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Original article

Analytical treatment of two-dimensional fractional Helmholtz equations

Salah Abuasad [a,*], Khaled Moaddy [b], Ishak Hashim [c]

[a] Faculty of Sciences, King Faisal University, 31982 Hofuf, Al-Hasa, Saudi Arabia
[b] Department of Mathematics, Faculty of Science and Arts, Shaqra University, Shaqra 11691, Saudi Arabia
[c] School of Mathematical Sciences, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor, Malaysia

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A B S T R A C T

In this paper, we propose a semi numerical-analytical method, called Fractional Reduced Differential Transform Method (FRDTM), for finding exact and approximate solutions of fractional Helmholtz equation with appropriate initial conditions. The fractional derivatives are demonstrated in the Caputo sense. The solutions are given in the form of series with easily computable terms, then with the help of Mittag-Leffler function, we find the exact solutions of the fractional Helmholtz equations. Three examples are given to demonstrate the applicability of FRDTM.

1. Introduction

Various methods for solving linear and nonlinear fractional partial differential equations based on different fractional derivatives. Morales-Delgado et al. (2016) presented an analysis based on a combination of the Laplace transform and homotopy methods in the Liouville-Caputo and Caputo-Fabrizio sense, while Bulut et al. (2016) studied the improved Bernoulli sub-equation function method and they apply it to the nonlinear time-fractional Burgers equation. Gomez-Aguilar et al. (2017) presented the homotopy perturbation transform method for nonlinear fractional partial differential equations of the Caputo-Fabrizio fractional operator. Atangana and Gomez-Aguilar (2017) studied the numerical approximation of fractional differentiation based on the Riemann-Liouville definition, from power-law kernel to the generalized Mittag-Leffler-law via exponential-decay-law. Also Yepez-Martinez et al. (2016) employed the fractional derivatives in the sense of the modified Riemann-Liouville derivative and the Feng's first integral method for solving the nonlinear coupled space-time fractional mKdV partial differential equation.

Differential Transform Method (DTM) was initially proposed by Zhou (1986), who solved linear and nonlinear problems in electrical circuit problems. Chen and Ho (1999) applied this method to partial differential equations. The essential definitions and applications of DTM in various types of differential equations were presented in Hassan (2002), Bildik and Konuralp (2006), Ayaz (2004), Arikoglu and Ozkol (2005). On the other hand, Keskin and Oturanc (2009) presented the Reduced Differential Transform Method (RDTM) for finding approximate analytical solutions of partial differential equations. Then, Keskin and Oturanc (2010) proposed the Fractional Reduced Differential Transform Method (FRDTM). The applicability of the recent method to several different types of fractional differential equations has been presented recently. For examples, Saravanan and Magesh (2016) presented numerical solutions of linear and nonlinear Fokker-Planck partial differential equations with space and time fractional derivatives and Gupta (2011) presented the approximate analytical solutions of Benney-Lin equation with fractional time derivative. Singh and Kumar (2016) applied FRDTM to find approximate solution of time-fractional order multi-dimensional Navier-Stokes equations and Singh and Srivastava (2015) gave FRDTM approximate series solution of the multi-dimensional (heat-like) diffusion equation with time-fractional-order. The FRDTM approximate solution of time-fractional Korteweg-de Vries equation was presented by Ebenezer et al. (2016). Rawashdeh (2017) employed FRDTM to solve nonlinear fractional partial differential equations such as

* Corresponding author.
E-mail addresses: abuasadsalah@yahoo.com (S. Abuasad), moaddy@yahoo.com (K. Moaddy), ishak_h@ukm.edu.my (I. Hashim).

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the space-time fractional Burgers equations and the time-fractional Cahn-Allen equations. An application of FRDTM to a system of linear and nonlinear fractional partial differential equations was done by Singh (2016). In Srivastava et al. (2014), FRDTM was used to obtain exact solution of a mathematical model for the generalised time fractional-order biological population model.

Helmholtz equation (or reduced wave equation) is an elliptic partial differential equation which can be derived directly from the wave equation. In the Cartesian coordinate system, consider the two-dimensional nonhomogeneous isotropic medium whose speed is c. The wave solution is $u(x,y)$ corresponding to a harmonic source $\Phi(x,y)$ vibrating at a given fixed frequency $\omega > 0$ satisfying the scalar Helmholtz equation on a given region R:

$$\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) + \lambda u(x,y) = -\Phi(x,y), \tag{1}$$

where $u(x,y)$ is a sufficiently differentiable function on the boundary of R, and $\Phi(x,y)$ is a given function, $\lambda > 0$ is a constant number, and $\sqrt{\lambda} = \omega/c$ is the wavenumber with wavelenght $2\pi/\sqrt{\lambda}$ (Thompson and Pinsky, 1995). If $\Phi(x,y) = 0$, then Eq. (1) is homogeneous Helmholtz equation. Many problems related to steady-state oscillations (mechanical, acoustical, thermal, electromagnetic) lead to the two-dimensional Helmholtz equation, if the plus sign (in front of the λ term) is switched to a minus sign, then this equation describes mass transfer processes with volume chemical reactions of the first order (Polyanin and Nazaikinskii, 2015). For example, in linear acoustics $\Phi(x,y)$ might represent a perturbation in pressure about a reference state (Thompson and Pinsky, 1995). Helmholtz equation in two dimensions has been studied by many authors with the finite difference method (FDM), Adomian decomposition method (ADM) (El-Sayed and Kaya, 2004), multiple theory (MT) method (Zheng et al., 1999), the finite element method (FEM) and the boundary element method (BEM). Thompson and Pinsky (1995) proposed Galerkin least-squares (GLS) finite element method for solving the Helmholtz equation. Also, wave simulation with the finite difference method for the Helmholtz equation based on the domain decomposition method was investigated by Zhang and Dai (2013). Gupta et al. (2012) obtained the approximate analytical solutions of a multidimensional fractional Helmholtz equation using homotopy perturbation method (HPM). Samuel and Thomas (2010) derived an analytic solution for the fractional Helmholtz equation in terms of the Mittag-Leffler function.

The aim of this paper is to apply FRDTM to the Helmholtz equation with x-space fractional order of the form:

$$\mathbf{D}_x^\alpha u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) + \lambda u(x,y) = -\Phi(x,y), \tag{2}$$

subject to the initial condition

$$u(0,y) = \psi(y). \tag{3}$$

Also, we can similarly apply FRDTM to the Helmholtz equation with y-space fractional order of the form:

$$\mathbf{D}_y^\alpha u(x,y) + \frac{\partial^2}{\partial x^2}u(x,y) + \lambda u(x,y) = -\Phi(x,y), \tag{4}$$

subject to the initial condition

$$u(x,0) = \psi(x), \tag{5}$$

where $\psi(x)$ is a given function and $1 < \alpha \leqslant 2$.

As we will show in the present work, the exact and approximate solutions using FRDTM of Helmholtz equations with the fractional order suggest new and promising interpretations for steady-state oscillations more than the integer-order derivatives; i.e., in the fractional derivatives we can find range of solutions depending on the fractional order $(1 < \alpha \leq 2)$ and this is actually one of the main reasons for generalizing the integer-order differential equations to fractional-order differential equations.

The paper is organized as follows: after presenting basic definitions and properties of fractional calculus in Section 2, we introduce the proposed method in Section 3. Section 4 presents the exact and approximate solutions of three examples of fractional Helmholtz equation. Section 5 concludes the study.

2. Preliminaries of fractional calculus

In this section, we present some useful definitions associated with fractional calculus. Firstly, we define the Mittag-Leffler function, which plays a major role in fractional calculus. There are several definitions of fractional derivatives, for examples, Riemann-Liouville, Caputo, Hadamard, Erdélyi-Kober, and Grünwald-Letnikov Oldham and Spanier (1974). A survey of many different applications which have emerged from fractional calculus was given by Podlubny (1999). In this work we use the Caputo fractional derivative.

Definition 2.1. Mittag-leffler function

The Mittag-Leffler function is a direct generalization of the exponential function, e^x. The two-parameter Mittag-Leffler function is defined in powers series by the formula:

$$E_{\gamma,\beta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\gamma k+\beta)}, \quad (\gamma > 0,\ \beta > 0). \tag{6}$$

The one-parameter Mittag-Leffler function is defined as:

$$E_{\gamma}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\gamma k+1)}, \quad (\gamma > 0). \tag{7}$$

For special choices of the values of the parameters γ, β we obtain well-known classical functions:

$$E_{1,1}(x) = e^x, \quad E_{1,2}(x) = \frac{e^x-1}{x},$$
$$E_{2,1}(x^2) = \cosh(x), \quad E_{2,2}(x^2) = \frac{\sinh(x)}{x}.$$

Definition 2.2. Caputo fractional derivative (Kilbas et al., 2006).

Let $a \in \mathbb{R}$, then the (Left-sided) Caputo fractional derivative $({}^cD_{a+}^\alpha y)(x)$ (the small c denotes the Caputo derivative) of order $\alpha \in \mathbb{R}^+$ is defined as:

$$({}^cD_{a+}^\alpha y)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}}\,\mathrm{d}t, \tag{8}$$

for $(n-1 < \alpha < n; x \geqslant a)$, $n \in \mathbb{N}$ and $\Gamma(x)$ is the well-known Gamma function.

For simplicity, we denote the Caputo fractional derivative as $\mathbf{D}_x^\alpha f(x)$.

3. Fractional Reduced Differential Transform Method (FRDTM)

In this section, we give the basic definitions and properties of FRDTM.

Consider a function of two variables $u(x,y)$, such that $u(x,y) = p(x)q(y)$, then from the properties of the one-dimensional differential transform method (DTM), we have

$$u(x,y) = \sum_{i=0}^{\infty}p(i)x^i\sum_{j=0}^{\infty}q(j)y^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}U(i,j)x^iy^j, \tag{9}$$

where $u(i,j) = p(i)q(j)$ is referred to as the spectrum of $u(x,y)$. Also, the lowercase $u(x,y)$ is used for the original function, while its fractional reduced transformed function is represented by the uppercase $U_k(y)$, which is called the T-function.

Definition 3.1. FRDTM (Singh and Srivastava, 2015; Srivastava et al., 2014).

Let $u(x,y)$ be an analytical and continuously differentiable with respect to two variables x and y in the domain of interest, then FRDTM of $u(x,y)$ is given by

$$U_k(y) = \frac{1}{\Gamma(k\alpha+1)}\left[\mathbf{D}_x^{\alpha k}(u(x,y))\right]_{x=x_0}, k = 0,1,2,\ldots \tag{10}$$

with x-space fractional derivative.

The inverse FRDTM of $U_k(y)$ is defined by

$$u(x,y) := \sum_{k=0}^{\infty} U_k(y)(x-x_0)^{k\alpha}. \tag{11}$$

From (10) and (11), we have

$$u(x,y) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\mathbf{D}_x^{\alpha k}(u(x,y))\right]_{x=x_0}(x-x_0)^{k\alpha}. \tag{12}$$

In particular, for $x_0 = 0$, the above equation becomes

$$u(x,y) = um_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\mathbf{D}_x^{\alpha k}(u(x,y))\right]_{x=x_0}x^{k\alpha}. \tag{13}$$

From the above definition, it can be found that the concept of FRDTM is derived from the power series expansion of a function. Then the inverse transformation of the set of values $\{U_k(y)\}_{k=0}^{n}$ gives approximate solution as

$$\tilde{u}_n(x,y) = \sum_{k=0}^{n} U_k(y)x^k, \tag{14}$$

where n is the order of approximate solution. Therefore, the exact solution is given by

$$u(x,y) = \lim_{n\to\infty}\tilde{u}_n(x,y). \tag{15}$$

In Table 1 we give some properties of FRDTM, where $\delta(k-m)$ is defined by

$$\delta(k-m) = \begin{cases} 1, & k = m \\ 0, & k \neq m. \end{cases} \tag{16}$$

4. Numerical examples

In this section we demonstrate the applicability of FRDTM via test examples.

Table 1
Fundamental operations of FRDTM.

Original function	Transformed function
$w(x,y) = c_1u(x,y) \pm c_2v(x,y)$	$W_k(y) = c_1U_k(y) \pm c_2V_k(y)$
$w(x,y) = u(x,y)v(x,y)$	$W_k(y) = \sum_{i=0}^{k}U_i(y)V_{k-i}(y)$
$w(x,y) = \mathbf{D}_x^{n\alpha}u(x,y)$	$W_k(y) = \frac{\Gamma(k\alpha+n\alpha+1)}{\Gamma(k\alpha+1)}U(k+n)(y)$
$w(x,y) = \frac{\partial^h u(x,y)}{\partial y^h}$	$W_k(y) = \frac{\partial^h U_k(y)}{\partial y^h}$
$w(x,y) = x^my^n$	$W_k(y) = y^n\delta(k-m)$
$w(x,y) = x^my^nu(x,y)$	$W_k(y) = y^nU_{k-m}(y)$
$w(x,y) = e^{\lambda x}$	$W_k(y) = \frac{\lambda^k}{k!}$
$w(x,y) = \sin(\omega x+\alpha)$	$W_k(y) = (\omega^k/k!)\sin[(\pi k/2!)+\alpha]$
$w(x,y) = \cos(\omega x+\alpha)$	$W_k(y) = (\omega^k/k!)\cos[(\pi k/2!)+\alpha]$

4.1. Example 1

Consider the following fractional homogeneous Helmholtz equation with x-space fractional derivative:

$$\mathbf{D}_x^{\alpha}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) - u(x,y) = 0, \tag{17}$$

where $1 < \alpha \leqslant 2$, with the initial condition

$$u(0,y) = y. \tag{18}$$

Applying the appropriate properties given in Table 1 to Eq. (17), we obtain the following recurrence relation:

$$U_{k+1}(y) = \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}\left(U_k(y) - \frac{\partial^2 U_k(y)}{\partial y^2}\right), \tag{19}$$

for $k = 0,1,2,\ldots$. From (19) we obtain the inverse transform coefficients of $x^{k\alpha}$ as follows:

$$U_0 = y, \quad U_1 = \frac{y}{\Gamma(\alpha+1)},$$
$$U_2 = \frac{y}{\Gamma(2\alpha+1)}, \quad U_3 = \frac{y}{\Gamma(3\alpha+1)}, \ldots.$$

Or in general,

$$U_k = \frac{y}{\Gamma(1+k\alpha)}, \quad \text{where } k \geqslant 0. \tag{20}$$

Continuing in the same manner and after a few iterations, the differential inverse transform of $\{U_k(y)\}_{k=0}^{\infty}$ will give the following series solution:

$$u(x,y) = \sum_{k=0}^{\infty}U_k(y)x^{k\alpha}$$
$$= y + \frac{y}{\Gamma(1+\alpha)}x^{\alpha} + \frac{y}{\Gamma(1+2\alpha)}x^{2\alpha} + \frac{y}{\Gamma(1+3\alpha)}x^{3\alpha} + \cdots,$$

which can be written in compact form,

$$u(x,y) = y\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}. \tag{21}$$

Now using the definition of Mittag-Leffler function, we obtain the exact solution of Eq. (17) subject to (18):

$$u(x,y) = yE_{\alpha}(x^{\alpha}), \tag{22}$$

where $1 < \alpha \leqslant 2$, and $E_{\alpha}(z)$ is the one-parameter Mittag-Leffler function (7).

In the case of $\alpha = 2$, and since the Hyperbolic cosine is a particular case of Mittag-Leffler function,

$$E_2(x^2) = \sum_{k=0}^{\infty}\frac{x^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!} = \cosh x. \tag{23}$$

Then, the exact solution of Eq. (17) when $\alpha = 2$ subject to (18) is

$$u(x,y) = y\cosh x. \tag{24}$$

In similar way, we can apply the FRDTM to fractional homogeneous Helmholtz equation with y-space fractional derivative:

$$\mathbf{D}_y^{\alpha}u(x,y) + \frac{\partial^2}{\partial x^2}u(x,y) - u(x,y) = 0, \tag{25}$$

where $1 < \alpha \leqslant 2$, with the initial condition

$$u(x,0) = x. \tag{26}$$

Then, the exact solution of Eq. (25) subject to (26) is

$$u(x,y) = xE_{\alpha}(y^{\alpha}). \tag{27}$$

Generally, if the plus sign (in front of the λ term) is switched to a minus sign, the exact solution of the homogeneous Helmholtz Eq. (2) subject to (3) with x-space fractional order is

$$u(x,y) = yE_\alpha(\lambda(x^\alpha)). \tag{28}$$

While, if the plus sign (in front of the λ term) is switched to a minus sign, the exact solution of the homogeneous Helmholtz Eq. (4) subject to (5) with y-space fractional order is

$$u(x,y) = xE_\alpha(\lambda(y^\alpha)). \tag{29}$$

The 3-dimensional plots of the FRDTM solutions of (17) with initial condition (18) are shown in Fig. 1 for different values of $\alpha = 2, 1.8, 1.6$. Fig. 2 depicts solutions in 2-dimensional plots for different values of $\alpha = 2, 1.8, 1.6, 1.4$ for $x \in [0,1]$ and $y = 1$. On the other hand, Fig. 3 depicts for different values of λ and constant $\alpha = 1.5$ the solutions in 2-dimensional plots; $y = 1$. In similar way, we can plots the figures for y-space fractional derivative.

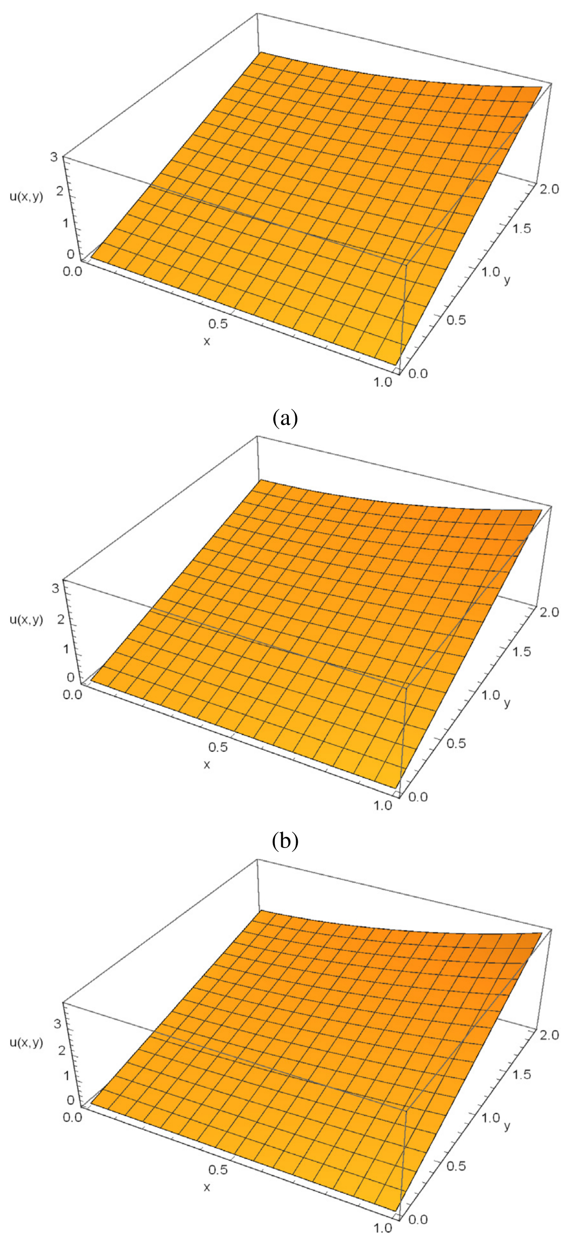

Fig. 1. The FRDTM solutions u: (a) $\alpha = 2$, (b) $\alpha = 1.8$ and (c) $\alpha = 1.5$.

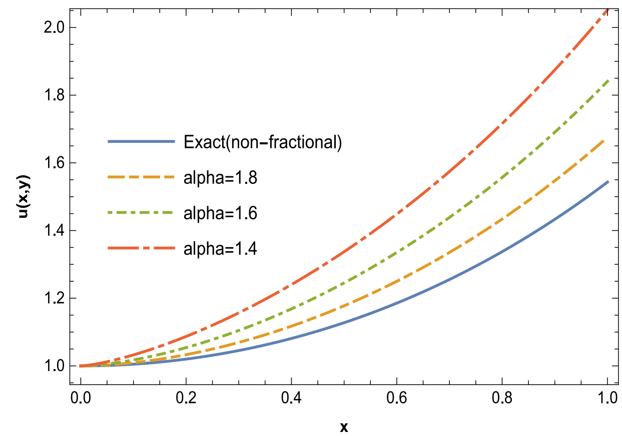

Fig. 2. The FRDTM solutions u for $\alpha = 2$ (exact), 1.8, 1.6, 1.4; $x \in [0,1]$ and $y = 1$.

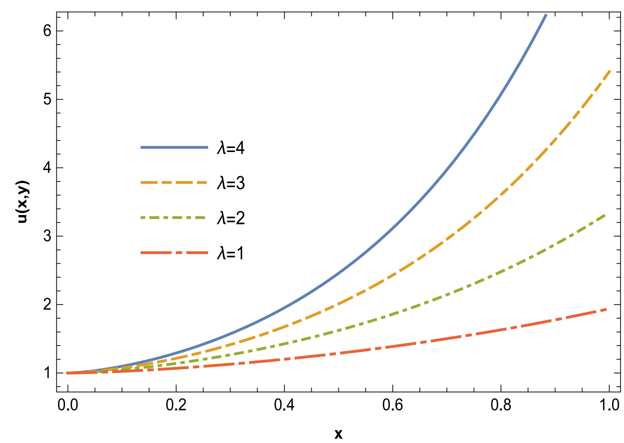

Fig. 3. The FRDTM solutions u for $\lambda = 4, 3, 2, 1$; $\alpha = 1.5$; $x \in [0,1]$ and $y = 1$.

4.2. Example 2

Now, we consider another especial case of fractional homogeneous Helmholtz equation with $\lambda = 5$ and x-space fractional derivative to illustrate the efficiency of FRDTM.

$$\mathbf{D}_x^\alpha u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) + 5u(x,y) = 0, \tag{30}$$

where $1 < \alpha \leqslant 2$ with the initial condition

$$u(0,y) = y. \tag{31}$$

Using the appropriate properties from Table 1 to Eq. (30), we obtain the following recurrence relation:

$$U_{k+1}(y) = \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}\left(-5U_k(y) - \frac{\partial^2 U_k(y)}{\partial y^2}\right), \tag{32}$$

where $k = 0, 1, 2, \ldots$. The inverse transform coefficients of $x^{k\alpha}$ are as follows:

$$U_0 = y, \quad U_1 = -\frac{5y}{\Gamma(\alpha+1)},$$
$$U_2 = \frac{25y}{\Gamma(2\alpha+1)}, \quad U_3 = -\frac{125y}{\Gamma(3\alpha+1)}, \ldots.$$

More generally,

$$U_k = \frac{(-5)^k}{\Gamma(1+k\alpha)}. \tag{33}$$

Again, if we continue in the same manner and after a few iterations, the differential inverse transform of $\{U_k(y)\}_{k=0}^{\infty}$ will give the following series solution:

$$u(x,y) = \sum_{k=0}^{\infty} U_k(y)x^{k\alpha}$$
$$= y - \frac{5y}{\Gamma(\alpha+1)}x^{\alpha} + \frac{25y}{\Gamma(2\alpha+1)}x^{2\alpha} - \frac{125y}{\Gamma(3\alpha+1)}x^{3\alpha} + \cdots.$$

In compact form,

$$u(x,y) = y\sum_{k=0}^{\infty}\frac{(-5x^{\alpha})^k}{\Gamma(1+k\alpha)}, \tag{34}$$

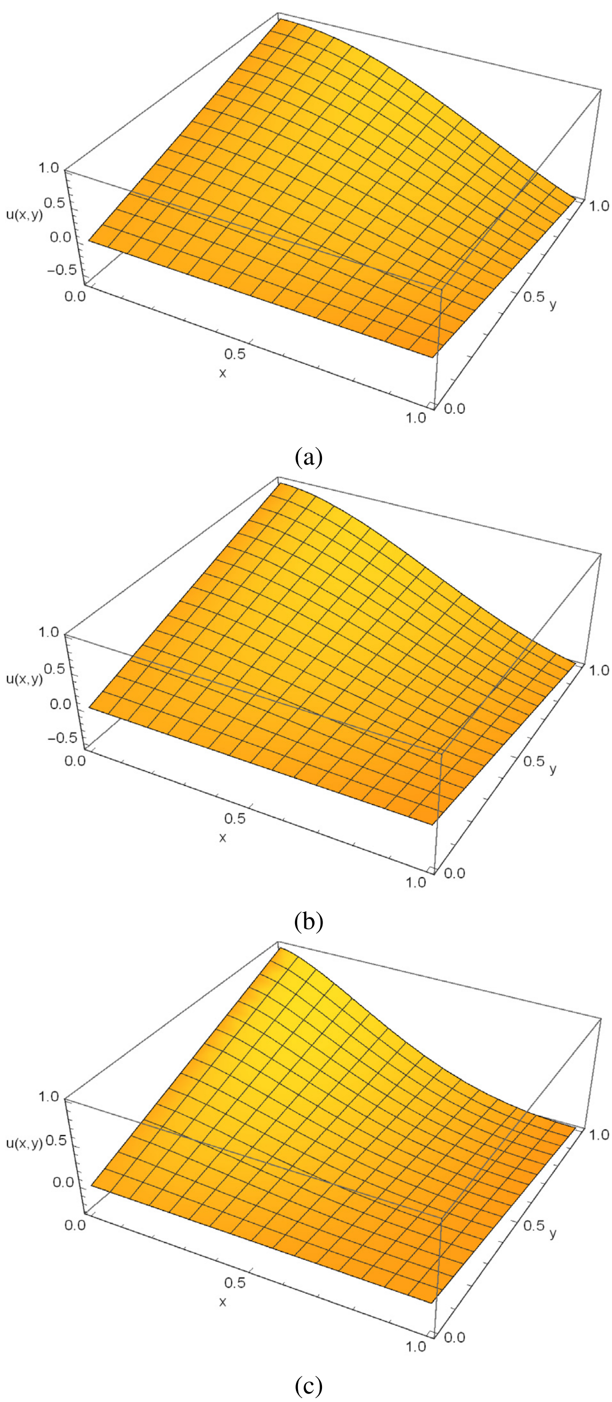

Fig. 4. The FRDTM solutions u: (a) $\alpha = 2$, (b) $\alpha = 1.8$ and (c) $\alpha = 1.5$.

and using the Mittag-Leffler function, we obtain the exact solution

$$u(x,y) = yE_{\alpha}(-5x^{\alpha}), \tag{35}$$

where $1 < \alpha \leqslant 2$, and $E_{\alpha}(z)$ is the one-parameter Mittag-Leffler function (7).

In the case of $\alpha = 2$, and since the cosine is a particular case of Mittag-Leffler function,

$$E_2(-5x^2) = \sum_{k=0}^{\infty}\frac{(-5x^2)^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k(\sqrt{5}x)^{2k}}{(2k)!} = \cos\sqrt{5}x. \tag{36}$$

Then, the exact solution of Eq. (30) when $\alpha = 2$ subject to (31) is

$$u(x,y) = y\cos\sqrt{5}x. \tag{37}$$

Also, the fractional homogeneous Helmholtz equation with $\lambda = 5$ and y-space fractional derivative can be easily solved using FRDTM

$$\mathbf{D}_y^{\alpha}u(x,y) + \frac{\partial^2}{\partial x^2}u(x,y) + 5u(x,y) = 0, \tag{38}$$

where $1 < \alpha \leqslant 2$ with the initial condition

$$u(x,0) = x. \tag{39}$$

So, the exact solution of Eq. (38) subject to (39)is

$$u(x,y) = yE_{\alpha}(-5x^{\alpha}), \tag{40}$$

Generally, the exact solution of the homogeneous Helmholtz Eq. (2) subject to (3) with x-space fractional order is

$$u(x,y) = yE_{\alpha}(-\lambda(x^{\alpha})). \tag{41}$$

While, the exact solution of the homogeneous Helmholtz Eq. (4) subject to (5) with y-space fractional order is

$$u(x,y) = xE_{\alpha}(-\lambda(y^{\alpha})). \tag{42}$$

Figs. 4 and 5 show the 3-dimensional and 2-dimensional plot of the FRDTM solutions respectively. While, Fig. 6 depicts the solutions for different values of λ.

4.3. Example 3

Now we consider the following two-dimensional inhomogeneous Helmholtz equation with $\lambda = -2$ and x-space fractional derivative:

$$\mathbf{D}_x^{\alpha}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) - 2u(x,y) = (12x^2 - 3x^4)\sin y, \tag{43}$$

where $1 < \alpha \leqslant 2, 0 \leqslant x \leq 1$, and $0 \leqslant y \leqslant 2\pi$, subject to the initial condition:

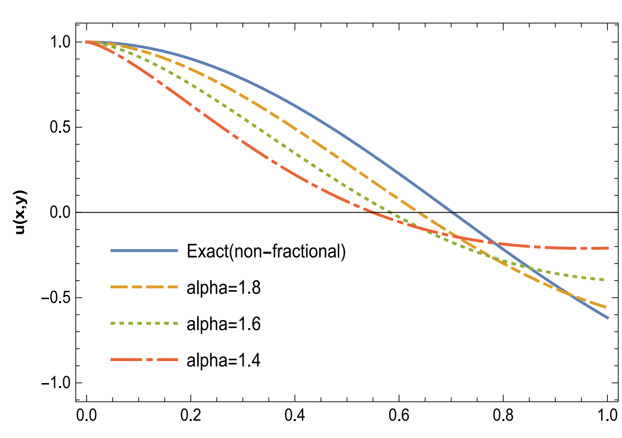

Fig. 5. The FRDTM solutions u for $\alpha = 2\,(\text{exact}), 1.8, 1.6, 1.4; x \in [0,1]$ and $y = 1$.

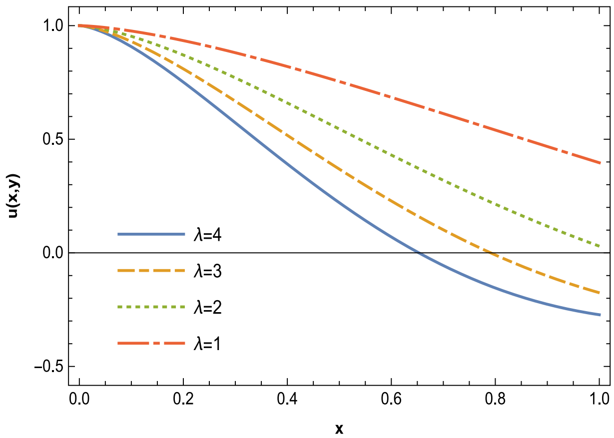

Fig. 6. The FRDTM solutions u for $\lambda = 4, 3, 2, 1; \alpha = 1.5; x \in [0,1]$ and $y = 1$.

$$u_0(x,y) = \left(x^4 - \frac{x^6}{10}\right)\sin y. \tag{44}$$

This fractional equation was solved using the homotopy perturbation method by Gupta et al. (2012) and the exact solution when $\alpha = 2$ is

$$u(x,y) = x^4 \sin y. \tag{45}$$

Using the appropriate properties from Table 1 to Eq. (30), we obtain the following recurrence relation:

$$U_{k+1}(y) = \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)} \times \left((2U_k(y) - \frac{\partial^2 U_k(y)}{\partial y^2} + 12\sin y\,\delta(k-2) - 3\sin y\,\delta(k-4)\right).$$

The initial conditions (44) yield

$$U_0(y) = \left(x^4 - \frac{x^6}{10}\right)\sin y. \tag{47}$$

The inverse transform coefficients of $x^{k\alpha}$ where $k = 0, 1, 2, \ldots$ are

$$U_0 = \left(x^4 - \frac{x^6}{10}\right)\sin y$$
$$U_1 = \frac{3\left(x^4 - \frac{x^6}{10}\right)\sin y}{\Gamma(\alpha+1)}$$
$$U_2 = \frac{9\left(x^4 - \frac{x^6}{10}\right)\sin y}{\Gamma(2\alpha+1)}$$
$$U_3 = \frac{3\sin y(40\Gamma(2\alpha+1) - 9x^6 + 90x^4)}{10\Gamma(3\alpha+1)}$$
$$U_4 = \frac{9\sin y(40\Gamma(2\alpha+1) - 9x^6 + 90x^4)}{10\Gamma(4\alpha+1)}$$
$$U_5 = \frac{3\sin y m(x,\alpha)}{\Gamma(5\alpha+1)}$$
$$U_6 = \frac{3^2\sin y m(x,\alpha)}{\Gamma(6\alpha+1)}, \ldots$$

where $m(x,\alpha)$ is defined as:

$$m(x,\alpha) = \frac{1}{10}(360\Gamma(2\alpha+1) - 10\Gamma(4\alpha+1) - 81(x^2 - 10)x^4). \tag{48}$$

More generally, for $k \geqslant 5$, we have:

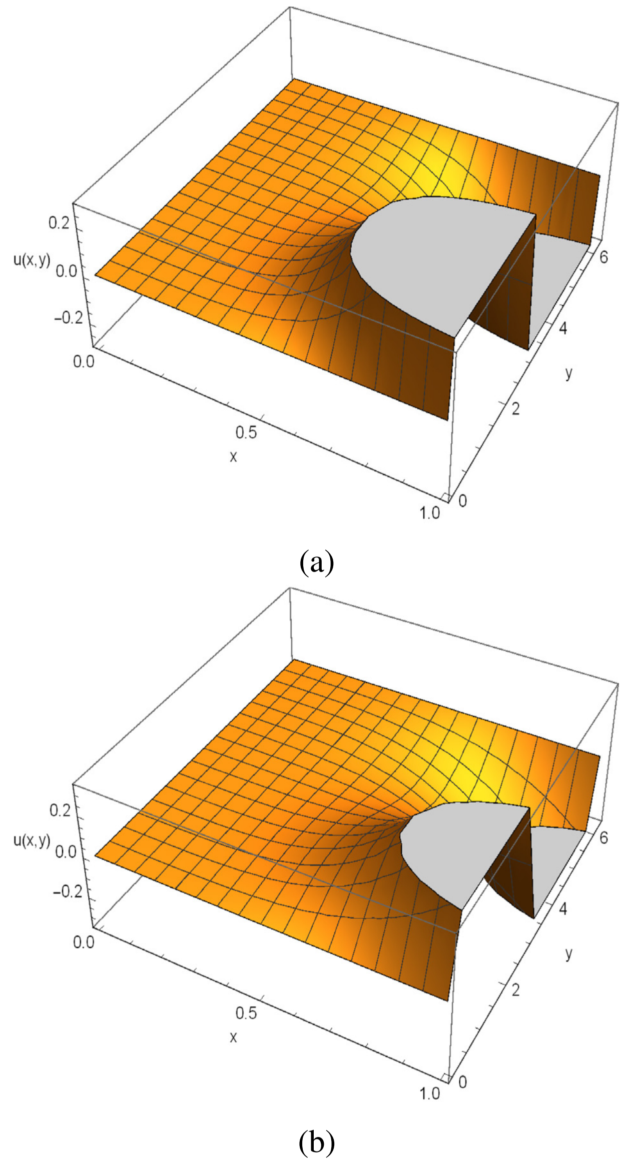

Fig. 7. (a) The FRDTM solution and (b) the corresponding exact solution (non-fractional) at $\alpha = 2$.

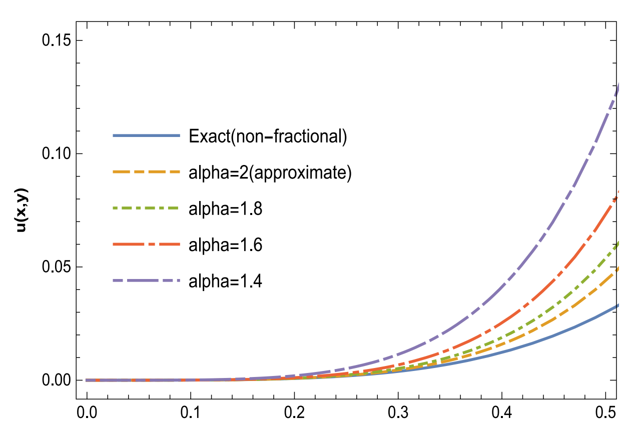

Fig. 8. The exact solution (non-fractional) and FRDTM approximate solutions for several values of α at $y = 0.5$.

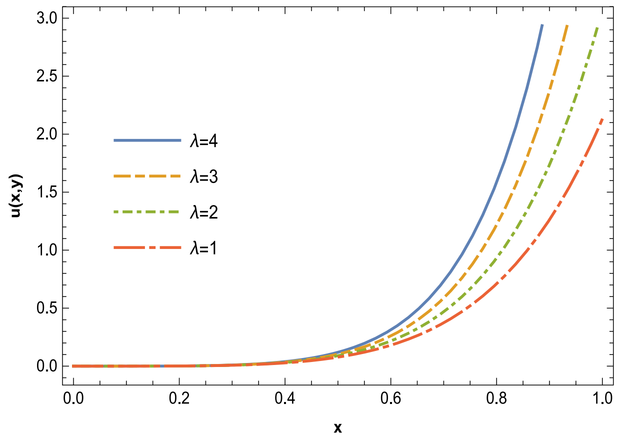

Fig. 9. The FRDTM approximate solutions for several values of λ at $y = 0.5$ and $\alpha = 1.5$.

$$U_k = \frac{3^{k-4}\sin y\ m(x,\alpha)}{\Gamma(k\alpha+1)} = \frac{\sin y\ m(x,\alpha)}{3^4}\left(\frac{3^k}{\Gamma(k\alpha+1)}\right).$$

If we continue in the same manner and after a few iterations, the differential inverse transform of $\{U_k(y)\}_{k=0}^{\infty}$ will give the following series solution:

$$\begin{aligned} u(x,y) &= \sum_{k=0}^{\infty} U_k(y)x^{k\alpha} \\ &= \left(1 + \frac{3x^{\alpha}}{\Gamma(\alpha+1)} + \frac{9x^{2\alpha}}{\Gamma(2\alpha+1)}\right)\left(x^4 - \frac{x^6}{10}\right)\sin y \\ &\quad + \left(\frac{3x^{3\alpha}}{10\Gamma(3\alpha+1)} + \frac{9x^{4\alpha}}{10\Gamma(4\alpha+1)}\right) \\ &\quad * \left(40\Gamma(2\alpha+1) - 9x^6 + 90x^4\right)\sin y \\ &\quad + \frac{\sin y m(x,\alpha)}{3^4}\sum_{k=5}^{\infty}\left(\frac{(3x^{\alpha})^k}{\Gamma(k\alpha+1)}\right). \end{aligned} \tag{49}$$

If we change the index of summation of the last term in (49), then we can use the definition of two-parameter Mittag-Leffler function (6) to arrive at the exact solution

$$\begin{aligned} u(x,y) &= \left(1 + \frac{3x^{\alpha}}{\Gamma(\alpha+1)} + \frac{9x^{2\alpha}}{\Gamma(2\alpha+1)}\right)\left(x^4 - \frac{x^6}{10}\right)\sin y \\ &\quad + \left(\frac{3x^{3\alpha}}{10\Gamma(3\alpha+1)} + \frac{9x^{4\alpha}}{10\Gamma(4\alpha+1)}\right) \\ &\quad * \left(40\Gamma(2\alpha+1) - 9x^6 + 90x^4\right)\sin y \\ &\quad + 3x^{5\alpha}m(x,\alpha)\sin y E_{\alpha,5\alpha+1}(3x^{\alpha}). \end{aligned}$$

Fig. 7 shows the comparison between the exact (non-fractional) solution and FRDTM approximate solution (when $\alpha = 2$) for $x \in [0,1]$ and $y \in [0, 2\pi]$. In Fig. 8 we depict the solution for different values of α and the exact solution (when $\alpha = 2$) of non-fractional order at $y = 0.5$. Fig. 9 depicts the approximate solutions for different values of $\lambda; \alpha = 1.5$ and $y = 0.5$.

5. Conclusion

An effective FRDTM has been introduced to find the exact and approximate solutions of fractional Helmholtz equations with appropriate initial conditions. For the computational cost of FRDTM, we can clearly note that this method involves two main steps: firstly, find the recurrence relation U_k, using the fundamental operations of FRDTM. Then, find the inverse transform coefficients of $x^{k\alpha}$ which leads to a series solution. In most cases, we can write the solution as a compact form and with the help of Mittag-Leffler function, we can find the exact or approximate solutions easily. In Examples 1 and 2 we note that if $\alpha = 2$, then the FRDTM is the same as the RDTM. In Example 3 we obtain the approximate solution (not exact) when $\alpha = 2$ and compare it with the exact solution (of non-fractional order).

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