Journal of King Saud University – Science 34 (2022) 101860

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

On central identities equipped with skew Lie product involving generalized derivations

Shakir Ali [a,*], Mohammad Salahuddin Khan [b], Mohammed Ayedh [a]

[a] Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
[b] Department of Applied Mathematics, Z. H. College of Engineering & Technology, Aligarh Muslim University, Aligarh 202002, India

ARTICLE INFO

Article history:
Received 19 February 2021
Revised 8 January 2022
Accepted 19 January 2022
Available online 31 January 2022

Mathematics Subject Classification (2020):
16N60
16W10
16W25

Keywords:
Prime ring
Involution
Skew Lie product
Generalized derivation

ABSTRACT

Let $\mathfrak{R}$ be a $*$-ring. For any $x, y \in \mathfrak{R}$, we denote the skew Lie product of x and y by $\triangledown[x, y] = xy - yx^*$. An additive mapping $\mathscr{F} : \mathfrak{R} \to \mathfrak{R}$ is called a generalized derivation if there exists a derivation d such that $\mathscr{F}(xy) = \mathscr{F}(x)y + xd(y)$ for all $x, y \in \mathfrak{R}$. The objective of this paper is to chracterize generalized derivations and to describe the structure of prime rings with involution $*$ involving skew Lie product. In particular, we prove that if $\mathfrak{R}$ is a 2-torsion free prime ring with involution $*$ of the second kind and admits a generalized derivation $(\mathscr{F}, d)$ such that $\triangledown[x, \mathscr{F}(x^*)] \pm \triangledown[x, x^*] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $\mathscr{F} = \mp I_{\mathfrak{R}}$, where $I_{\mathfrak{R}}$ is the identity mapping of $\mathfrak{R}$. Moreover, some related results are also obtained. Finally, we provide two examples to prove that the assumed restrictions on our main results are not superfluous.

1. Introduction

Let $\mathfrak{R}$ be an associative ring and $d, \mathscr{F} : \mathfrak{R} \to \mathfrak{R}$ be additive mappings on $\mathfrak{R}$. Recall that d is called a derivations if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in \mathfrak{R}$. An additive mapping $\mathscr{F} : \mathfrak{R} \to \mathfrak{R}$ is said to be left multiplier if $\mathscr{F}(xy) = \mathscr{F}(x)y$ holds for all $x, y \in \mathfrak{R}$. An additive mapping $\mathscr{F}$ is called a generalized derivation if there is a derivation on $\mathfrak{R}$ satisfying $\mathscr{F}(xy) = \mathscr{F}(x)y + xd(y)$ for all $x, y \in \mathfrak{R}$. The generalized derivation $\mathscr{F}$ with an associated derivation d is denoted by $(\mathscr{F}, d)$. Obviously, any derivation is a generalized derivation, but the converse is not true in general. A significant example is a map of the form $\mathscr{F}(x) = ax + xb$ for all $x \in \mathfrak{R}$, where a and b are fixed elements of $\mathfrak{R}$. Moreover, the concept of generalized derivations includes both the concepts of derivations and left multipliers. Hence, the concept of generalized derivation is a natural generalization of the concept of derivation and left multiplier. Further, generalized derivations have been primarily studied on operator algebras. Therefore, any investigation from the algebraic point of view might be interesting (see for example Hvala, 1998; Lee, 1999 where further references can be looked).

This research is motivated by the recent work's of Abbasi et al. (2020) and Qi and Zhang (2018). However, our approach is different from those authors (Qi and Zhang, 2018). A ring $\mathfrak{R}$ with an involution $*$ is called a $*$-ring or ring with involution. Throughout, we let $\mathfrak{R}$ be a ring with involution $*$ and $Z(\mathfrak{R})$, the center of the ring $\mathfrak{R}$. Moreover, the sets of all hermitian and skew-hermitian elements of $\mathfrak{R}$ will be denoted by $H(\mathfrak{R})$ and $S(\mathfrak{R})$, respectively. The involution is called the first kind if $Z(\mathfrak{R}) \subseteq H(\mathfrak{R})$, otherwise $S(\mathfrak{R}) \cap Z(\mathfrak{R}) \neq (0)$ (see Herstein, 1969 for details). A ring $\mathfrak{R}$ is said to be 2-torsion free if $2x = 0$ (where $x \in \mathfrak{R}$) implies $x = 0$. A ring $\mathfrak{R}$ is called prime if $a\mathfrak{R}b = (0)$ (where $a, b \in \mathfrak{R}$) implies $a = 0$ or $b = 0$. For any $x, y \in \mathfrak{R}$, the symbol $[x, y]$ will denote the Lie product $xy - yx$ and the symbol $\triangledown[x, y]$ will denote the skew Lie product $xy - yx^*$, where $*$ is an involution on $\mathfrak{R}$.

In 1995, Bell and Daif (1995) showed that if $\mathfrak{R}$ is a prime ring admitting a nonzero derivation d such that $d([x, y]) = 0$ for all

* Corresponding author.
E-mail address: shakir.ali.mm@amu.ac.in (S. Ali).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2022.101860
1018-3647/

$x, y \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative. Ali et al. (2016), studied the above mentioned result in the settings of prime rings with involution by taking x^* instead of y. Recently, Alahmadi et al. (2017) extended this result to the class of generalized derivations by proving that: *let $\mathfrak{R}$ be a prime ring with involution of the second kind such that char($\mathfrak{R}$) $\neq 2$. If $\mathfrak{R}$ admits a generalized derivation $\mathscr{F}$ such that $\mathscr{F}([x, x^*]) = 0$ for all $x \in \mathfrak{R}$, then either $\mathscr{F} = 0$ or $\mathfrak{R}$ is commutative.* Most recently, Idrissi and Oukhtite (2019) studied this problem in more general setting. In fact, they established the following result: *let $\mathfrak{R}$ be a 2-torsion free prime ring with involution of the second kind. If $\mathfrak{R}$ admits a nonzero generalized derivation $\mathscr{F}$ associated with a derivation d, then $\mathfrak{R}$ is commutative if and only if $\mathscr{F}([x, x^*]) \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$* (see also Ali and Dar, 2014; Ali et al., 2021 for recent results).

The main purpose of this paper is to study generalized derivations involving skew Lie product on prime rings with involution. Further, we investigate the impact of these mappings and describe the structure of prime $*$-rings which satisfy certain $*$-differential identities. In particular, we prove that if $\mathfrak{R}$ is a 2-torsion free prime ring with involution $*$ of the second kind and admits a generalized derivation $(\mathscr{F}, d)$ such that $\triangledown[x, \mathscr{F}(x^*)] \pm \triangledown[x, x^*] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $\mathscr{F} = \mp I_{\mathfrak{R}}$, where $I_{\mathfrak{R}}$ is the identity mapping of $\mathfrak{R}$. Moreover, some related results are also obtained. In fact, our results extend and unify some recent results proved by several authors (viz.; Abbasi et al., 2020; Ali and Abbasi, 2020; Mozumder et al., 2021 where further references can be found). Finally, we provide two examples to prove that the assumed restrictions on our main results are not superfluous.

2. Main results

We start this section by recalling some useful lemmas which are needed in the proof of our results.

Lemma 1 (*[2020, Abbasi et al Lemma 4]*). *Let $\mathfrak{R}$ be a prime ring with involution of the second kind such that char($\mathfrak{R}$) $\neq 2$. If $\triangledown[x, x^*] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Lemma 2 (*[2017, Nejjar et al Lemma 2.2]*). *Let $\mathfrak{R}$ be a prime ring with involution of the second kind. Then $x \circ x^* \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$ if and only if $\mathfrak{R}$ is commutative.*

Lemma 3 (*[2017, Nejjar et al Lemma 2.1]*). *Let $\mathfrak{R}$ be a prime ring with involution of the second kind. Then $[x, x^*] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$ if and only if $\mathfrak{R}$ is commutative.*

Lemma 4. [Posner, 1957, Lemma 3] *Let $\mathfrak{R}$ be a prime ring and I a nonzero left ideal. If $\mathfrak{R}$ admits a nonzero derivation d such that $[d(x), x] \in Z(\mathfrak{R})$ for all $x \in I$, then $\mathfrak{R}$ is commutative.*

Now, we begin our discussions with the following theorem.

Theorem 1. *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits a generalized derivation $(\mathscr{F}, d)$ such that $\mathscr{F}(\triangledown[x, x^*]) \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $\mathscr{F} = 0$*

Proof. By the assumption, we have

$$\mathscr{F}(\triangledown[x, x^*]) \in Z(\mathfrak{R}) \text{ for all } x \in \mathfrak{R}. \tag{2.1}$$

By linearizing (2.1), we get

$$\mathscr{F}(\triangledown[x, y^*]) + \mathscr{F}(\triangledown[y, x^*]) \in Z(\mathfrak{R}) \text{ for all } x, y \in \mathfrak{R}. \tag{2.2}$$

Replacing x by xh in (2.2), where $0 \neq h \in H(\mathfrak{R}) \cap Z(\mathfrak{R})$, we obtain

$$(\mathscr{F}(\triangledown[x, y^*]) + \mathscr{F}(\triangledown[y, x^*]))h + (\triangledown[x, y^*] + \triangledown[y, x^*])d(h) \in Z(\mathfrak{R}). \tag{2.3}$$

Application of (2.2) gives

$$(\triangledown[x, y^*] + \triangledown[y, x^*])d(h) \in Z(\mathfrak{R}). \tag{2.4}$$

This implies that $\triangledown[x, y^*] + \triangledown[y, x^*] \in Z(\mathfrak{R})$ for all $x, y \in \mathfrak{R}$ or $d(h) = 0$ for all $h \in S(\mathfrak{R}) \cap Z(\mathfrak{R})$. Suppose $\triangledown[x, y^*] + \triangledown[y, x^*] \in Z(\mathfrak{R})$ for all $x, y \in \mathfrak{R}$. In particular, for $y = x$, we have $2\triangledown[x, x^*] \in Z(\mathfrak{R})$. Since $\mathfrak{R}$ is 2-torsion free, so $\triangledown[x, x^*] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$. Therefore, $\mathfrak{R}$ is commutative by Lemma 1.

Now, if $d(h) = 0$ for all $h \in H(\mathfrak{R}) \cap Z(\mathfrak{R})$, then $d(k) = 0$ for all $k \in S(\mathfrak{R}) \cap Z(\mathfrak{R})$. Thus from (2.2), we have

$$\mathscr{F}(xy^* - y^*x^*) + \mathscr{F}(yx^* - x^*y^*) \in Z(\mathfrak{R}) \text{ for all } x, y \in \mathfrak{R}. \tag{2.5}$$

Replacing x by xk in (2.5), we get

$$(\mathscr{F}(xy^* + y^*x^*) - \mathscr{F}(yx^* - x^*y^*))k \in Z(\mathfrak{R}).$$

The primeness of $\mathfrak{R}$ yields that

$$\mathscr{F}(xy^* + y^*x^*) - \mathscr{F}(yx^* - x^*y^*) \in Z(\mathfrak{R}) \text{ for all } x, y \in \mathfrak{R}. \tag{2.6}$$

Combining (2.5), (2.6) and using the fact that $\mathfrak{R}$ is 2-torsion free, we obtain

$$\mathscr{F}(xy^*) \in Z(\mathfrak{R}) \text{ for all } x, y \in \mathfrak{R}. \tag{2.7}$$

Replacing y by h in (2.7), where $0 \neq h \in H(\mathfrak{R}) \cap Z(\mathfrak{R})$, we get $\mathscr{F}(x) \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$. Hence in view of Hvala (1998, Lemma 3), we conclude that $\mathfrak{R}$ is commutative or $\mathscr{F} = 0$. This completes the proof of the theorem.

Corollary 1 (*[Abbasi et al., 2020, Theorem 2]*). *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits a nonzero left centralizer $\mathscr{T}$ such that $\mathscr{T}(\triangledown[x, x^*]) \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Corollary 2. *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits generalized derivations $(\mathscr{F}, d)$ and $(\mathscr{G}, g)$ such that $\mathscr{F}(\triangledown[x, x^*]) \pm \mathscr{G}(\triangledown[x, x^*]) \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $\mathscr{F} = \mp\mathscr{G}$.*

Proof. By the hypothesis, we have

$$\mathscr{F}(\triangledown[x, x^*]) \pm \mathscr{G}(\triangledown[x, x^*]) \in Z(\mathfrak{R}) \text{ for all } x \in \mathfrak{R}.$$

This implies that

$$(\mathscr{F} \pm \mathscr{G})(\triangledown[x, x^*]) \in Z(\mathfrak{R}) \text{ for all } x \in \mathfrak{R}.$$

Set $\mathscr{H} = \mathscr{F} \pm \mathscr{G}$. Then, we have

$$\mathscr{H}(\triangledown[x, x^*]) \in Z(\mathfrak{R}) \text{ for all } x \in \mathfrak{R}.$$

Since $\mathscr{H} = \mathscr{F} \pm \mathscr{G}$ is generalized derivation of $\mathfrak{R}$ associated with the derivation $d \pm g$ respectively. Therefore, in view of Theorem 1, we conclude that $\mathfrak{R}$ is commutative or $\mathscr{F} = \mp\mathscr{G}$.

Corollary 3 (*[Ali and Abbasi, 2020, Theorem 2]*). *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits a generalized derivation $(\mathscr{F}, d)$ such that $\mathscr{F}(\triangledown[x, x^*]) \pm \triangledown[x, x^*] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $\mathscr{F} = \mp I_{\mathfrak{R}}$ where $I_{\mathfrak{R}}$ is the identity mapping of $\mathfrak{R}$.*

Proof. If $\mathscr{F} = 0$, then $\triangledown[x, x^*] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$. Thus $\mathfrak{R}$ is commutative by Lemma 1. Now, we may assume that $\mathscr{F} \neq 0$. By the assumption, we have

$\mathscr{F}(\nabla[x,x^*]) \pm \nabla[x,x^*] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$.

This can be written as

$(\mathscr{F} \pm I_{\mathfrak{R}})(\nabla[x,x^*]) \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$,

where $I_{\mathfrak{R}}$ is the identity mapping of $\mathfrak{R}$. Since $\mathscr{F} \pm I_{\mathfrak{R}}$ is generalized derivation of $\mathfrak{R}$ associated with the derivation d, so from Theorem 1, we get the required result.

Corollary 4 (*[Mozumder et al., 2021, Theorem 1.11]). Let $\mathfrak{R}$ be a prime ring with involution of the second kind such that char$(\mathfrak{R}) \neq 2$. If $\mathfrak{R}$ admits a derivation d such that $d(\nabla[x,x^*]) \pm \nabla[x,x^*] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Theorem 2. *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits a generalized derivation $(\mathscr{F},d)$ such that $\nabla[x,\mathscr{F}(x)] \in Z(\mathfrak{R})$, then $\mathfrak{R}$ is commutative or $\mathscr{F} = 0$.*

Proof. By the assumption, we have

$$\nabla[x,\mathscr{F}(x)] \in Z(\mathfrak{R}) \quad (2.8)$$

for all $x \in \mathfrak{R}$. By linearizing (2.8), we get

$\nabla[x,\mathscr{F}(y)] + \nabla[y,\mathscr{F}(y)] \in Z(\mathfrak{R})$

for all $x, y \in \mathfrak{R}$. This implies that

$$x\mathscr{F}(y) - \mathscr{F}(y)x^* + y\mathscr{F}(x) - \mathscr{F}(x)y^* \in Z(\mathfrak{R}). \quad (2.9)$$

Replacing x by xh in (2.9), where $0 \neq h \in H(\mathfrak{R}) \cap Z(\mathfrak{R})$ and using it, we obtain

$$(yx - xy^*)d(h) \in Z(\mathfrak{R}). \quad (2.10)$$

The primeness of $\mathfrak{R}$ yields that $yx - xy^* \in Z(\mathfrak{R})$ or $d(h) = 0$. In the first case, we have $\nabla[x,x^*] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$. Therefore, $\mathfrak{R}$ is commutative in view of Lemma 1. Now, if $d(h) = 0$ for all $h \in H(\mathfrak{R}) \cap Z(\mathfrak{R})$, then $d(k) = 0$ for all $k \in S(\mathfrak{R}) \cap Z(\mathfrak{R})$. Replacing x by xk in (2.9) and using the fact that $d(k) = 0$, where $k \in S(\mathfrak{R}) \cap Z(\mathfrak{R})$, we get

$(x\mathscr{F}(y) + \mathscr{F}(y)x^* + y\mathscr{F}(x) - \mathscr{F}(x)y^*)k \in Z(\mathfrak{R})$.

By the primeness of $\mathfrak{R}$, we obtain

$$x\mathscr{F}(y) + \mathscr{F}(y)x^* + y\mathscr{F}(x) - \mathscr{F}(x)y^* \in Z(\mathfrak{R}) \text{ for all } x,y \in \mathfrak{R}. \quad (2.11)$$

Subtracting (2.11) from (2.9), we have

$$2\mathscr{F}(y)x^* \in Z(\mathfrak{R}) \quad (2.12)$$

for all $x, y \in \mathfrak{R}$. Since $\mathfrak{R}$ is 2-torsion free, we deduce that

$$\mathscr{F}(y)x^* \in Z(\mathfrak{R}). \quad (2.13)$$

Putting h in place of x, in (2.13) and using the primeness of $\mathfrak{R}$, we get

$$\mathscr{F}(y) \in Z(\mathfrak{R}) \text{ for all } y \in \mathfrak{R}. \quad (2.14)$$

Therefore, in view of Lemma 3 in Hvala (1998), we conclude that either $\mathfrak{R}$ is commutative or $\mathscr{F} = 0$. Thereby the proof is completed.

Corollary 5. *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits nonzero generalized derivations $(\mathscr{F},d)$ and $(\mathscr{G},g)$ such that $\nabla[x,\mathscr{F}(x)] \pm \nabla[x,\mathscr{G}(x)] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then either $\mathfrak{R}$ is commutative or $\mathscr{F} = \mp\mathscr{G}$.*

Proof. By the hypothesis, we have

$\nabla[x,\mathscr{F}(x)] \pm \nabla[x,\mathscr{G}(x)] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$.

This implies that

$\nabla[x,(\mathscr{F} \pm \mathscr{G})(x)] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$.

Set $\mathscr{H} = \mathscr{F} \pm \mathscr{G}$, then we obtain

$\nabla[x,\mathscr{H}(x)] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$.

Since $\mathscr{H} = \mathscr{F} \pm \mathscr{G}$ is generalized derivation of $\mathfrak{R}$ associated with the derivation $d \pm g$ respectively. Hence application of Theorem 2 yields that either $\mathfrak{R}$ is commutative or $\mathscr{F} = \mp\mathscr{G}$.

Corollary 6. *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits a generalized derivation $(\mathscr{F},d)$ such that $\nabla[x,\mathscr{F}(x)] \pm \nabla[x,x] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then either $\mathfrak{R}$ is commutative or $\mathscr{F} = \mp I_{\mathfrak{R}}$, where $I_{\mathfrak{R}}$ is the identity mapping of $\mathfrak{R}$.*

Proof. By the assumption, we have

$\nabla[x,\mathscr{F}(x)] \pm \nabla[x,x] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$.

This can be written as

$\nabla[x,(\mathscr{F} \pm I_{\mathfrak{R}})(x)]$ for all $x \in \mathfrak{R}$,

where $I_{\mathfrak{R}}$ is the identity mapping of $\mathfrak{R}$. Since $\mathscr{F} \pm I_{\mathfrak{R}}$ are generalized derivations of $\mathfrak{R}$ associated with the derivation d, so from Theorem 2, we get either $\mathfrak{R}$ is commutative or $\mathscr{F} = \mp I_{\mathfrak{R}}$, where $I_{\mathfrak{R}}$ is the identity mapping of $\mathfrak{R}$.

Theorem 3. *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits generalized derivations $(\mathscr{F},d)$ and $(\mathscr{G},g)$ such that $\mathscr{F}(\nabla[x,x^*]) + \nabla[x,\mathscr{G}(x)] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $\mathscr{F} = \mathscr{G} = 0$.*

Proof. By the assumption, we have

$$\mathscr{F}(\nabla[x,x^*] + \nabla[x,\mathscr{G}(x)] \in Z(\mathfrak{R}) \text{ for all } x \in \mathfrak{R}. \quad (2.15)$$

Expansion of (2.15) gives that

$$\mathscr{F}(x)x^* + xd(x^*) - \mathscr{F}(x^*)x^* - (x^*)d(x^*) + x\mathscr{G}(x) - \mathscr{G}(x)x^* \in Z(\mathfrak{R}). \quad (2.16)$$

Replacing x by xh in (2.16), where $0 \neq h \in H(\mathfrak{R}) \cap Z(\mathfrak{R})$, we get

$$(2xx^* - (x^*)^2 - x^2)d(h) + (x^2 - xx^*)g(h) \in Z(\mathfrak{R}). \quad (2.17)$$

Substituting xk in place of x in (2.17), where $0 \neq k \in S(\mathfrak{R}) \cap Z(\mathfrak{R})$, we obtain

$((-2xx^* - (x^*)^2 - x^2)d(h) + (x^2 + xx^*)g(h))k^2 \in Z(\mathfrak{R})$.

This implies that

$$(-2xx^* - (x^*)^2 - x^2)d(h) + (x^2 + xx^*)g(h) \in Z(\mathfrak{R}). \quad (2.18)$$

Subtracting (2.18) from (2.17), we arrive

$xx^*(2d(h) - g(h)) \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$.

This implies that either $xx^* \in Z(\mathfrak{R})$ or $2d(h) - g(h) = 0$. If $xx^* \in Z(\mathfrak{R})$. Also $x^*x \in Z(\mathfrak{R})$, hence $x \circ x^* \in Z(\mathfrak{R})$, $\mathfrak{R}$ is commutative by Lemma 2.

Now, if $(2d - g)(h) = 0$, then $(2d - g)(z) = 0$ for all $z \in Z(\mathfrak{R})$ and so

$$2d(z) = g(z) \text{ for all } z \in Z(\mathfrak{R}). \quad (2.19)$$

Application of (2.19) in (2.17) gives that

$(x^2 - (x^*)^2)d(h) \in Z(\mathfrak{R})$.

This implies that $x^2-(x^*)^2 \in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$ or $d(h)=0$ for all $h\in H(\mathfrak{R})\cap Z(\mathfrak{R})$. Consider the first case $x^2-(x^*)^2\in Z(\mathfrak{R})$. Linearizing the last relation, we get

$$xy+yx-x^*y^*-y^*x^*\in Z(\mathfrak{R}) \text{ for all } x,y\in\mathfrak{R}. \tag{2.20}$$

Replacing x by xk in (2.20) and $0\neq k\in S(\mathfrak{R})\cap Z(\mathfrak{R})$, we obtain

$$(xy+yx+x^*y^*+y^*x^*)k\in Z(\mathfrak{R}) \text{ for all } x,y\in\mathfrak{R}.$$

This gives that

$$xy+yx+x^*y^*+y^*x^*\in Z(\mathfrak{R}) \text{ for all } x,y\in\mathfrak{R}. \tag{2.21}$$

From (2.20) and (2.21), we conclude that

$$xy+yx\in Z(\mathfrak{R}) \text{ for all } x,y\in\mathfrak{R}.$$

In particular, $x\circ x^*\in Z(\mathfrak{R})$. Therefore, $\mathfrak{R}$ is commutative by Lemma 2. Now, if $d(h)=0$, then $g(h)=0$ and so $d(k)=g(k)=0$ for all $k\in S(\mathfrak{R})\cap Z(\mathfrak{R})$ and hence for all $z\in Z(\mathfrak{R})$. Now, replacing x by xk in (2.16), we have

$$(-\mathscr{F}(x)x^*-xd(x^*)-\mathscr{F}(x^*)x^*-x^*d(x^*)+x\mathscr{G}(x)+\mathscr{G}(x)x^*)k^2\in Z(\mathfrak{R}).$$

Primeness of $\mathfrak{R}$ forces that

$$-\mathscr{F}(x)x^*-xd(x^*)-\mathscr{F}(x^*)x^*-x^*d(x^*)+x\mathscr{G}(x)+\mathscr{G}(x)x^*\in Z(\mathfrak{R}) \tag{2.22}$$

for all $x\in\mathfrak{R}$. Combining (2.16) and (2.22), we obtain

$$\mathscr{F}(x^*)x^*+x^*d(x^*)-x\mathscr{G}(x)\in Z(\mathfrak{R}) \text{ for all } x\in\mathfrak{R}. \tag{2.23}$$

Linearizing (2.23) and using it, we get

$$\mathscr{F}(x^*)y^*+\mathscr{F}(y^*)x^*+x^*d(y^*)+y^*d(x^*)-x\mathscr{G}(y)-y\mathscr{G}(x)\in Z(\mathfrak{R}). \tag{2.24}$$

Replacing x by xk in (2.24), we get

$$(-\mathscr{F}(x^*)y^*-\mathscr{F}(y^*)x^*-x^*d(y^*)-y^*d(x^*)-x\mathscr{G}(y)-y\mathscr{G}(x))k\in Z(\mathfrak{R}).$$

That is,

$$-\mathscr{F}(x^*)y^*-\mathscr{F}(y^*)x^*-x^*d(y^*)-y^*d(x^*)-x\mathscr{G}(y)-y\mathscr{G}(x)\in Z(\mathfrak{R}) \tag{2.25}$$

for all $x,y\in\mathfrak{R}$. Adding (2.24) and (2.25), we have

$$x\mathscr{G}(y)+y\mathscr{G}(x)\in Z(\mathfrak{R}) \text{ for all } x,y\in\mathfrak{R}. \tag{2.26}$$

Put $x=y=h$ in (2.26), where $0\neq h\in H(\mathfrak{R})\cap Z(\mathfrak{R})$ we have

$$2h\mathscr{G}(h)\in Z(\mathfrak{R}).$$

This implies that

$$\mathscr{G}(h)\in Z(\mathfrak{R}) \text{ for all } h\in H(\mathfrak{R})\cap Z(\mathfrak{R}). \tag{2.27}$$

Replacing y by h in (2.26) where $0\neq h\in H(\mathfrak{R})\cap Z(\mathfrak{R})$, we get

$$x\mathscr{G}(h)+h\mathscr{G}(x)\in Z(\mathfrak{R}).$$

Taking the Lie product of above with x, we arrive at $h[\mathscr{G}(x),x]=0$ and hence

$$[\mathscr{G}(x),x]=0 \text{ for all } x\in\mathfrak{R}. \tag{2.28}$$

A linearization of (2.28) gives that

$$[\mathscr{G}(x),y]+[\mathscr{G}(y),x]=0 \text{ for all } x,y\in\mathfrak{R}. \tag{2.29}$$

Replacing x by hy in (2.29) and using (2.28), we get

$$h[g(y),y]=0,$$

which implies

$$[g(y),y]=0 \text{ for all } y\in\mathfrak{R}.$$

Therefore, in view of Lemma 4 either $g=0$ or $\mathfrak{R}$ is commutative. We proceed the proof for the case $g=0$, that is, $\mathfrak{R}$ is noncommutative. Substitute xw for x in (2.29), we get

$$\mathscr{G}(x)[w,y]+x[\mathscr{G}(y),w]=0 \text{ for all } x,y,w\in\mathfrak{R}. \tag{2.30}$$

Replacing x by tx in (2.30), we obtain

$$\mathscr{G}(t)x[w,y]+tx[\mathscr{G}(y),w]=0 \text{ for all } x,y,w,t\in\mathfrak{R}. \tag{2.31}$$

Left multiplying by t in (2.30), we have

$$t\mathscr{G}(x)[w,y]+tx[\mathscr{G}(y),w]=0 \text{ for all } x,y,w,t\in\mathfrak{R}. \tag{2.32}$$

Subtracting (2.32) from (2.31), we get

$$(\mathscr{G}(t)x-t\mathscr{G}(x))[w,y]=0 \text{ for all } x,y,w,t\in\mathfrak{R}.$$

Replacing w by rw in last relation, we get

$$(\mathscr{G}(t)x-t\mathscr{G}(x))r[w,y]=0 \text{ for all } x,y,w,t,r\in\mathfrak{R}.$$

Invoking the primeness of $\mathfrak{R}$, we deduce that

$$\mathscr{G}(t)x=t\mathscr{G}(x) \text{ for all } x,t\in\mathfrak{R}. \tag{2.33}$$

By using (2.33) in (2.26), we have

$$2y\mathscr{G}(x)\in Z(\mathfrak{R}) \text{ for all } x,y\in\mathfrak{R}.$$

In particular for $y=h$, we have

$$\mathscr{G}(x)\in Z(\mathfrak{R}) \text{ for all } x\in\mathfrak{R}. \tag{2.34}$$

Thus $\mathscr{G}=0$ by Hvala (1998, Lemma 3). Now, from (2.15), we have $\nabla[x,\mathscr{F}(x)]\in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$. Therefore, in view of Theorem 1, $\mathscr{F}=0$. This completes the proof of the theorem.

Corollary 7. *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits a generalized derivation $\mathscr{F}$ such that $\mathscr{F}(\nabla[x,x^*])+\nabla[x,\mathscr{F}(x)]\in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$, then $\mathfrak{R}$ is commutative or $\mathscr{F}=0$.*

Theorem 4. *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits a generalized derivation $(\mathscr{F},d)$ such that $\nabla[x,\mathscr{F}(x^*)]\pm\nabla[x,x^*]\in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$, then $\mathfrak{R}$ is commutative or $\mathscr{F}=\mp I_{\mathfrak{R}}$, where $I_{\mathfrak{R}}$ is the identity mapping of $\mathfrak{R}$.*

Proof. Firstly, we suppose that

$$\nabla[x,\mathscr{F}(x^*)]+\nabla[x,x^*]\in Z(\mathfrak{R}) \text{ for all } x\in\mathfrak{R}. \tag{2.35}$$

If $\mathscr{F}=0$, then $\nabla[x,x^*]\in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$ and hence $\mathfrak{R}$ is commutative by Lemma 1. Next, we assume that $\mathscr{F}\neq 0$. By linearizing (2.35), we get

$$\nabla[x,\mathscr{F}(y^*)]+\nabla[y,\mathscr{F}(x^*)]+\nabla[x,y^*]+\nabla[y,x^*]\in Z(\mathfrak{R}).$$

This can be written as

$$x\mathscr{F}(y^*)-\mathscr{F}(y^*)x^*+y\mathscr{F}(x^*)-\mathscr{F}(x^*)y^*+xy^*-y^*x^*+yx^*-x^*y^*\in Z(\mathfrak{R}). \tag{2.36}$$

Replacing x by xh in (2.36), where $0\neq h\in H(\mathfrak{R})\cap Z(\mathfrak{R})$, we get

$$(x\mathscr{F}(y^*)-\mathscr{F}(y^*)x^*+y\mathscr{F}(x^*)-\mathscr{F}(x^*)y^*+xy^*-y^*x^*+yx^*-x^*y^*)h + (yx^*-x^*y^*)d(h)\in Z(\mathfrak{R}).$$

By using (2.36) in last relation, we have

$$(yx^*-x^*y^*)d(h)\in Z(\mathfrak{R}).$$

This implies that $(yx^*-x^*y^*)\in Z(\mathfrak{R})$ or $d(h)=0$. Now, if $(yx^*-x^*y^*)\in Z(\mathfrak{R})$ for all $x,y\in\mathfrak{R}$, then $\mathfrak{R}$ is commutative by Lemma 1. Now, if $d(h)=0$ then we have $d(k)=0$ for all $k\in S(\mathfrak{R})\cap Z(\mathfrak{R})$. Replacing x by xk in (2.36), where $0\neq k\in S(\mathfrak{R})\cap Z(\mathfrak{R})$ and using the fact that $d(k)=0$, we get

$$(x\mathscr{F}(y^*)+\mathscr{F}(y^*)x^*-y\mathscr{F}(x^*)+\mathscr{F}(x^*)y^*+xy^*+y^*x^*-yx^*+x^*y^*)k \in Z(\mathfrak{R}).$$

This gives

$$x\mathscr{F}(y^*)+\mathscr{F}(y^*)x^*-y\mathscr{F}(x^*)+\mathscr{F}(x^*)y^*+xy^*+y^*x^*-yx^*+x^*y^* \in Z(\mathfrak{R}).$$

Combining the last expression with (2.36), we arrive at

$$2x\mathscr{F}(y^*)+2xy^* \in Z(\mathfrak{R})$$

for all $x,y\in\mathfrak{R}$. Since $\mathfrak{R}$ is 2-torsion free prime ring, we deduce that

$$x\mathscr{F}(y^*)+xy^* \in Z(\mathfrak{R}) \text{ for all } x,y\in\mathfrak{R}. \tag{2.37}$$

Replacing x by h and y by y^* in (2.37), where $0\neq h\in H(\mathfrak{R})\cap Z(\mathfrak{R})$, we get

$$\mathscr{F}(y)+y\in Z(\mathfrak{R}) \text{ for all } y\in\mathfrak{R}.$$

That is,

$$(\mathscr{F}+I_{\mathfrak{R}})(y)\in Z(\mathfrak{R}) \text{ for all } y\in\mathfrak{R},$$

where $I_{\mathfrak{R}}$ is the identity mapping of $\mathfrak{R}$. Since $\mathscr{F}+I_{\mathfrak{R}}$ is a generalized derivation of $\mathfrak{R}$, in view of Hvala (1998, Lemma 3) either $\mathfrak{R}$ is commutative or $\mathscr{F}=-I_{\mathfrak{R}}$. Now consider the case $\triangledown[x,\mathscr{F}(x^*)]-\triangledown[x,x^*]\in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$. Proceeding the same lines with necessary variations, we get the required result.

Corollary 8 (*[Abbasi et al., 2020, Theorem 3]*). *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits a left centralizer $\mathscr{T}$ such that $\triangledown[x,\mathscr{T}(x^*)]\pm\triangledown[x,x^*]\in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$, then either $\mathscr{T}$ is a centralizer or $\mathfrak{R}$ is commutative.*

Corollary 9 (*[Mozumder et al., 2021, Theorem 1.5]*). *Let $\mathfrak{R}$ be a prime ring with involution of the second kind such that char$(\mathfrak{R})\neq 2$. If $\mathfrak{R}$ admits a derivation d such that $\triangledown[x,d(x^*)]\pm\triangledown[x,x^*]\in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Theorem 5. *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits a generalized derivation $(\mathscr{F},d)$ such that $\triangledown[x,\mathscr{F}(x)]+\triangledown[x,x^*]\in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Proof. We have

$$\triangledown[x,\mathscr{F}(x)]+\triangledown[x,x^*]\in Z(\mathfrak{R}) \text{ for all } x\in\mathfrak{R}. \tag{2.38}$$

Expansion of (2.38) gives that

$$x\mathscr{F}(x)-\mathscr{F}(x)x^*+xx^*-(x^*)^2\in Z(\mathfrak{R}) \text{ for all } x\in\mathfrak{R}. \tag{2.39}$$

Replacing x by xh in (2.39), where $0\neq h\in H(\mathfrak{R})\cap Z(\mathfrak{R})$, we get

$$(x\mathscr{F}(x)-\mathscr{F}(x)x^*+xx^*-(x^*)^2)h^2+(x^2-xx^*)d(h)\in Z(\mathfrak{R}).$$

By using (2.39) in last relation, we have

$$(x^2-xx^*)d(h)\in Z(\mathfrak{R}).$$

The primeness of $\mathfrak{R}$ yields that $x^2-xx^*\in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$ or $d(h)=0$ for all $h\in H(\mathfrak{R})\cap Z(\mathfrak{R})$. First we consider the case

$$x^2-xx^*\in Z(\mathfrak{R}) \text{ for all } x\in\mathfrak{R}. \tag{2.40}$$

Substituting xk in place of x in (2.40), where $0\neq k\in S(\mathfrak{R})\cap Z(\mathfrak{R})$, we find that

$$(x^2+xx^*)k^2\in Z(\mathfrak{R}) \text{ for all } x\in\mathfrak{R},$$

which gives that

$$x^2+xx^*\in Z(\mathfrak{R}) \text{ for all } x\in\mathfrak{R}. \tag{2.41}$$

Subtracting (2.40) from (2.41) and using the fact that $\mathfrak{R}$ is 2-torsion free, we obtain $xx^*\in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$. This further implies $x\circ x^*\in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$. Therefore $\mathfrak{R}$ is commutative by Lemma 2.

On the other hand, if $d(h)=0$, then $d(k)=0$ for all $k\in S(\mathfrak{R})\cap Z(\mathfrak{R})$ and so $d(z)=0$ for all $z\in Z(\mathfrak{R})$. Replacing x by xk in (2.39), we get

$$(x\mathscr{F}(x)+\mathscr{F}(x)x^*-xx^*-(x^*)^2)k^2\in Z(\mathfrak{R}) \text{ for all } x\in\mathfrak{R}.$$

This implies that

$$x\mathscr{F}(x)+\mathscr{F}(x)x^*-xx^*-(x^*)^2\in Z(\mathfrak{R}) \text{ for all } x\in\mathfrak{R}. \tag{2.42}$$

Combining (2.39) and (2.42), we get

$$x\mathscr{F}(x)-(x^*)^2\in Z(\mathfrak{R}) \text{ for all } x\in\mathfrak{R}. \tag{2.43}$$

Linearization of (2.43) gives that

$$x\mathscr{F}(y)+y\mathscr{F}(x)-x^*y^*-y^*x^*\in Z(\mathfrak{R}) \text{ for all } x,y\in\mathfrak{R}. \tag{2.44}$$

Replacing x by xk in last relation, where $0\neq k\in S(\mathfrak{R})\cap Z(\mathfrak{R})$, we get

$$(x\mathscr{F}(y)+y\mathscr{F}(x)+x^*y^*+y^*x^*)k\in Z(\mathfrak{R}) \text{ for all } x,y\in\mathfrak{R}.$$

By the primeness of $\mathfrak{R}$, we have

$$x\mathscr{F}(y)+y\mathscr{F}(x)+x^*y^*+y^*x^*\in Z(\mathfrak{R}) \text{ for all } x,y\in\mathfrak{R}. \tag{2.45}$$

Combining (2.44), (2.45) and using the fact that $\mathfrak{R}$ is 2-torsion free, we obtain $x^*y^*+y^*x^*\in Z(\mathfrak{R})$ for all $x,y\in\mathfrak{R}$. In particular, we have $x\circ x^*\in Z(\mathfrak{R})$. Therefore, $\mathfrak{R}$ is commutative by Lemma 2.

Theorem 6. *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits a generalized derivation $(\mathscr{F},d)$ such that $\triangledown[x,\mathscr{F}(x)]+[x,x^*]\in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Proof. Assume that

$$\triangledown[x,\mathscr{F}(x)]+[x,x^*]\in Z(\mathfrak{R}) \text{ for all } x\in\mathfrak{R}. \tag{2.46}$$

By linearizing (2.46), we get

$$\triangledown[x,\mathscr{F}(y)]+\triangledown[y,\mathscr{F}(y)]+[x,y^*]+[y,x^*]\in Z(\mathfrak{R}) \text{ for all } x,y\in\mathfrak{R}.$$

Expansion of the above expression gives that

$$x\mathscr{F}(y)-\mathscr{F}(y)x^*+y\mathscr{F}(x)-\mathscr{F}(x)y^*+xy^*+yx^*-x^*y-y^*x \in Z(\mathfrak{R}) \tag{2.47}$$

for all $x,y\in\mathfrak{R}$. Replacing x by xh in (2.47), where $0\neq h\in H(\mathfrak{R})\cap Z(\mathfrak{R})$, we get

$$(yx-xy^*)d(h)\in Z(\mathfrak{R}) \text{ for all } x,y\in\mathfrak{R} \text{and} h\in H(\mathfrak{R})\cap Z(\mathfrak{R}). \tag{2.48}$$

This implies $yx-xy^*\in Z(\mathfrak{R})$ or $d(h)=0$. In first case, we have $\triangledown[x,x^*]\in Z(\mathfrak{R})$ for all $x\in\mathfrak{R}$ and hence $\mathfrak{R}$ is commutative in view of Lemma 1.

Now, if $d(h)=0$, then $d(k)=0$ for all $k\in S(\mathfrak{R})\cap Z(\mathfrak{R})$. Replacing x by xk, where $0\neq k\in S(\mathfrak{R})\cap Z(\mathfrak{R})$ in (2.47), we obtain

$$(x\mathscr{F}(y)+\mathscr{F}(y)x^*+y\mathscr{F}(x)-\mathscr{F}(x)y^*+xy^*-yx^*+x^*y-y^*x)k\in Z(\mathfrak{R}).$$

Invoking the primeness of $\mathfrak{R}$, we have

$$x\mathscr{F}(y)+\mathscr{F}(y)x^*+y\mathscr{F}(x)-\mathscr{F}(x)y^*+xy^*-yx^*+x^*y-y^*x\in Z(\mathfrak{R}) \tag{2.49}$$

for all $x,y\in\mathfrak{R}$. Subtracting (2.49) from (2.47), we arrive

$$2\mathscr{F}(y)x^*-2yx^*+2x^*y\in Z(\mathfrak{R}) \tag{2.50}$$

for all $x,y\in\mathfrak{R}$. Since $\mathfrak{R}$ is 2-torsion free prime ring, we deduce that

$$\mathscr{F}(y)x^* - yx^* + x^*y \in Z(\mathfrak{R}). \tag{2.51}$$

for all $x, y \in \mathfrak{R}$. Putting h in place of x^* in (2.51), we get

$$\mathscr{F}(y)h \in Z(\mathfrak{R}) \text{ for all } y \in \mathfrak{R},$$

which implies

$$\mathscr{F}(y) \in Z(\mathfrak{R}) \text{ for all } y \in \mathfrak{R}.$$

In view of Hvala (1998, Lemma 3), we conclude that either $\mathfrak{R}$ is commutative or $\mathscr{F} = 0$. In the latter case from (2.46), we have $[x, x^*] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, hence $\mathfrak{R}$ is commutative by Lemma 3. This completes the proof.

Theorem 7. *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits a generalized derivation $(\mathscr{F}, d)$ such that $\nabla[x, \mathscr{F}(x^*)] \pm (x \circ x^*) \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Proof. First, we consider the case

$$\nabla[x, \mathscr{F}(x^*)] + (x \circ x^*) \in Z(\mathfrak{R}) \text{ for all } x \in \mathfrak{R}. \tag{2.52}$$

If $\mathscr{F} = 0$, then $x \circ x^* \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$ and hence $\mathfrak{R}$ is commutative by Lemma 2. Now, assume that $\mathscr{F} \neq 0$. Linearization of (2.52) yields that

$$\nabla[x, \mathscr{F}(y^*)] + \nabla[x, \mathscr{F}(x^*)] + (x \circ y^*) + (y \circ x^*) \in Z(\mathfrak{R})$$

for all $x, y \in \mathfrak{R}$. This can be further written as

$$x\mathscr{F}(y^*) - \mathscr{F}(y^*)x^* + y\mathscr{F}(x^*) - \mathscr{F}(x^*)y^* + xy^* + y^*x + yx^* + x^*y \in Z(\mathfrak{R}) \tag{2.53}$$

for all $x, y \in \mathfrak{R}$. Replacing x by xh in (2.53), where $0 \neq h \in H(\mathfrak{R}) \cap Z(\mathfrak{R})$, we get

$$(x\mathscr{F}(y^*) - \mathscr{F}(y^*)x^* + y\mathscr{F}(x^*) - \mathscr{F}(x^*)y^* + xy^* + y^*x + yx^* + x^*y)h + (yx^* - x^*y^*)d(h) \in Z(\mathfrak{R}).$$

By using (2.53) in last relation, we have

$$(yx^* - x^*y^*)d(h) \in Z(\mathfrak{R}),$$

which implies $(yx^* - x^*y^*) \in Z(\mathfrak{R})$ or $d(h) = 0$. Now, if $(yx^* - x^*y^*) \in Z(\mathfrak{R})$ then $\mathfrak{R}$ is commutative by Lemma 1. Now, if $d(h) = 0$ then we have $d(k) = 0$ for all $k \in S(\mathfrak{R}) \cap Z(\mathfrak{R})$. Replacing x by xk in (2.53), where $0 \neq k \in S(\mathfrak{R}) \cap Z(\mathfrak{R})$, we get

$$(x\mathscr{F}(y^*) + \mathscr{F}(y^*)x^* - y\mathscr{F}(x^*) + \mathscr{F}(x^*)y^* + xy^* + y^*x - yx^* - x^*y)k \in Z(\mathfrak{R}).$$

This implies that

$$x\mathscr{F}(y^*) + \mathscr{F}(y^*)x^* - y\mathscr{F}(x^*) + \mathscr{F}(x^*)y^* + xy^* + y^*x - yx^* - x^*y \in Z(\mathfrak{R}).$$

Combining (2.53) with the above expression, we get

$$2x\mathscr{F}(y^*) + 2xy^* + 2y^*x \in Z(\mathfrak{R})$$

for all $x, y \in \mathfrak{R}$. Since $\mathfrak{R}$ is 2-torsion free prime ring, we deduce that

$$x\mathscr{F}(y^*) + xy^* + y^*x \in Z(\mathfrak{R}) \text{ for all } x, y \in \mathfrak{R}. \tag{2.54}$$

Replacing x by h and y by y^* in (2.54), where $0 \neq h \in H(\mathfrak{R}) \cap Z(\mathfrak{R})$, we get

$$\mathscr{F}(y) + 2y \in Z(\mathfrak{R}) \text{ for all } y \in \mathfrak{R}.$$

This can be written as

$$(\mathscr{F} + 2I_{\mathfrak{R}})(y) \in Z(\mathfrak{R}) \text{ for all } y \in \mathfrak{R},$$

where $I_{\mathfrak{R}}$ is the identity mapping of $\mathfrak{R}$. Since $\mathscr{F} + 2I_{\mathfrak{R}}$ is a generalized derivation of $\mathfrak{R}$, so by Hvala (1998, Lemma 3) $\mathfrak{R}$ is commutative or $\mathscr{F} = -2I_{\mathfrak{R}}$. Suppose if $\mathscr{F} = -2I_{\mathfrak{R}}$, then from (2.52) we have

$$\nabla[x, -2x^*] + (x \circ x^*) \in Z(\mathfrak{R}) \text{ for all } x \in \mathfrak{R}.$$

This implies

$$2(x^*)^2 - [x, x^*] \in Z(\mathfrak{R}) \text{ for all } x \in \mathfrak{R}. \tag{2.55}$$

Replacing x by kx in (2.55), where $0 \neq k \in S(\mathfrak{R}) \cap Z(\mathfrak{R})$, we find that

$$(2(x^*)^2 + [x, x^*])k^2 \in Z(\mathfrak{R}) \text{ for all } x \in \mathfrak{R}.$$

The primeness of $\mathfrak{R}$ gives that

$$2(x^*)^2 + [x, x^*] \in Z(\mathfrak{R}) \text{ for all } x \in \mathfrak{R}. \tag{2.56}$$

Subtracting (2.55) from (2.56) and using the fact that $\mathfrak{R}$ is 2-torsion free, we obtain $[x, x^*] \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$. Therefore, $\mathfrak{R}$ is commutative in view of Lemma 3. The proof in case we have the relation $\nabla[x, \mathscr{F}(x^*)] - (x \circ x^*) \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$ goes through similarly. This proves the theorem completely.

Corollary 10 (*[Abbasi et al., 2020, Theorem 4]*). *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution $*$ of the second kind. If $\mathfrak{R}$ admits a left centralizer $\mathscr{T}$ such that $\nabla[x, \mathscr{T}(x^*)] \pm (x \circ x^*) \in Z(\mathfrak{R})$ for all $x \in \mathfrak{R}$, then either $\mathscr{T}$ is a centralizer or $\mathfrak{R}$ is commutative.*

The following example justifies the fact that the condition of second kind involution in Theorems 2, 6 and 7 is indispensable.

Example 1. Let $\mathfrak{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$. Then $\mathfrak{R}$ is a 2-torsion free prime ring under the usual matrix addition and multiplication. Define mappings $\mathscr{F}, * : \mathfrak{R} \to \mathfrak{R}$ such that

$$\mathscr{F}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Clearly, $X^* = X$ for all $X \in Z(\mathfrak{R})$ i.e, $Z(\mathfrak{R}) \subseteq H(\mathfrak{R})$ hence ′ $*$ ′ is of the first kind involution. Further, it can be easily check that $\mathscr{F}$ is a generalized derivation of $\mathfrak{R}$ induced by the derivation $d = \mathscr{F}$ and satisfies the identities $\nabla[X, \mathscr{F}(X)] \in Z(\mathfrak{R})$, $\nabla[X, \mathscr{F}(X)] + [X, X^*] \in Z(\mathfrak{R})$ and $\nabla[X, \mathscr{F}(X^*)] \pm (X \circ X^*) \in Z(\mathfrak{R})$ for all $X \in \mathfrak{R}$. But, $\mathfrak{R}$ is not commutative and $\mathscr{F}$ is a nonzero non-identity mapping of $\mathfrak{R}$. Therefore, the assumption of second kind involution in Theorem 2, Theorem 6 and Theorem 7 is essential.

Next example shows that primeness of the ring in our results is an essential condition.

Example 2. Let $\mathfrak{T} = \mathfrak{R} \times \mathbb{C}$, where $\mathfrak{R}$ is same as in Example 1 with involution $*_1$ and generalized derivation f same as in above example, $\mathbb{C}$ is the ring of complex numbers with conjugate involution $*_2$. Hence, $\mathfrak{T}$ is a 2-torsion free noncommutative semiprime ring. Now define an involution $*$ on $\mathfrak{T}$, as $(x, y)^* = (x^{*_1}, y^{*_2})$. Clearly, $*$ is an involution of the second kind. Further, we define the mappings $\mathscr{F}$ and $\mathscr{G}$ from $\mathfrak{T}$ to $\mathfrak{T}$ such that $\mathscr{F}(x, y) = (f(x), 0)$ and $\mathscr{G}(x, y) = a(0, y)$ for all $(x, y) \in \mathfrak{T}$, where a is a nonzero fixed element of $\mathfrak{T}$. It can be easily checked that $\mathscr{F}$ and $\mathscr{G}$ are generalized derivations of $\mathfrak{T}$ and satisfying the following identities

(i) $\mathscr{G}(\nabla[X, X^*]) \in Z(\mathfrak{T})$ for all $X \in \mathfrak{T}$;
(ii) $\nabla[X, \mathscr{F}(X)] \in Z(\mathfrak{T})$ for all $X \in \mathfrak{T}$;
(iii) $\mathscr{G}(\nabla[X, X^*]) + \nabla[X, \mathscr{F}(X)] \in Z(\mathfrak{T})$ for all $X \in \mathfrak{T}$;
(iv) $\nabla[X, \mathscr{F}(X)] + [X, X^*] \in Z(\mathfrak{T})$ for all $X \in \mathfrak{T}$;
(v) $\nabla[X, \mathscr{F}(X^*)] \pm (X \circ X^*) \in Z(\mathfrak{T})$ for all $X \in \mathfrak{T}$.

However, $\mathfrak{T}$ is not commutative and $\mathscr{F}, \mathscr{G}$ are nonzero non identity mappings of $\mathfrak{T}$. Hence, in Theorems 1, 2, 3, 6 & 7, the hypothesis of primeness is essential.

3. Conclusion

In the present paper, we studied the action of generalized derivations on prime rings with involution involving skew Lie product. Moreover, we describe the structures of prime rings and generalized derivations.

Disclosure of funding

The research of first named author is supported by SERB-DST MATRIC Project (Grant No.: MTR/2019/000603), India.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors are deeply indebted to the learned referees(s) for their careful reading of the manuscript and constructive comments.

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, athttps://doi.org/10.1016/j.jksus.2022.101860.

References

Abbasi, A., Mozumder, M.R., Dar, N.A., 2020. A note on skew Lie product of prime ring with involution. Miskolc Math. Notes 21 (1), 3–18.
Alahmadi, A., Alhazmi, H., Ali, S., Khan, A.N., 2017. Generalized derivations on prime rings with involution. Commun. Math. Appl. 8 (1), 87–97.
Ali, S., Abbasi, A., 2020. On ∗-differential identities equipped with skew Lie product. Mathematics Today 36 (2), 29–34.
Ali, S., Dar, N.A., 2014. On *)centralizing mappings in rings with involution. Georgain Math. J. 21, 25–28.
Ali, S., Dar, N.A., Asci, M., 2016. On derivations and commutativity of prime rings with involution. Georgian Math. J. 23 (1), 9–14.
Ali, S., Mozumder, M.R., Khan, M.S., Abbasi, A., 2021. On N-skew Lie product on prime rings with involution involving derivations. Kyungpook Math. J. 61(4) (2022)(To, Appear).
Bell, H.E., Daif, M.N., 1995. On derivations and commutativity of prime rings. Acta Math. Hungar. 66, 337–343.
Herstein, I.N., 1969. Rings with involution. University of Chicago Press, Chicago.
Hvala, B., 1998. Generalized derivations in rings. Comm. Algebra 26 (4), 1147–1166.
Idrissi, M.A., Oukhtite, L., 2019. Some commutativity theorems for rings with involution involving generalized derivations. Asian-European J. Math. 12 (1), 11. 1950001.
Lee, T.-K., 1999. Generalized derivations of left faithful rings. Comm. Algebra 27 (8), 4057–4073.
Mozumder, M.R., Dar, N.A., Khan, M.S., Abbasi, A., 2021. On the skew Lie product and derivations of prime rings with involution. Discuss. Math. - Gen. Algebra Appl. 41 (1), 183–194.
Nejjar, B., Kacha, Mamouni, A., Oukhtite, L., 2017. Commutativity theorems in rings with involution. Comm. Algebra 45(2), 698–708..
Posner, E.C., 1957. Derivations in prime rings. Proc. Amer. Math. Soc. 8 (6), 1093–1100.
Qi, X., Zhang, Y., 2018. k-skew Lie products on prime rings with involution. Comm. Algebra 46 (3), 1001–1010.