Journal of King Saud University – Science (2011) **23**, 407–411

King Saud University
Journal of King Saud University – Science
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

Application of He's variational iteration method for solution of the family of Kuramoto–Sivashinsky equations

Mehdi Gholami Porshokouhi *, Behzad Ghanbari

Department of Mathematics, Faculty of Science Islamic Azad University, Takestan Branch, Iran

Received 20 June 2010; accepted 22 July 2010
Available online 30 July 2010

KEYWORDS
He's variational iteration method;
Kuramoto–Sivashinsky equations

Abstract In this paper, a kind of analytical technique for a non-linear problem called the variational iteration method VIM is used to give approximate solutions for the Kuramoto–Sivashinsky equations. The VIM is to construct correction functionals using general Lagrange multipliers identified optimally via the variational theory. This method constructs a convergent sequence of functions, which approximates the exact solution of problems. Comparisons of the obtained results with exact solutions reveal that this method is very effective and simple and could be applied for non-linear problems.

1. Introduction

In recent years, He's variational iteration method has been favorably applied to various kinds of problems; for example, this scheme is used for solving the fractional KdV–Burgers–Kuramoto equation (Safari et al., 2009). This technique computes the exact solution of equations using the initial condition only. It is also important to note that the present method does not require discretization of the equation. Therefore, it is not affected by computation round-off errors and one is not faced with the necessity of large computer memory and time. Furthermore, using this idea we do not need to solve any linear or non-linear system of equations. In Biazar and Ghazvini (2007) VIM is employed to solve fourth-order parabolic equations. Also, this method is employed in He (1997b) to solve delay differential equations. The interested reader is referred to Dehghan and Tatari (2006), Wazwaz (2007), Sweilam and Khader (2007) for some other applications of the method.

The Kuramoto–Sivashinsky equation is a non-linear evolution equation and has many applications in a variety of physical phenomena such as reaction diffusion systems (Kuramoto and Tsuzuki, 1976), long waves on the interface between two viscous fluids (Hooper and Grimshaw, 1985) and thin hydrodynamics films (Sivashinsky, 1983). The Kuramoto–Sivashinsky equation has been studied numerically by many authors (Akrivis and Smyrlis, 2004; Manickam et al., 1998; Uddin et al., 2009).

* Corresponding author.
E-mail addresses: m_gholami_p@yahoo.com (M.G. Porshokouhi), b.ghanbary@yahoo.com (B. Ghanbari).

Peer-review under responsibility of King Saud University.
doi:10.1016/j.jksus.2010.07.023

Production and hosting by Elsevier

Consider the Kuramoto–Sivashinsky equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^3 u}{\partial x^3} + \beta\frac{\partial^4 u}{\partial x^4} = 0. \tag{1}$$

Subject to the initial condition

$$u(x,0) = f(x), \quad a \leqslant x \leqslant b. \tag{2}$$

And boundary conditions

$$u(a,t) = g_1(t), \quad u(b,t) = g_2(t), \quad t > 0.$$

In this paper we use VIM for solving this equation, which was first proposed by He (1997a), and the convergence of the method is systematically discussed by Tatari and Dehghan(2007).

2. Using VIM to solve Kuramoto–Sivashinsky equations

To clarify the basic ideas of VIM, we consider the following differential equation

$$Lu(t) + Nu(t) = g(t),$$

where L is a linear operator, N is a non-linear operator and $g(t)$ an inhomogeneous term.

According to VIM, we can write down a correction functional as following

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi)(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))d\xi.$$

where λ is a Lagrange multiplier which could be identified optimally via variational theory, u_n is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

In this section the application of the VIM is discussed for solving Eq. (1). According to the correction functional in t – direction in the following form

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left\{\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{u}_n(x,\xi)\frac{\partial \tilde{u}_n(x,\xi)}{\partial x} + \alpha\frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} + \gamma\frac{\partial^3 \tilde{u}_n(x,\xi)}{\partial x^3} + \beta\frac{\partial^4 \tilde{u}_n(x,\xi)}{\partial x^4}\right\}d\xi.$$

Taking the variation with respect to the independent variable u_n and noticing that $\delta u_n(0) = 0$, we get

$$\delta u_{n+1} = \delta u_n + \lambda\delta u_n|_{\xi=t} - \int_0^t \lambda'\delta u_n d\xi = 0.$$

Then we apply the following stationary conditions:

$$\lambda'(\xi)|_{\xi=t} = 0, \quad 1 + \lambda(\xi)|_{\xi=t} = 0.$$

The general Lagrange multiplier, therefore, could be readily identified:

$$\lambda = -1.$$

And as result, we obtain the following iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{\frac{\partial u_n(x,\xi)}{\partial \xi} + u_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} + \alpha\frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \gamma\frac{\partial^3 u_n(x,\xi)}{\partial x^3} + \beta\frac{\partial^4 u_n(x,\xi)}{\partial x^4}\right\}d\xi.$$

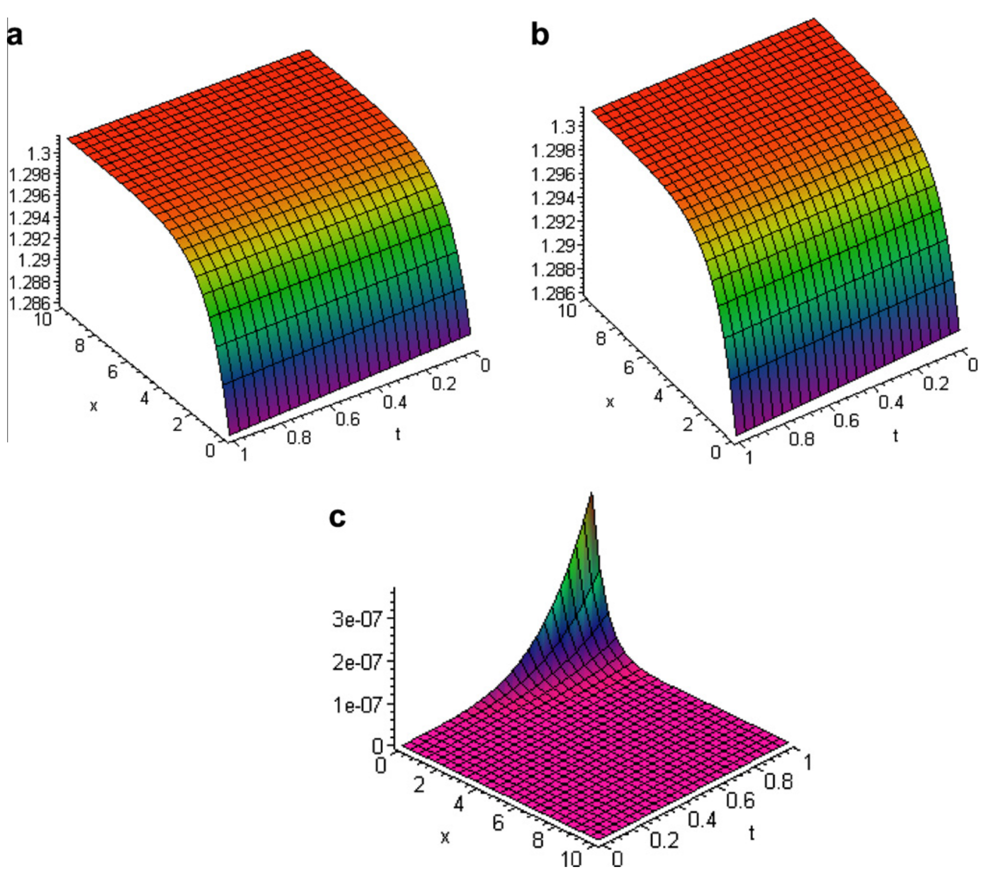

Figure 1 The surface shows the solution $u(x,t)$ for Example 1 when $c = 0.1$, $k = \frac{1}{2}\sqrt{\frac{11}{19}}$, $x_0 = -10$: (a) exact solution (3), (b) 3th order of approximate solution (4) and (c) the absolute error between exact and numerical solutions.

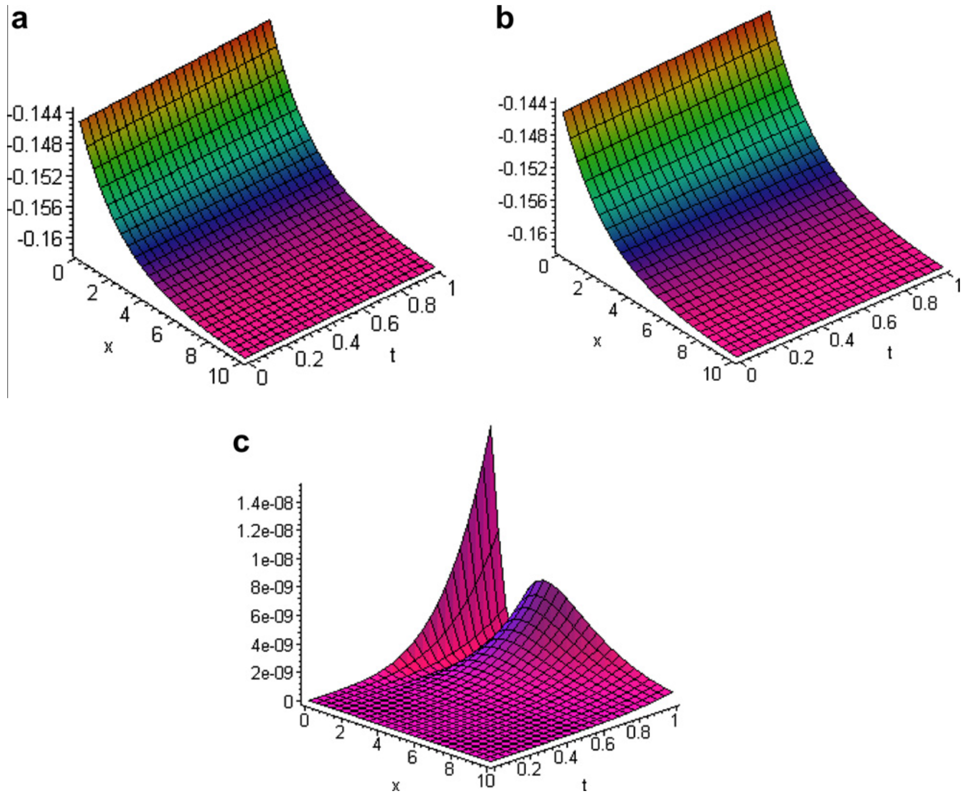

Figure 2 The surface shows the solution $u(x,t)$ for Example 2 when $c = 0.2$, $k = \frac{1}{2\sqrt{19}}$, $x_0 = -10$: (a) exact solution (5), (b) 3th order of approximate solution (6) and (c) the absolute error between exact and numerical solutions.

We start with the initial approximation of $u(x,0)$ given by Eq. (2). Using the above iteration formula, we can obtain the other components as follows

$$u_0(x,t) = u(x,0) = f(x),$$

$$u_1(x,t) = u_0(x,t) - \int_0^t \left\{ \frac{\partial u_0(x,\xi)}{\partial \xi} + u_0(x,\xi)\frac{\partial u_0(x,\xi)}{\partial x} + \alpha \frac{\partial^2 u_0(x,\xi)}{\partial x^2} + \gamma \frac{\partial^3 u_0(x,\xi)}{\partial x^3} + \beta \frac{\partial^4 u_0(x,\xi)}{\partial x^4} \right\} d\xi,$$

$$u_2(x,t) = u_1(x,t) - \int_0^t \left\{ \frac{\partial u_1(x,\xi)}{\partial \xi} + u_1(x,\xi)\frac{\partial u_1(x,\xi)}{\partial x} + \alpha \frac{\partial^2 u_1(x,\xi)}{\partial x^2} + \gamma \frac{\partial^3 u_1(x,\xi)}{\partial x^3} + \beta \frac{\partial^4 u_1(x,\xi)}{\partial x^4} \right\} d\xi,$$

$$\vdots$$

3. Numerical results

In this section, we apply the technique discussed in the previous section to find numerical solution of the family of Kuramoto–Sivashinsky equations and compare our results with exact solutions. The results reveal that this method is very effective and simple.

Example 1. Consider the Kuramoto–Sivashinsky equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = 0.$$

With initial condition

$$u(x,0) = c + \frac{5}{19}\sqrt{\frac{11}{19}}\left[11\tanh^3(k(x-x_0)) - 9\tanh(k(x-x_0))\right].$$

Exact solution of problem is given by

$$u(x,t) = c + \frac{5}{19}\sqrt{\frac{11}{19}}[11\tanh^3(k(x-ct-x_0)) - 9\tanh(k(x-ct-x_0))]. \tag{3}$$

For solving by VIM we obtain the recurrence relation

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ \frac{\partial u_n(x,\xi)}{\partial \xi} + u_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} + \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \frac{\partial^4 u_n(x,\xi)}{\partial x^4} \right\} d\xi. \tag{4}$$

Starting with the initial approximation $u_0(x,t) = u(x,0)$ in (4), successive approximations $u_i(x,t)$'s will be achieved.

The plot of exact solution (3), the 3th order of approximate solution obtained using the VIM and comparison between the exact and numerical solutions of this example are shown in Fig. 1.

Example 2. Let us have the following form of Kuramoto–Sivashinsky equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = 0.$$

With initial condition

$$u(x,0) = c + \frac{15}{19\sqrt{19}}\left[\tanh^3(k(x-x_0)) - 3\tanh(k(x-x_0))\right].$$

The solution of the above problem is recognized

$$u(x,t) = c + \frac{15}{19\sqrt{19}}\left[\tanh^3(k(x-ct-x_0)) - 3\tanh(k(x-ct-x_0))\right].$$

For solving by VIM we obtain the recurrence relation

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{\frac{\partial u_n(x,\xi)}{\partial \xi} + u_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \frac{\partial^4 u_n(x,\xi)}{\partial x^4}\right\} d\xi. \tag{6}$$

Starting with the initial approximation $u_0(x,t) = u(x,0)$ in (6), the next iterates $u_i(x,t)$'s will be obtained.

The plot of exact solution (5), the 3th order of approximate solution obtained using the VIM and comparison between the exact and numerical solutions are shown in Fig. 2.

Example 3. As the last example, we try the case of $\alpha = 1,\ \gamma = 4,\ \beta = 1$ in (1) as:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^3 u}{\partial x^3} + \frac{\partial^4 u}{\partial x^4} = 0.$$

Its exact solution of above problem reads

$$u(x,t) = c + 9 - 15\left[\tanh(k(x-ct-x_0)) + \tanh^2(k(x-ct-x_0)) - \tanh^3(k(x-ct-x_0))\right]. \tag{7}$$

With initial condition

$$u(x,0) = c + 9 - 15\left[\tanh(k(x-x_0)) + \tanh^2(k(x-x_0)) - \tanh^3(k(x-x_0))\right].$$

For solving by VIM we obtain the recurrence relation

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{\frac{\partial u_n(x,\xi)}{\partial \xi} + u_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} + \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + 4\frac{\partial^3 u_n(x,\xi)}{\partial x^3} + \frac{\partial^4 u_n(x,\xi)}{\partial x^4}\right\} d\xi. \tag{8}$$

Using the initial approximation $u_0(x,t) = u(x,0)$ in (8), approximations $u_i(x,t)$'s will be calculated, successively.

The plot of exact solution (7), the 3th order of approximate solution obtained using the VIM and comparison between the exact and numerical solutions of this example are shown in Fig. 3.

4. Conclusion

In this paper the He's variational iteration method is used to solve the Kuramoto–Sivashinsky equations. We described the method, used it on three test problems, and compared

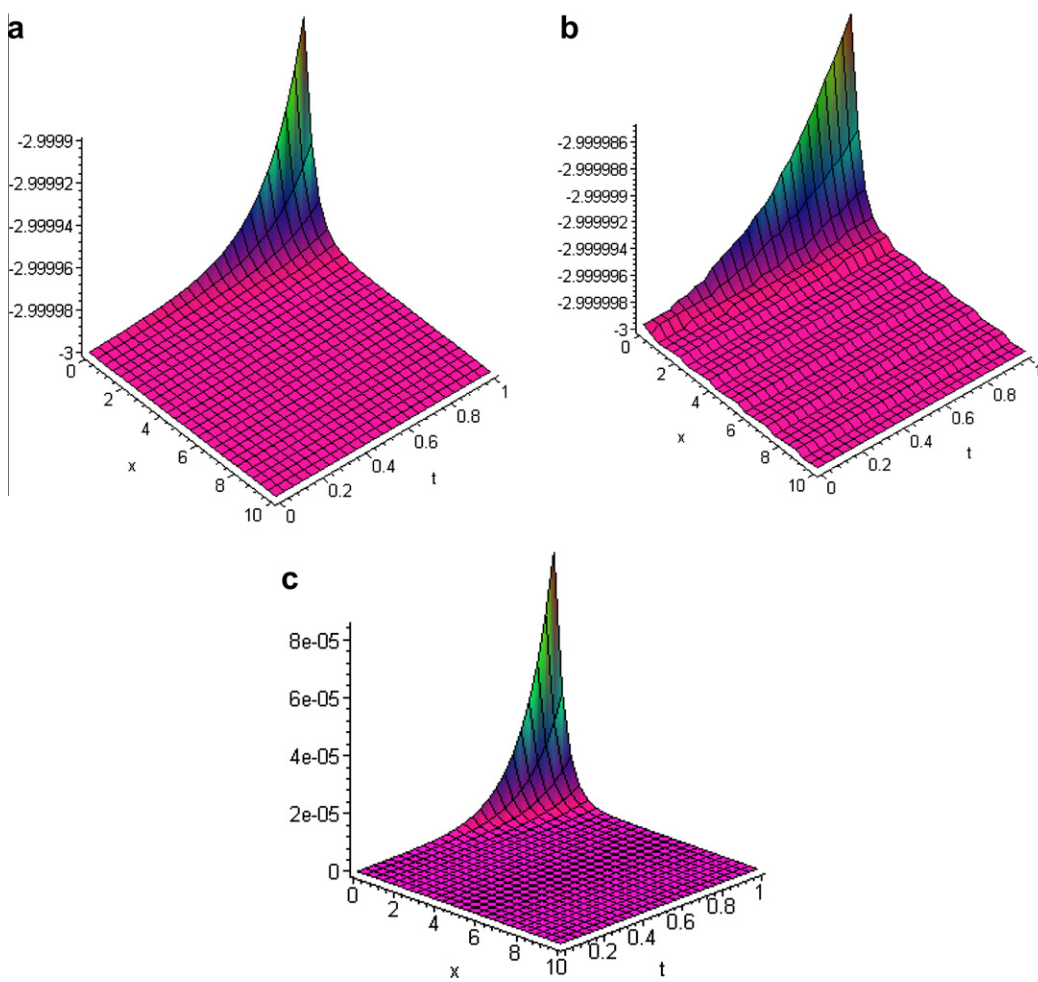

Figure 3 The surface shows the solution $u(x,t)$ for Example 3 when $c = 3$, $k = \frac{1}{2}$, $x_0 = -10$: (a) exact solution (7), (b) 3th order of approximate solution (8) and (c) the absolute error between exact and numerical solutions.

the results with their exact solutions in order to demonstrate the validity and applicability of the method. Moreover, only a small number of iterations are needed to obtain a satisfactory result. The given numerical examples support this claim.

We use the Maple Package to calculate the functions obtained from the variational iteration method

References

Akrivis, Georgios, Smyrlis, Yiorgos-sokratis, 2004. Implicit-explicit BDF methods for the Kuramoto–Sivashinsky equation. Appl. Numer. Math. 51, 151–169.

Biazar, J., Ghazvini, H., 2007. He's variational iteration method for fourth-order parabolic equations. Comput. Math. Appl. 54, 1047–1054.

Dehghan, M., Tatari, M., 2006. The use of He's variational iteration method for solving the Fokker–Planck equation. Phys. Scr. 74, 310–316.

He, J.H., 1997a. A new approach to nonlinear partial differential equations. Commun. Nonlinear Sci. Numer. Simul. 2 (4), 203–205.

He, J.H., 1997b. Variational iteration method for delay differential equations. Commun. Nonlinear Sci. Numer. Simul. 2 (4), 235–236.

Hooper, A.P., Grimshaw, R., 1985. Nonlinear instability at the interface between two viscous fluids. Phys. Fluids 28, 37–54.

Kuramoto, Y., Tsuzuki, T., 1976. Persistent propagation of concentration waves in dissipative media far from thermal equilibrium. Pron. Theor. Phys. 55, 356.

Manickam, A.V., Moudgalya, K.M., Pani, A.K., 1998. Second-order splitting combined with orthogonal cubic spline collocation method for the Kuramoto–Sivashinsky equation. Comput. Math. Appl. 35, 5–25.

Safari, M., Ganji, D.D., Moslemi, M., 2009. Application of He's variational iteration method and Adomian's decomposition method to the fractional, KdV–Burgers–Kuramoto equation. Comput. Math. Appl. 58, 2091–2097.

Sivashinsky, G.L., 1983. Instabilities, pattern-formation, and turbulence in flames. Ann. Rev. Fluid Mech. 15, 179–199.

Sweilam, N.H., Khader, M.M., 2007. Variational iteration method for one dimensional nonlinear thermo elasticity. Chaos, Solitons Fractals 32, 145–149.

Tatari, M., Dehghan, M., 2007. On the convergence of He's variational iteration method. J. Comput. Appl. Math. 207, 121–128.

Uddin, M., Haq, S., Siraj-ul-Islam, 2009. A mesh-free numerical method for solution of the family of Kuramoto–Sivashinsky equations. Appl. Math. Comput. 212, 458–469.

Wazwaz, A.M., 2007. A comparison between the variational iteration method and Adomian decomposition method. J. Comput. Appl. Math. 207, 129–136.