform to solving integral and differential equations. Proc. Estonian Acad. Sci. Phys. Math. 56, 28–46.
Lv, Xueqin, Gao, Yue, 2012. The RKHSM for solving neutral functional-differential equations with proportional delays. Math. Methods Appl. Sci. https://doi.org/10.1002/mma.2612.
Oruç, Ömer, 2017a. A computational method based on Hermite wavelets for two-dimensional Sobolev and regularized long wave equations in fluids. Numer. Methods Partial Diff. Eq.;00:1-23.
Oruç, Ömer, 2017b. A numerical procedure based on Hermite wavelets for two-dimensional hyperbolic telegraph equation. Eng. Comput. https://doi.org/10.1007/s00366-017-0570-6.
Oruç, Ömer, 2018. A haar wavelet approximation for two-dimensional time fractional reaction-subdiffusion equation. Eng. Comput. https://doi.org/10.1007/s00366-018-0584-8.
Sapna, Pandit, Manoj, Kumar, 2012. Wavelet transform and wavelet-based numerical methods: an introduction. Int. J. Nonlinear Sci. 13, 325–345.
Sapna, Pandit, Manoj, Kumar, 2014. Haar wavelet approach for numerical solution of two parameters singularly perturbed boundary value problems. Appl. Math. Inf. Sci. 8 (6), 2965–2974.
Lepik, Ulo, Hein, Helle, 2014. Haar Wavelet With Applications. Springer, 7–44.
Wang, L., Chen, X., 2010. The variational iteration method for solving a neutral functional-differential equation with proportional delays. Comput. Math. Appl. 59, 2696–2702.
Wang, W., Zhang, Y., Li, S., 2009a. Stability of continuous Runge–Kutta-type methods for nonlinear neutral delay-differential equations. Appl. Math. Model. 33 (8), 3319–3329.
Wang, W., Qin, T., Li, S., 2009b. Stability of one-leg Θ-methods for nonlinear neutral differential equations with proportional delay. Appl. Math. Comput. 213 (1), 177–183.
Yves, Nievergelt, 1999. Wavelets Made Easy, 978-1-4612-0573-9. Springer.