ORIGINAL ARTICLE

Fuzzy simple expansion

H.M. Abu-Donia [a,c,*], A.S. Salama [b,c]

[a] *Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt*
[b] *Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt*
[c] *Department of Mathematics, Faculty of Science, Shaqra University, Al-Dawadmi, Saudi Arabia*

Abstract   In this paper we generalize the concept of simple expansion due to Levine (1964) to the fuzzy setting. Also, we introduce new classes of fuzzy sets, namely fuzzy η-preopen sets, fuzzy weakly η-open sets and fuzzy weakly η-preopen sets. This families not only depend on the fuzzy topology τ but also on it simple expansion and we study their fundamental properties.

© 2010 King Saud University. All rights reserved.

1. Introduction

The concept of fuzzy operations and fuzzy set operations was first introduced by Zadeh (1965). A fuzzy set on a non-empty set X is a mapping from X into the unit interval $I = [0, 1]$. The null set 0_X is the mapping from X into I which assumes the only the value 0 and the whole fuzzy set 1_X is the mapping from X into I which takes value 1 only. A family τ of fuzzy sets of X is called a fuzzy topology (Chang, 1968) on X if 0_X and 1_X belong to τ and τ closed under arbitrary union and finite intersection. The members of τ are called fuzzy open sets and their complements are fuzzy closed sets.

* Corresponding author at: Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt. Tel.: + 20 0597895288.

In 1982, Mashhour et al. (1982) defined preopen sets in topological spaces. In 1991, Singal and Prakash (1991) generalized the concept of preopen sets to the fuzzy setting. In 1991, Bin Shahna (1991) introduced the concept of preopen function between two fuzzy topological spaces. In 1985, Rose (1984) defined weakly open functions in topological spaces. In 1997, Park et al. (1997) introduced the notion of weakly open functions between two fuzzy topological spaces. In 2004, Caldas et al. (2004) introduced the concept of fuzzy weakly preopen functions which is weaker than fuzzy preopen sets. Simple expansion was first introduced in 1964 by Levine (1964) as "Let (X, τ) is a topological space and $B \subset X$, $B \notin \tau$, then the topology $\tau(B) = \{O \cup (O' \cap B) : O, O' \in \tau\}$ is called simple expansion of τ by B". Also, a similar concept discussed by Hewit in 1943 (Hewit, 1943) as "If (X, τ) be a topological space and $B \subset X$, $B \notin \tau$, then the class $\tau \cup \{B\}$ is a subbase for a topology τ^* finer than τ, τ^* is called a simple expansion of τ by B". In this paper we generalized the concept of simple expansion due to Levine (1964) to the fuzzy setting. Also, we introduced new classes of fuzzy sets, namely fuzzy η-preopen sets, fuzzy weakly η-open sets and fuzzy weakly η-preopen sets. This families not only depend on the fuzzy topology τ but also on it simple expansion and we study their fundamental properties.

Throughout this paper (X, τ) and (Y, σ) or simply X, Y, respectively, denoted by fuzzy topological spaces (fts, short) on which no separation axioms are assumed unless explicitly stated. If μ is any fuzzy subset of a fts X (for short, $\mu \in I^X$), then $cl(\mu)$, $int(\mu)$, $1 - \mu$ denote the fuzzy closure, the fuzzy interior and the complement of fuzzy set μ in fts X, respectively. We denote the set of all fuzzy open (resp. fuzzy closed) sets by $FO(X)$ (resp. $FC(X)$).

2. Expansion of fuzzy topology

In this paper we generalize the concept of simple expansion due to Levine (1964) to the fuzzy setting as the following:

Definition 2.1. Let (X, τ) be a fuzzy topological space, $\eta \in I^X$ such that $\eta \notin \tau$. Then the fuzzy topology $\tau_\eta = \{\rho \vee (\rho' \wedge \eta) : \rho, \rho' \in \tau\}$ is called a simple expansion of τ by η.

Example 2.1. Let

$$a(x) = \begin{cases} x, & \text{if } 0 \leqslant x \leqslant \frac{1}{2} \\ 1 - x, & \text{if } \frac{1}{2} \leqslant x \leqslant 1 \end{cases}, \quad b(x) = \begin{cases} 1 - 2x, & \text{if } 0 \leqslant x \leqslant \frac{1}{2} \\ \frac{1}{2}, & \text{if } \frac{1}{2} \leqslant x \leqslant 1 \end{cases},$$

$$c(x) = \begin{cases} \frac{3}{2}x, & \text{if } 0 \leqslant x \leqslant \frac{1}{2} \\ \frac{3}{4}, & \text{if } \frac{1}{2} \leqslant x \leqslant 1 \end{cases}, \quad d(x) = \begin{cases} 4x, & \text{if } 0 \leqslant x \leqslant \frac{1}{4} \\ \frac{4}{3}(1 - x), & \text{if } \frac{1}{4} \leqslant x \leqslant 1 \end{cases}.$$

Let $\tau = \{0_X, 1_X, a\}$ be a fuzzy topology on X. Then $\tau_b = \{0_X, 1_X, a, b, a \wedge b, a \vee b\}$, $\tau_c = \{0_X, 1_X, a, c\}$ and $\tau_d = \{0_X, 1_X, a, d\}$ be three expansions for a fuzzy topology τ by using the fuzzy sets b, c and d, respectively.

Theorem 2.1. If (X, τ) is a fuzzy topological space and $\eta \in I^X$, $\eta \notin \tau$, then the topology $\tau_\eta = \{\rho \vee (\rho' \wedge \eta) : \rho, \rho' \in \tau\}$ coincides with the fuzzy topology τ^* generated by the class $\tau \cup \eta$.

Proof. Let $\zeta \in \tau^*$. Then $\zeta \in \bigvee_{i \in \triangle}(\bigwedge_{j \in \triangle_0} \lambda_{ij})$, $\lambda_{ij} \in \tau \cup \{\eta\}$ and $\triangle(\triangle_0)$ is an arbitrary (finite) index set. If $\lambda_{ij} \in \tau$ for each $i \in \triangle, j \in \triangle_0$, then $\zeta \in \tau \subset \tau_\eta$. Assume that there exist $i_0 \in \triangle, j_0 \in \triangle_0$ such that $\lambda_{i_0 j_0} = \eta \wedge \rho$ for some $\rho \in \tau$. Therefore $\zeta = [\bigvee_{i \in \triangle - \{i_0\}}(\bigwedge_{j \in \triangle_0 - \{j_0\}} \lambda_{ij})] \vee (\eta \wedge \rho)$, clearly this implies that $\zeta = \rho' \vee (\rho \wedge \eta)$, $\rho, \rho' \in \tau$. Hence $\zeta \in \tau_\eta$.

Conversely, if $\zeta \in \tau_\eta$, then there exist $\rho, \rho' \in \tau$ such that $\zeta = \rho' \vee (\rho \wedge \eta)$. Therefore $\zeta \in \tau^*$. $\square$

Definition 2.2 Singal and Prakash, 1991. Let μ be any fuzzy set of a fts (X, τ)

  (1) μ is called fuzzy preopen set of X if $\mu \leqslant int(cl(\mu))$.
  (2) μ is called fuzzy preclosed set of X if $\mu \geqslant cl(int(\mu))$.

We denote the set of all fuzzy preopen (resp. fuzzy preclosed) sets by $FPO(X)$ (resp. $FPC(X)$).

Definition 2.3 Singal and Prakash, 1991. Fuzzy preclosure and fuzzy preinterior of a fuzzy set μ of a fts (X, τ) are defined as follows:

  (1) $pcl(\mu) = \bigwedge\{\rho : \rho \text{ is a fuzzy preclosed and } \mu \leqslant \rho\}$.
  (2) $pint(\mu) = \bigvee\{v : v \text{ is a fuzzy preopen and } \mu \geqslant v\}$.

In this paper we introduce new classes of fuzzy sets called fuzzy η-preopen sets. This family of fuzzy sets not only depends on the fuzzy topology τ but also on τ_η. We study some of its properties and its characterizations.

Definition 2.4. For any fts (X, τ). A fuzzy subset μ is called a fuzzy η-preopen if $\mu \leqslant int_{\tau_\eta}(cl(\mu))$.

The complement of fuzzy η-preopen set is called fuzzy η-preclosed set. The family of all fuzzy η-preopen sets is denoted by $FP_\eta O(X)$ and the family of all fuzzy η-preclosed sets is denoted by $FP_\eta C(X)$.

The following example show that a fuzzy preopen set is a fuzzy η-preopen but not conversely.

Example 2.2. Consider the following fuzzy sets

$$a(x) = \begin{cases} x - \frac{1}{4}, & \text{if } \frac{1}{4} \leqslant x \leqslant \frac{1}{2} \\ 0, & \text{otherwise} \\ \frac{3}{4} - x, & \text{if } \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \end{cases}, \quad b(x) = \begin{cases} x, & \text{if } 0 \leqslant x \leqslant \frac{1}{4} \\ \frac{1}{4}, & \text{if } \frac{1}{4} \leqslant x \leqslant 1 \end{cases},$$

$$c(x) = \begin{cases} \frac{3}{2}x, & \text{if } 0 \leqslant x \leqslant \frac{1}{2} \\ \frac{3}{4}, & \text{if } \frac{1}{2} \leqslant x \leqslant 1 \end{cases}, \quad \eta(x) = \begin{cases} 3x, & \text{if } 0 \leqslant x \leqslant \frac{1}{4} \\ \frac{3}{4}, & \text{if } \frac{1}{4} \leqslant x \leqslant 1 \end{cases}.$$

Let $\tau = \{0_X, 1_X, a, b\}$, $\eta \notin \tau$. Then $\tau_\eta = \{0_X, 1_X, a, b, \eta\}$. We have the fuzzy set c is a fuzzy η-preopen but it is not fuzzy preopen, since $int(cl(c)) = b \ngeq c$.

Theorem 2.2. Let (X, τ) be a fts, η be a fuzzy set such that $\eta \notin \tau$. We have:

  (1) An arbitrary union of fuzzy η-preopen sets is a fuzzy η-preopen set.
  (2) An arbitrary intersection of fuzzy η-preclosed sets is a fuzzy η-preclosed set.

Proof. (1) Let $\{V_\alpha\}$ be a collection of fuzzy η-preopen sets. Then, for each α, $V_\alpha \leqslant int_{\tau_\eta}(cl(V_\alpha))$. Now,

$$\bigvee V_\alpha \leqslant \bigvee int_{\tau_\eta}(cl(V_\alpha)) \leqslant int_{\tau_\eta} \bigvee(cl(V_\alpha)) = int_{\tau_\eta}(cl(\bigvee V_\alpha))$$

Hence $\bigvee V_\alpha$ is a fuzzy η-preopen set.
  (2) Follows easily by taking complements. $\square$

Definition 2.5. Let (X, τ) be a fts and η be a fuzzy set on X, $\eta \notin \tau$. Fuzzy η-preclosure $(p_\eta cl)$ and fuzzy η-preinterior $(p_\eta int)$ of a fuzzy set ϱ are defined as follows:

  (1) $p_\eta cl(\varrho) = \bigwedge\{\mu : \mu \in FP_\eta C(X, \tau) \text{ and } \varrho \leqslant \mu\}$.
  (2) $p_\eta int(\varrho) = \bigvee\{v : v \in FP_\eta O(X, \tau) \text{ and } \varrho \geqslant v\}$.

Theorem 2.3. Let ϱ be any fuzzy set in a fts (X, τ), η be a fuzzy set such that $\eta \notin \tau$. Then $p_\eta - cl(1 - \varrho) = 1 - p_\eta - int(\varrho)$ and $p_\eta - int(1 - \varrho) = 1 - p_\eta - cl(\varrho)$.

Proof. We see that a fuzzy η-preopen set $\mu \leqslant \varrho$ is precisely the complement of fuzzy η-preopen set $v \geqslant 1 - \varrho$. Thus

$$p_\eta int(\varrho) = \bigvee\{1 - v : v \in FP_\eta C(X,\tau) \text{ and } v \geqslant 1 - \varrho\}$$
$$= 1 - \bigwedge\{v : v \in FP_\eta C(X,\tau) \text{ and } v \geqslant 1 - \varrho\}$$
$$= 1 - p_\eta cl(1 - \varrho)$$

Similarly, $p_\eta cl(\varrho) = 1 - p_\eta int(1 - \varrho)$. $\square$

Theorem 2.4. *For a fts* (X,τ), η *be a fuzzy set such that* $\eta \notin \tau$, *then* μ *is called a fuzzy* η-*preclosed (resp. fuzzy* η-*preopen) if and only if* $\mu = p_\eta cl(\mu)$ *(resp.* $\mu = p_\eta int(\mu)$*).*

Proof. Let $\mu \in FP_\eta C(X,\tau)$. Since $\mu \leqslant \mu$, $\mu \in \{f : f \in FP_\eta C(X,\tau) \text{ and } \mu \leqslant f\}$, and $\mu \leqslant f$ implies that $\mu = \bigwedge\{f : f \in FP_\eta C(X,\tau) \text{ and } \mu \leqslant f\}$, i.e., $\mu = p_\eta cl(\mu)$. Conversely, suppose that $\mu = p_\eta cl(\mu)$, i.e., $\mu = \bigwedge\{f : f \in FP_\eta C(X,\tau) \text{ and } \mu \leqslant f\}$. This implies that $\mu \in \{f : f \in FP_\eta C(X,\tau) \text{ and } \mu \leqslant f\}$. Hence $\mu \in FP_\eta C(X,\tau)$. $\square$

Theorem 2.5. *For a fts* (X,τ), η *is a fuzzy set such that* $\eta \notin \tau$, *the following hold for* η-*preclosure*

(1) $p_\eta cl(O_X) = O_X$.
(2) $p_\eta cl(\mu) \in FP_\eta C(X,\tau)$.
(3) $p_\eta cl(\mu) \leqslant p_\eta cl(v)$ *if* $\mu \leqslant v$.
(4) $p_\eta cl(p_\eta cl(\mu)) = p_\eta cl(\mu)$.

Proof. Obvious. $\square$

Theorem 2.6. *For a fts* (X,τ), η *be a fuzzy set such that* $\eta \notin \tau$, *the following are hold*

(1) $p_\eta cl(\mu \vee v) \geqslant p_\eta cl(\mu) \vee p_\eta cl(v)$.
(2) $p_\eta cl(\mu \wedge v) \leqslant p_\eta cl(\mu) \wedge p_\eta cl(v)$.

Proof. Obvious. $\square$

Definition 2.6. Let (X,τ) and (Y,σ) be two fts's. A function $f : (X,\tau) \rightarrow (Y,\sigma)$ is called:
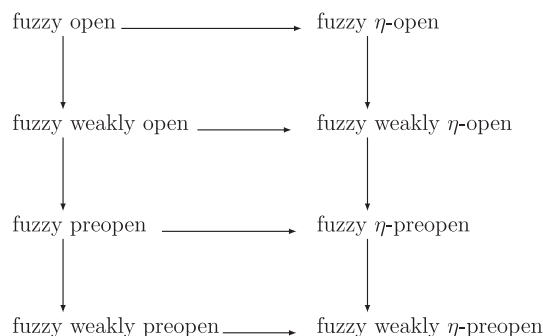
(1) Fuzzy preopen (Bin Shahna, 1991) if $f(\lambda)$ is fuzzy preopen subset of Y, for each fuzzy open set λ in X.
(2) Fuzzy open (Chang, 1968) if $f(\lambda)$ is fuzzy open subset of Y, for each fuzzy open set λ in X.
(3) Fuzzy weakly open (Park et al., 1997) if $f(\lambda) \leqslant (int(f(cl(\lambda))))$, for each fuzzy open set λ in X.
(4) Fuzzy contra open (Caldas et al., 2004) (resp. fuzzy contra closed) if $f(\lambda)$ is fuzzy closed (resp. fuzzy open) set of Y, for each fuzzy open (resp. fuzzy closed) set λ in X.
(5) Fuzzy weakly preopen (Caldas et al., 2004) if $f(\lambda) \leqslant (pint(f(cl(\lambda))))$, for each fuzzy open set λ in X.

3. New forms of fuzzy functions

Now, we define the generalized forms of preopen function in fuzzy setting.

Definition 3.1. Let (X,τ) and (Y,σ) be two fts's, η be a fuzzy set such that $\eta \notin \sigma$. A function $f : (X,\tau) \rightarrow (Y,\sigma)$ is called:

(1) Fuzzy η-preopen if $f(\lambda)$ is a fuzzy η-preopen subset of Y, for each fuzzy open set λ in X.
(2) Fuzzy η-open if $f(\lambda)$ is fuzzy η-open subset of Y, for each fuzzy open set λ in X.
(3) Fuzzy weakly η-open if $f(\lambda) \leqslant (\sigma_\eta int(f(cl(\lambda))))$, for each fuzzy open set λ in X.
(4) Fuzzy weakly η-preopen if $f(\lambda) \leqslant (p_\eta int(f(cl(\lambda))))$, for each fuzzy open set λ in X.

fuzzy open $\longrightarrow$ fuzzy η-open
$\downarrow$ $\downarrow$
fuzzy weakly open $\longrightarrow$ fuzzy weakly η-open
$\downarrow$ $\downarrow$
fuzzy preopen $\longrightarrow$ fuzzy η-preopen
$\downarrow$ $\downarrow$
fuzzy weakly preopen $\longrightarrow$ fuzzy weakly η-preopen

Theorem 3.1. *Let* (X,τ) *and* (Y,σ) *be two fts's,* η *be a fuzzy set such that* $\eta \notin \sigma$. *For a function* $f : (X,\tau) \rightarrow (Y,\sigma)$ *the following are equivalent.*

(1) f *is a fuzzy weakly* η-*preopen.*
(2) $f(int(\mu)) \leqslant p_\eta int(f(\mu)) \forall \mu \in FC(X)$.
(3) $f(int(cl(\lambda))) \leqslant p_\eta int(f(cl(\lambda))) \forall \lambda \in FO(X)$.
(4) $f(\lambda) \leqslant p_\eta int(f(cl(\lambda))) \forall \lambda \in FPO(X)$.

Proof. Obvious. $\square$

Definition 3.2 Caldas et al., 2004. A function $f : (X,\tau) \rightarrow (Y,\sigma)$ is said to be fuzzy strongly continuous, if for every fuzzy subset λ of X, $f(cl(\lambda)) \leqslant f(\lambda)$.

Theorem 3.2. *Let* (X,τ) *and* (Y,σ) *be two fts's,* η *be a fuzzy set such that* $\eta \notin \sigma$. *If a function* $f : (X,\tau) \rightarrow (Y,\sigma)$ *is a fuzzy weakly* η-*preopen and fuzzy strongly continuous. Then it is a fuzzy* η-*preopen.*

Proof. Let λ be a fuzzy subset of X. Since f is fuzzy weakly η-preopen, then we have $f(\lambda) \leqslant p_\eta int(f(cl(\lambda)))$. However, because f is strongly continuous, $f(\lambda) \leqslant p_\eta int(f(\lambda))$, i.e., $f(\lambda) = p_\eta int(f(\lambda))$. Consequently, $f(\lambda) \in FP_\eta O(Y)$. Therefore f is fuzzy η-preopen function. $\square$

Theorem 3.3. *Let* (X,τ) *be a fuzzy regular space and* (Y,σ) *be a fts,* η *be a fuzzy set such that* $\eta \notin \sigma$. *Then the function* $f : (X,\tau) \rightarrow (Y,\sigma)$ *is a fuzzy weakly* η-*preopen if and only if* f *is a fuzzy* η-*preopen.*

Proof. The sufficiency is clear. For the necessity, let λ be a non-null fuzzy open subset of X. For each x_p fuzzy point in λ, let μ_{x_p} be a fuzzy open set such that $x_p \in \mu_{x_p} \leqslant cl(\mu_{x_p}) \leqslant \lambda$. Hence we obtain that $\lambda = \bigcup\{\mu_{x_p} : x_p \in \lambda\} = \bigcup\{cl(\mu_{x_p}) : x_p \in \lambda\}$ and

$$f(\lambda) = \bigcup\{f(\mu_{x_p}) : x_p \in \lambda\}$$
$$\leqslant \bigcup\{p_\eta int(f(cl(\mu_{x_p}))) : x_p \in \lambda\}$$
$$\leqslant p_\eta int(f(\bigcup\{cl(\mu_{x_p}))) : x_p \in \lambda\})) = p_\eta int(f(\lambda)).$$

Thus f is a fuzzy η-preopen function. $\square$

Definition 3.3 Pu and Liu, 1980. An fuzzy set μ is said be quasi-coincident (q-coincident) with an fuzzy set λ, if there exists at least one point $x \in X$ such that $\mu(x) + \lambda(x) > 1$. It is denoted by $\mu q\lambda$. $\mu\bar{q}\lambda$ means that μ and λ are not q-coincident. For two fuzzy μ and λ. $\mu \leqslant \lambda$ iff $\mu(x) \leqslant \lambda(x)$ for each $x \in X$. Note that $\mu \leqslant \lambda$ iff $\mu\bar{q}X \setminus \lambda$.

Definition 3.4 Caldas et al., 2004. Two non-empty fuzzy sets λ and β in a fuzzy topological space (X, τ) (i.e., neither λ nor β is 0_X) are said to be fuzzy preseparated if $\lambda\bar{q}pcl(\beta)$ and $\beta\bar{q}pcl(\lambda)$ or equivalently if there exist two fuzzy preopen sets μ and v such that $\lambda \leqslant \mu$, $\beta \leqslant v$, $\lambda\bar{q}v$ and $\beta\bar{q}\mu$.

Definition 3.5. Two non-empty fuzzy sets λ and β in a fuzzy topological space (X, τ) (i.e., neither λ nor β is 0_X) are said to be fuzzy η-preseparated if $\lambda\bar{q}p_\eta cl(\beta)$ and $\beta\bar{q}p_\eta cl(\lambda)$ or equivalents if there exist two fuzzy η-preopen sets μ and v such that $\lambda \leqslant \mu$, $\beta \leqslant v$, $\lambda\bar{q}v$ and $\beta\bar{q}\mu$.

A fuzzy topological space which cannot be expressed as the union of two fuzzy preseparated (fuzzy η-preseparated) sets is said to be a fuzzy preconnected (fuzzy η-preconnected).

Theorem 3.4. *Let (X, τ) and (Y, σ) are the fuzzy topological spaces, η be a fuzzy set such that $\eta \notin \sigma$. If $f : (X, \tau) \to (Y, \sigma)$ is an injective fuzzy weakly η-preopen function of a space X onto a fuzzy η-preconnected space Y, then X is fuzzy connected.*

Proof. Let X be not fuzzy connected. Then there exist fuzzy separated sets β and γ in X, such that $X = \beta \cup \gamma$. Since β and γ are fuzzy separated. There exists two fuzzy open sets μ and v such that $\beta \leqslant \mu$, $\gamma \leqslant v$, $\beta\bar{q}v$ and $\gamma\bar{q}\mu$. Hence we have $f(\beta) \leqslant f(\mu)$, $f(\gamma) \leqslant f(v)$, $f(\beta)\bar{q}f(v)$ and $f(\gamma)\bar{q}f(\mu)$. Since f is fuzzy weakly η-preopen, we have $f(\mu) \leqslant p_\eta int(f(cl(\mu)))$ and $f(v) \leqslant p_\eta int(f(cl(v)))$ and since μ and v are fuzzy open and also fuzzy closed, we have $f(cl(\mu)) = f(\mu)$, $f(cl(v)) = f(v)$. Hence $f(\mu)$ and $f(v)$ are fuzzy η-preopen in Y. Therefore, $f(\beta)$ and $f(\gamma)$ are fuzzy η-preseparated sets in Y and $Y = f(X) = f(\beta \cup \gamma) = f(\beta) \cup f(\gamma)$. Hence this contrary to the fact that Y is fuzzy η-preconnected. Thus X is fuzzy connected. $\square$

Definition 3.6 Caldas et al., 2004. Space X is said to be fuzzy hyper connected if every non-null fuzzy open subset of X is fuzzy dense in X.

Theorem 3.5. *Let (X, τ) and (Y, σ) be two fts's, η be a fuzzy set such that $\eta \notin \sigma$. If X is a fuzzy hyper connected space. Then $f : (X, \tau) \to (Y, \sigma)$ is fuzzy weakly η-preopen if and only if $f(X)$ is fuzzy η-preopen set in Y.*
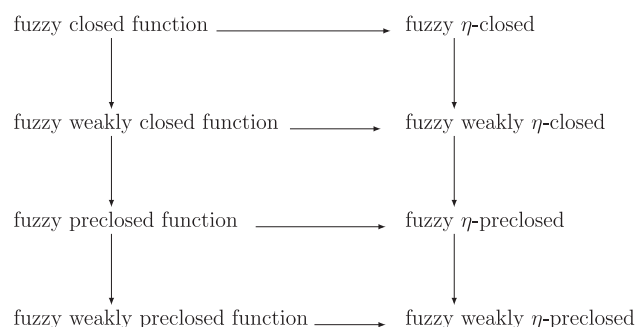
Proof. The sufficiency is trivial. For the necessity observe that for any fuzzy open subset λ os X, $f(\lambda) \leqslant f(X) = p_\eta int(f(X)) = p_\eta int(f(cl(\lambda)))$. $\square$

Definition 3.7. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be fuzzy weakly η-preclosed if $p_\eta cl(f(int(\beta))) \leqslant f(\beta)$ for each fuzzy closed subset β of X.

Definition 3.8. Let (X, τ) and (Y, σ) be two fts's, η be a fuzzy set such that $\eta \notin \sigma$. A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(1) Fuzzy η-preclosed if $f(\lambda)$ is a fuzzy η-preclosed subset of Y, for each fuzzy closed set λ in X.
(2) Fuzzy η-closed if $f(\lambda)$ is fuzzy η-closed subset of Y, for each fuzzy closed set λ in X.
(3) Fuzzy weakly η-closed if $(\sigma_\eta cl(f(int(\lambda)))) \leqslant f(\lambda)$, for each fuzzy open set λ in X.
(4) Fuzzy weakly η-preclosed if $(p_\eta cl(f(int(\lambda)))) \leqslant f(\lambda)$, for each fuzzy open set λ in X.

We have the following diagram and the converse of these implication do not hold.

fuzzy closed function	$\longrightarrow$	fuzzy η-closed
fuzzy weakly closed function	$\longrightarrow$	fuzzy weakly η-closed
fuzzy preclosed function	$\longrightarrow$	fuzzy η-preclosed
fuzzy weakly preclosed function	$\longrightarrow$	fuzzy weakly η-preclosed

Theorem 3.6. *For a function $f : (X, \tau) \to (Y, \sigma)$, $\eta \notin \sigma$, the following conditions are equivalent:*

(1) f is fuzzy weakly η-preclosed.
(2) $p_\eta cl(f(\lambda)) \leqslant f(cl(\lambda))$ for every open set λ in X.

Proof. $(1) \Rightarrow (2)$ Let λ be any fuzzy open subset of X. Then

$$p_\eta cl(f(\lambda)) = p_\eta cl(f(int(\lambda))) \leqslant p_\eta cl(f(int(cl(\lambda)))) \leqslant f(cl(\lambda)).$$

$(2) \Rightarrow (1)$ Let β be any fuzzy closed subset of X. Then

$$p_\eta cl(f(int(\beta))) \leqslant f(cl(int(\beta))) \leqslant f(cl(\beta)) = f(\beta). \quad \square$$

Theorem 3.7. *For a function $f : (X, \tau) \to (Y, \sigma)$, $\eta \notin \sigma$, the following conditions are equivalent:*

(1) f is fuzzy weakly η-preclosed.
(2) $p_\eta cl(f(\beta)) \leqslant f(cl(\beta))$ for each fuzzy open subset β of X.
(3) $p_\eta cl(f(int(\beta))) \leqslant f(\beta)$ for each fuzzy closed subset β of X.

Proof. Obvious. $\square$

Theorem 3.8. *Let (X, τ) and (Y, σ) are two fts's, $\eta \notin \sigma$. Then:*

(1) If $f : (X,\tau) \rightarrow (Y,\sigma)$ is fuzzy preclosed and fuzzy contra closed, then f is fuzzy weakly η-preclosed.

(2) If $f : (X,\tau) \rightarrow (Y,\sigma)$ is fuzzy contra open, then f is fuzzy weakly η-preclosed.

Proof. (1) Let β be a fuzzy closed subset of X. Since f is a fuzzy preclosed, then $cl(int(f(\beta))) \leqslant f(\beta)$ and since f is fuzzy contra closed, $f(\beta)$ is fuzzy open. Therefore $p_\eta cl(f(int(\beta))) \leqslant p_\eta cl(f(\beta)) \leqslant cl(int(f(\beta))) \leqslant f(\beta)$. Consequently f is a fuzzy weakly η-preclosed.

(2) Let β be a fuzzy closed subset of X. Then $p_\eta cl(f(int(\beta))) \leqslant (f(int(\beta))) \leqslant f(\beta)$. $\square$

References

Bin Shahna, A.S., 1991. On fuzzy strong semi continuity and fuzzy precontinuity. Fuzzy Sets Syst. 44, 303–308.

Caldas, M., Navalagi, G., Saraf, R., 2004. Weakly preopen and weakly preclosed functions in fuzzy topological spaces. Electron. Busefal 89, 1–18.

Chang, C.L., 1968. Fuzzy topological spaces. J. Math. Anal. Appl. 24, 182–192.

Hewit, E., 1943. A problem of set theoretic topology. Duke Math. J. (10), 309–333.

Levine, N., 1964. Simple extension of topologies. Amer. Math. Monthly 71, 22–25.

Mashhour, A.S., Abd El-Monsef, M.E., El-Deeb, S.N., 1982. On precontinuous and weak precontinuous mappings. Proc. Math. Phys. Soc. Egypt 53, 47–53.

Pu, Pao-Ming, Liu, Ying-Ming, 1980. Fuzzy topology 1. Neighborhood structure of a fuzzy point and Moore–Smith convergence. J. Math. Anal. Appl. 76, 571–599.

Park, J.H., Park, Y.B., Park, J.S., 1997. Fuzzy weakly open mappings and fuzzy RS-compact sets. Far East J. Math. Sci. Special vol. (Part II), 201–212.

Rose, D.A., 1984. On weak openness and almost openness. Internat. J. Math. Math. Sci. 7, 35–40.

Singal, M.K., Prakash, Niti, 1991. Fuzzy preopen sets and fuzzy preseparation axioms. Fuzzy Sets Syst. 44, 273–281.

Zadeh, L.A., 1965. Fuzzy sets. Inform. Control 8, 338–353.