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A hybrid method for solving fuzzy Volterra integral equations of separable type kernels

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ABSTRACT

The paper deals with the computation of solutions of fuzzy Volterra integral equations with degenerate kernel by applying a hybrid method. The proposed method is built on Laplace transform coupled with Adomian Decomposition Method; Laplace Adomian Decomposition Method abbreviated as (LADM). In the considered equation the unknown function has a solution in terms of infinite series expansion and hence LADM becomes more accurate to give the exact solution. Firstly, using the fuzzy number in term of parametric form, the fuzzy Volterra integral equation is converted to two crisp integral equations and then LADM is applied to get the exact fuzzy solutions of fuzzy linear Volterra integral equations. For the illustration, some examples of considered equations are solved to highlight the robustness, efficiency and the applicability of the developed scheme. The obtained results play an important role in developing the theory of fuzzy analytical dynamic equations.

1. Introduction

The study of fuzzy integral equations is of more interest and fast growing, especially the relation to fuzzy control, which has been developed recently. Mostly, mathematical models are applied in various problems of Chemistry, Engineering, Biology, Physics, and other many fields are based on integral equations, should be Mostly, mathematical models are applied in various problems of Chemistry, Engineering, Biology, Physics, and other many fields are based on differential, fractional order differential and integral equations see Atangana (2018), Atangana (2020, 2017). Particularly, the Volterra linear integral equations of second kind (Bede, 2004; Congxin and Ma, 1990; Chang and Zadeh, 1972; Friedman et al., 1996, 1997) is one of the most powerful types of integral equations to deal with such type of problems. Clearly, each model has some parameters which may be inherited to some vagueness. To solve these models we should mention such studied problems under uncertainty which leads to the presentation of fuzzy concept. The numerical idea for solving the Fredholm fuzzy linear integral equations of second kind by using the Adomian method were presented by Babolian et al. (2005). Also Friedman et al. (1999) presented the embedding method for the solution of fuzzy Volterra and Fredholm integral equations. In another numerical scheme, which was proposed by Abbasbandy et al. (2007) for the solutions of second kind fuzzy linear Fredholm integral equations, the authors applied the parametric form of fuzzy number to transform the fuzzy linear Fredholm integral equations into two systems of linear integral equation in deterministic cases. Recently, to solve the second kind of non-linear fuzzy Volterra–Fredholm integral equations by using the algorithm of homotopy perturbation method was presented by Attari et al. (2011). Molabahrami et al. (2011) used fuzzy parametric pair to transform the fuzzy Fredholm linear integral equation into two systems of linear integral equations as in crisp form, then they applied homotopy analysis method for obtaining the approximate solutions of the systems. To get the solutions of fuzzy Volterra and Fredholm integral equations, there are many research papers which presented the integration methods numerically for the solution of fuzzy-valued function

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(Friedman et al., 1999; Goetschel et al., 1986; Maltok, 1987; Nanda, 1989; Park et al., 1995; Wu, 1999, 2000). Moreover, there are some analytical and computational methods for the solution of fuzzy Volterra and fuzzy Volterra integro-differential equation (Salahshour and Allahviranloo, 2013; Alikhani et al., 2012). The classical Volterra integral equation is given by

$$v(y) = h(y) + \lambda_1 \int_{\beta_1}^{y} k(y,s)v(s)ds. \quad (1)$$

Here we remark that we investigate the given problem in Eq. (1) under fuzzy concept for analytical solution. The fuzzy form of the given equation is provided by

$$v(y,\alpha_1) = h(y,\alpha_1) + \lambda_1 \int_{\beta_1}^{y} k(y,s)v(s,\alpha_1)ds, \quad (2)$$

where the unknown fuzzy parametric function $v(y,\alpha_1) = (\underline{v}(y,\alpha_1), \overline{v}(y,\alpha_1))$ is to be determined. Further, $h(y,\alpha_1) = \left(\underline{h}(y,\alpha_1), \overline{h}(y,\alpha_1)\right)$ is known fuzzy parametric form function and the real-valued function $k(y,s)$ is also known and is said to be the kernel of integral equation and $\lambda_1 \in R, y \in [\beta_1, \beta_2]$ and $\beta_2 < \infty$ where $\beta_1,\ \beta_2$ are real constants. This article shows the simplicity of LADM to solve the linear fuzzy Volterra integra equation given in Eq. (2). Here we remark that the symbol α_1 designates the α-cut in the given problem. In many problems which of differential equations which are converted to their corresponding integral equations, the kernel $k(y,s)$, may be singular or nonsingular and depend on the nature of differential operator of the investigated fuzzy dynamic equation (Kilbas et al., 2006; Caputo and Fabrizio, 2015; Atangana and Baleanu, 2016; Atangana and Koca, 2019; Atangana and Gómez-Aguilar, 2019; Gupta and Dangar, 2014; Abdeljawad, 2019). If the fuzzy dynamic equation is integer order its corresponding fuzzy integral equation contains kernel of integer order. The deterministic system build on ordinary differential equation is the most acceptable mathematical instrument when modeled utilize fractional order differential operator, a hallmark of which is their powerful memory report. This is noticeable from a number of recent studies (Qureshi, 2020; Qureshi and Memon, 2020; Qureshi and Atangana, 2019) wherein it is convey that in contrast to classical differential operators, fractional order operator better analyze the chaotic pattern of various diseases. Therefore in recent times in most cases, the mentioned non integer order differential equations are converted to integral equations of Volterra or Fredholm type to study them. In addition, various engineering and physical systems build upon deterministic nature of governing equations have recently been investigated under non integer order differential operators (see Qureshi et al., 2019; Qureshi and Kumar, 2019; Atangana and Qureshi, 2019; Qureshi and Aziz, 2020;Akgül et al., 2019; Owolabi and Atangana, 2019; Akgüla, 2019; Akgül, 2018; Atangana and Hammouch, 2019; Atangana and Akgül, 2020). In Least Square Approximation (LSA), the convergence series depend upon the operator defined and for the existing numbers c_i which minimize the function. The necessary condition for convergence of LSA is that the partial derivative of function is zero (see Ameri and Nezhad, 2017). On the other side the hybrid method solution for the given problem obtain in a series form show the higher convergence of the algorithm. The mention algorithm need not any type of linearization or discretization and also need not an extra parameter as needed in LSA which catch the solution of the method. Further the qualitative theory of existence of solutions to Volterra integral equations in one and two dimensions has also investigated, (for detail see Bica and Ziari, 2019; Bica, 2013; Song et al., 1999; Subrahmanyam and Sudarsanam, 1996; Park and Jeong, 2000). Further numerical solutions for the said problem via using wavelet (Sahu and Ray, 2017) have been established. Here we remark that wavelet technique need discretization of date by using collocation points techniques which need extra memory and waste time. The mentioned numerical techniques are more powerful to handle nonlinear problems rather than linear problems. Further, our study is devoted to analytical aspect and hence we are interesting to find the analytical solution for the considered problem.

This paper is arranged as follow: Section 1 is the introduction part, Section 2 consists of some basic necessary preliminaries about the fuzzy calculus. Section 3, contains the main work and in Section 4, some examples are handled by using the suggested method. Finally, a brief conclusion is given in Section 5.

2. Preliminaries

Some basic definitions are given which used throughout the paper.

Definition 1 Goetschel et al., 1986. A fuzzy number is a mapping $v : R \to [0,1]$, which is an upper semi-continuous, fuzzy convex, normal and closure(supp v) is compact, where $suppv = \{x \in R :\ v(x) > 0\}$ represent the support of v.

Let E_1 be the set of all fuzzy numbers on R. The α_1-level set of v, is denoted by

$$[v]^{\alpha_1} = \begin{cases} \{x \in R :\ v(x) \geqslant \alpha_1\} & if\ 0 < \alpha_1 \leq 1, \\ (cl(suppv) if\ \alpha_1 = 0, \end{cases}$$

and is a closed bounded interval $[\underline{v}(\alpha_1), \overline{v}(\alpha_1)]$, where, $\underline{v}(\alpha_1)$ is the left-hand endpoint and $\overline{v}(\alpha_1)$ the right-hand endpoint of $[v]^{\alpha_1}$ respectively.

Definition 2 Friedman et al., 1996. A pair of functions

$$(\underline{v}(\alpha_1), \overline{v}(\alpha_1)) \quad 0 \leqslant \alpha_1 \leq 1,$$

is a parametric form of fuzzy number, which has the given properties:

(i) $\underline{v}(\alpha_1)$ is a non-decreasing bounded left continuous in $(0,1]$ and at 0 right continuous.
(ii) $\overline{v}(\alpha_1)$ is a non-increasing bounded left continuous in $(0,1]$ and at 0 right continuous.
(iii) $\underline{v}(\alpha_1) \leqslant \overline{v}(\alpha_1), 0 \leqslant \alpha_1 \leq 1$.

For any different fuzzy numbers

$$v = (\underline{v}(\alpha_1), \overline{v}(\alpha_1)), \quad \omega = (\underline{\omega}(\alpha_1), \overline{\omega}(\alpha_1)),$$

and for arbitrary scaler κ_1, the various operations are defined as follow,

(i) Addition: $\left(\underline{v(\alpha_1) + \omega(\alpha_1)}, \overline{v(\alpha_1) + \omega(\alpha_1)}\right) = \left(\underline{v(\alpha_1)} + \underline{\omega(\alpha_1)}, \overline{v(\alpha_1)} + \overline{\omega(\alpha_1)}\right)$.
(ii) Subtraction: $\left(\underline{v(\alpha_1) - \omega(\alpha_1)}, \overline{v(\alpha_1) - \omega(\alpha_1)}\right) = \left(\underline{v(\alpha_1)} - \underline{\omega(\alpha_1)}, \overline{v(\alpha_1)} - \overline{\omega(\alpha_1)}\right)$.
(iii) Scaler multiplication: $\kappa_1 \cdot v(\alpha_1) = \begin{cases} (\kappa_1 \underline{v}(\alpha_1), \kappa_1 \overline{v}(\alpha_1)) & \kappa_1 \geqslant 0, \\ (\kappa_1 \overline{v}(\alpha_1), \kappa_1 \underline{v}(\alpha_1)) & \kappa_1 < 0. \end{cases}$

Definition 3 Friedman et al., 1996. Let $D_1 : E_1 \times E_1 \to R_+ \cup \{0\}$ be a mapping, $v = (\underline{v}(\alpha_1), \overline{v}(\alpha_1))$ and $\omega = (\underline{\omega}(\alpha_1), \overline{\omega}(\alpha_1))$ are any two fuzzy numbers in parametric form. Then the Hausdorff distance between (v, ω) are defined as:

$$D_1(v,\omega) = \sup_{\alpha_1 \in [0,1]} \max\{|\underline{v}(\alpha_1) - \underline{\omega}(\alpha_1)|, |\overline{v}(\alpha_1) - \overline{\omega}(\alpha_1)|\}.$$

In E_1, a metric D_1 as defined above have following properties (see Puri and Ralescu, 1986):

(i) $D_1(\nu+\upsilon,\omega+\upsilon)=D_1(\nu,\omega)$ for all $\nu,\upsilon,\omega\in E_1$;
(ii) $D_1(\kappa_1\cdot\nu,\kappa_1\cdot\omega)=|\kappa_1|D_1(\nu,\omega)$ for all $\kappa_1\in R,\nu,\omega\in E_1$;
(iii) $D_1(\nu+\mu,\omega+\upsilon)\leqslant D_1(\nu,\omega)+D_1(\mu,\upsilon)$ for all $\nu,\omega,\mu,\upsilon\in E_1$;
(iv) (E_1,D_1) is a complete metric space.

Definition 4 Allahviranloo and Barkhordari Ahmadi, 2010. Suppose that $y_1,y_2\in E_1$. If there exist $y_3\in E_1$ such that

$$y_1=y_2+y_3$$

then y_3 is said to be H-difference of y_1 and y_2 and denoted as $y_1\ominus y_2$.

Definition 5 Friedman et al., 1999. Consider the fuzzy function $h:R\to E_1$. Then h is said to be continuous if for any rooted $y_0\in[\beta_1,\beta_2]$, if for every $\epsilon>0$, there exist $\delta>0$ such that if $|y-y_0|<\delta$ which implies that

$$D_1(h(y),h(y_0))<\epsilon.$$

Definition 6 Park et al., 1999. A level wise continuous mapping $h:[\beta_1,\beta_2]\subset R\to E_1$ is defined at $a\in[\beta_1,\beta_2]$, if the set-valued mapping $h_{\alpha_1}(y)=[h(y)]^{\alpha_1}$ is continuous at $y=a$ with respect to the Hausdorff metric D_1 for all $\alpha_1\in[0,1]$.

Theorem 1 Park et al., 1999. *Consider*

(i) *$h(y)$ is a levelwise continuous function on $[a,a+y_0]$, $y_0>0$,*
(ii) *$k(y,s)$ is a levelwise continuous function on $\triangle: a\leqslant s\leqslant y\leqslant a+y_0$ and $D_1(v(y),h(y_0))<y_1$, where $y_1>0$*
(iii) *For any $(y,s,v(s)),(y,s,\omega(s))\in\triangle$, we have*

$$D_1\left([k(y,s,v(s))]^{\alpha_1},[k(y,s,\omega(s))]^{\alpha_1}\right)\leqslant MD_1\left([v(s)]^{\alpha_1},[\omega(s)]^{\alpha_1}\right),$$

where the constant $M>0$ is given and for any $\alpha_1\in[0,1]$. Then, the levelwise continuous solution $v(y)$ exist and unique for Eq. (2) and defined for $y\in(a,a+\theta)$, where $\theta=\min\{y_0,\frac{y_1}{N}\}$, and $N=D_1(k(y,s,v(s)),(y,s,\omega(s)))\in\triangle$.

Theorem 2 (*Fuzzy Convolution Theorem*). Salahshour et al., 2012 *Let ϕ_1,ϕ_2 are fuzzy valued function of exponential order p, which are piecewise continuous on $[0,\infty)$, then*

$$L[(\phi_1*\phi_2)(s)]=L[\phi_1(s)]\cdot L[\phi_2(s)], \tag{3}$$

where L represent the Laplace transform.

3. Basic idea

In this section, we propose a basic idea to solve the fuzzy Volterra linear integral equations with convolution class kernel applying LADM. Suppose that a kernel function $k(y,s)$ is in the form of $(y-s)$ which is a separable kernel i.e. $k(y,s)=\sum_{i=0}^{n}h_i(y)g_i(s)$ and satisfies the conditions of Theorem 1 such that the solution of Eq. (2) exists and may be unique. We assume that for any $\beta_1\leqslant y,s\leqslant\beta_2$ in $k(y,s)$. Let the parametric form of Eq. (2) are written as

$$\begin{cases}\underline{v}(y,\alpha_1)=\underline{h}(y,\alpha_1)+\lambda_1\int_{\beta_1}^{y}(k(y,s)\underline{v}(s,\alpha_1))ds,\\ \overline{v}(y,\alpha_1)=\overline{h}(y,\alpha_1)+\lambda_1\int_{\beta_1}^{y}(k(y,s)\overline{v}(s,\alpha_1))ds,\end{cases} \tag{4}$$

considered $k(y,s)=y-s$ and $\lambda_1=1$, Eq. (4) becomes

$$\begin{cases}\underline{v}(y,\alpha_1)=\underline{h}(y,\alpha_1)+\int_{\beta_1}^{y}(y-s)\underline{v}(s,\alpha_1)ds,\\ \overline{v}(y,\alpha_1)=\overline{h}(y,\alpha_1)+\int_{\beta_1}^{y}(y-s)\overline{v}(s,\alpha_1)ds.\end{cases} \tag{5}$$

Now applying the Laplace transform and Theorem 2 to Eq. (5), one has

$$\begin{cases}L[\underline{v}(y,\alpha_1)]=L[\underline{h}(y,\alpha_1)]+L\left[\int_{\beta_1}^{y}(y-s)\underline{v}(s,\alpha_1)ds\right],\\ L[\overline{v}(y,\alpha_1)]=L\left[\overline{h}(y,\alpha_1)\right]+L\left[\int_{\beta_1}^{y}(y-s)\overline{v}(s,\alpha_1)ds\right],\end{cases}$$

$$\begin{cases}L[\underline{v}(y,\alpha_1)]=L[\underline{h}(y,\alpha_1)]+L[y]\cdot L[\underline{v}(y,\alpha_1)],\\ L[\overline{v}(y,\alpha_1)]=L\left[\overline{h}(y,\alpha_1)\right]+L[y]\cdot L[\overline{v}(y,\alpha_1)],\end{cases}$$

$$\begin{cases}L[\underline{v}(y,\alpha_1)]=L[\underline{h}(y,\alpha_1)]+\frac{1}{s^2}L[\underline{v}(y,\alpha_1)],\\ L[\overline{v}(y,\alpha_1)]=L\left[\overline{h}(y,\alpha_1)\right]+\frac{1}{s^2}L[\overline{v}(y,\alpha_1)].\end{cases} \tag{6}$$

Now applying the inverse Laplace transform on Eq. (6), and simplify one has

$$\begin{cases}\underline{v}(y,\alpha_1)=\underline{h}(y,\alpha_1)+L^{-1}\left[\frac{1}{s^2}L[\underline{v}(y,\alpha_1)]\right],\\ \overline{v}(y,\alpha_1)=\overline{h}(y,\alpha_1)+L^{-1}\left[\frac{1}{s^2}L[\overline{v}(y,\alpha_1)]\right].\end{cases} \tag{7}$$

Let the solution of Eq. (2) can be written in series as

$$\begin{cases}\underline{v}(y,\alpha_1)=\sum_{i=0}^{\infty}\underline{v}_i(y,\alpha_1),\\ \overline{v}(y,\alpha_1)=\sum_{i=0}^{\infty}\overline{v}_i(y,\alpha_1)\end{cases} \tag{8}$$

Putting Eq. (8) in Eq. (7), we get

$$\begin{cases}\sum_{i=0}^{\infty}\underline{v}_i(y,\alpha_1)=\underline{h}(y,\alpha_1)+L^{-1}\left[\frac{1}{s^2}L\left[\sum_{i=0}^{\infty}\underline{v}_i(y,\alpha_1)\right]\right],\\ \sum_{i=0}^{\infty}\overline{v}_i(y,\alpha_1)=\overline{h}(y,\alpha_1)+L^{-1}\left[\frac{1}{s^2}L\left[\sum_{i=0}^{\infty}\overline{v}_i(y,\alpha_1)\right]\right].\end{cases} \tag{9}$$

Comparing both side of Eq. (9) termwise respectively, we get

$$\begin{cases}\underline{v}_0(y,\alpha_1)=\underline{h}(y,\alpha_1),\\ \overline{v}_0(y,\alpha_1)=\overline{h}(y,\alpha_1),\end{cases}$$

$$\begin{cases}\underline{v}_1(y,\alpha_1)=L^{-1}\left[\frac{1}{s^2}L[\underline{v}_0(y,\alpha_1)]\right],\\ \overline{v}_1(y,\alpha_1)=L^{-1}\left[\frac{1}{s^2}L[\overline{v}_0(y,\alpha_1)]\right],\end{cases}$$

$$\begin{cases}\underline{v}_2(y,\alpha_1)=L^{-1}\left[\frac{1}{s^2}L[\underline{v}_1(y,\alpha_1)]\right],\\ \overline{v}_2(y,\alpha_1)=L^{-1}\left[\frac{1}{s^2}L[\overline{v}_1(y,\alpha_1)]\right].\end{cases}$$

and so on and we write general terms as

$$\begin{cases}\underline{v}_{n+1}(y,\alpha_1)=L^{-1}\left[\frac{1}{s^2}L[\underline{v}_n(y,\alpha_1)]\right],\\ \overline{v}_{n+1}(y,\alpha_1)=L^{-1}\left[\frac{1}{s^2}L[\overline{v}_n(y,\alpha_1)]\right],\end{cases} \tag{10}$$

where $n\geqslant 0$. The initial guess $h(y,\alpha_1)$ in above iteration scheme are very important because the solution of Eq. (2) are more rapidly convergence to the exact solution.

4. Application

In this section, three examples are given to described the application of proposed algorithm.

Example 1. Considered the fuzzy linear Volterra integral equation Ameri and Nezhad, 2017 of second kind

$$v(y,\alpha_1) = h(y,\alpha_1) + \int_0^y (y-s)v(s)ds, \tag{11}$$

for a known function $h(y,\alpha_1) = [3+\alpha_1, 8-2\alpha_1], 0 \leqslant y \leq 1$ and the given integral equation has exact solution is $v(y,\alpha_1) = ([3+\alpha_1, 8-2\alpha_1]\cosh y)$. To solve this integral equation by applying LADM.

Parametric form of Eq. (11) can be written as

$$\begin{cases} \underline{v}(y,\alpha_1) = (3+\alpha_1) + \int_0^y (y-s)\underline{v}(s,\alpha_1)ds, \\ \overline{v}(y,\alpha_1) = (8-2\alpha_1) + \int_0^y (y-s)\overline{v}(s,\alpha_1)ds. \end{cases} \tag{12}$$

Applying the Laplace transform and convolution Theorem 2 to Eq. (12), one has

$$\begin{cases} L[\underline{v}(y,\alpha_1)] = L[(3+\alpha_1)] + L\left[\int_0^y (y-s)\underline{v}(s,\alpha_1)ds\right], \\ L[\overline{v}(y,\alpha_1)] = L[(8-2\alpha_1)] + L\left[\int_0^y (y-s)\overline{v}(s,\alpha_1)ds\right], \end{cases}$$

$$\begin{cases} L[\underline{v}(y,\alpha_1)] = L[(3+\alpha_1)] + L[y]\cdot L[\underline{v}(y,\alpha_1)], \\ L[\overline{v}(y,\alpha_1)] = L[(8-2\alpha_1)] + L[y]\cdot L[\overline{v}(y,\alpha_1)], \end{cases}$$

$$\begin{cases} L[\underline{v}(y,\alpha_1)] = L[(3+\alpha_1)] + \frac{1}{s^2}L[\underline{v}(y,\alpha_1)], \\ L[\overline{v}(y,\alpha_1)] = L[(8-2\alpha_1)] + \frac{1}{s^2}L[\overline{v}(y,\alpha_1)]. \end{cases}$$

Now applying the inverse Laplace transform, and simplify, we get

$$\begin{cases} \underline{v}(y,\alpha_1) = (3+\alpha_1) + L^{-1}\left[\frac{1}{s^2}L[\underline{v}(y,\alpha_1)]\right]. \\ \overline{v}(y,\alpha_1) = (8-2\alpha_1) + L^{-1}\left[\frac{1}{s^2}L[\overline{v}(y,\alpha_1)]\right]. \end{cases} \tag{13}$$

Let the series solution of Eq. (11) be as

$$\begin{cases} \underline{v}(y,\alpha_1) = \sum_{i=0}^{\infty} \underline{v}_i(y,\alpha_1), \\ \overline{v}(y,\alpha_1) = \sum_{i=0}^{\infty} \overline{v}_i(y,\alpha_1). \end{cases} \tag{14}$$

Putting Eq. (14) in Eq. (13)

$$\begin{cases} \sum_{i=0}^{\infty} \underline{v}_i(y,\alpha_1) = (3+\alpha_1) + L^{-1}\left[\frac{1}{s^2}L\left[\sum_{i=0}^{\infty}\underline{v}_i(y,\alpha_1)\right]\right], \\ \sum_{i=0}^{\infty} \overline{v}_i(y,\alpha_1) = (8-2\alpha_1) + L^{-1}\left[\frac{1}{s^2}L\left[\sum_{i=0}^{\infty}\overline{v}_i(y,\alpha_1)\right]\right], \end{cases}$$

which may written as

$$\begin{cases} \underline{v}_0(y,\alpha_1) + \underline{v}_1(y,\alpha_1) + \underline{v}_2(y,\alpha_1) + \cdots \\ \quad = (3+\alpha_1) + L^{-1}\left[\frac{1}{s^2}L[\underline{v}_0(y,\alpha_1)]\right] + L^{-1}\left[\frac{1}{s^2}L[\underline{v}_1(y,\alpha_1)]\right] + \cdots \\ \overline{v}_0(y,\alpha_1) + \overline{v}_1(y,\alpha_1) + \overline{v}_2(y,\alpha_1) + \cdots \\ \quad = (8-2\alpha_1) + L^{-1}\left[\frac{1}{s^2}L[\overline{v}_0(y,\alpha_1)]\right] + L^{-1}\left[\frac{1}{s^2}L[\overline{v}_1(y,\alpha_1)]\right] + \cdots. \end{cases} \tag{15}$$

Comparing termwise of Eq. (15) on both side respectively, we obtain

$$\begin{cases} \underline{v}_0(y,\alpha_1) = 3+\alpha_1, \\ \overline{v}_0(y,\alpha_1) = 8-2\alpha_1, \end{cases}$$

$$\begin{cases} \underline{v}_1(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\underline{v}_0(y,\alpha_1)]\right], \\ \overline{v}_1(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\overline{v}_0(y,\alpha_1)]\right], \end{cases}$$

$$\begin{cases} \underline{v}_2(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\underline{v}_1(y,\alpha_1)]\right], \\ \overline{v}_2(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\overline{v}_1(y,\alpha_1)]\right]. \end{cases}$$

The general terms are given by

$$\begin{cases} \underline{v}_{n+1}(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\underline{v}_n(y,\alpha_1)]\right], \\ \overline{v}_{n+1}(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\overline{v}_n(y,\alpha_1)]\right], \end{cases} \tag{16}$$

where $n \geqslant 0$. Taking the lower limit solution of Eq. (11), and simplify

$$\begin{cases} \underline{v}_0(y,\alpha_1) = 3+\alpha_1, \\ \underline{v}_1(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\underline{v}_0(y,\alpha_1)]\right] = (3+\alpha_1)\frac{y^2}{2!}, \\ \underline{v}_2(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\underline{v}_1(y,\alpha_1)]\right] = (3+\alpha_1)\frac{y^4}{4!}, \\ \underline{v}_3(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\underline{v}_2(y,\alpha_1)]\right] = (3+\alpha_1)\frac{y^6}{6!}, \\ \underline{v}_4(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\underline{v}_3(y,\alpha_1)]\right] = (3+\alpha_1)\frac{y^8}{8!} \end{cases} \tag{17}$$

and so on. The other terms may be in same fashion computed. Now to find the upper limit solution of Eq. (11), we get

$$\begin{cases} \overline{v}_0(y,\alpha_1) = 8-2\alpha_1, \\ \overline{v}_1(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\overline{v}_0(y,\alpha_1)]\right] = (8-2\alpha_1)\frac{y^2}{2!}, \\ \overline{v}_2(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\overline{v}_1(y,\alpha_1)]\right] = (8-2\alpha_1)\frac{y^4}{4!}, \\ \overline{v}_3(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\overline{v}_2(y,\alpha_1)]\right] = (8-2\alpha_1)\frac{y^6}{6!}, \\ \overline{v}_4(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\overline{v}_3(y,\alpha_1)]\right] = (8-2\alpha_1)\frac{y^8}{8!} \end{cases} \tag{18}$$

and so on. Putting Eqs. (17) and (18) in Eq. (14) and simplifying, we achieve the closed form as

$$\begin{cases} \underline{v}(y,\alpha_1) = (3+\alpha_1)\cosh y, \\ \underline{v}(y,\alpha_1) = (8-2\alpha_1)\cosh y, \end{cases}$$

which is the exact solution of Eq. (11) and can be written also as

$v(y,\alpha_1) = [3+\alpha_1, 8-2\alpha_1]\cosh y$.

Next in Fig. 1, we provide graphical presentation of fuzzy solutions at the given values of uncertainty α_1 as.

Example 2. Let the fuzzy linear Volterra integral equation (Ameri and Nezhad, 2017)

$$v(y,\alpha_1) = h(y,\alpha_1) + \int_0^y (y-s)v(s,\alpha_1)ds, \tag{19}$$

where $h(y,\alpha_1) = \left([\alpha_1, 2-\alpha_1]\left(1-y-\frac{y^2}{2}\right)\right), 0 \leqslant y \leq 1$ and the exact solution is $([\alpha_1, 2-\alpha_1](1-\sinh y))$.

To solve the Eq. (19) by LADM.

Considered the parametric form of Eq. (19) which is given by

$$\begin{cases} \underline{v}(y,\alpha_1) = \alpha_1\left(1-y-\frac{y^2}{2}\right) + \int_0^y (y-s)\underline{v}(s,\alpha_1)ds, \\ \overline{v}(y,\alpha_1) = (2-\alpha_1)\left(1-y-\frac{y^2}{2}\right) + \int_0^y (y-s)\overline{v}(s,\alpha_1)ds. \end{cases} \tag{20}$$

Applying the Laplace transform together with convolution Theorem 2 and after the inverse Laplace transform, we get

$$\begin{cases} \underline{v}(y,\alpha_1) = \alpha_1\left(1-y-\frac{y^2}{2}\right) + L^{-1}\left[\frac{1}{s^2}L[\underline{v}(y,\alpha_1)]\right], \\ \overline{v}(y,\alpha_1) = (2-\alpha_1)\left(1-y-\frac{y^2}{2}\right) + L^{-1}\left[\frac{1}{s^2}L[\overline{v}(y,\alpha_1)]\right]. \end{cases} \tag{21}$$

Let the solution of Eq. (19) in the form of infinite series as

$$\begin{cases} \underline{v}(y,\alpha_1) = \sum_{i=0}^{\infty} \underline{v}_i(y,\alpha_1), \\ \overline{v}(y,\alpha_1) = \sum_{i=0}^{\infty} \overline{v}_i(y,\alpha_1). \end{cases} \tag{22}$$

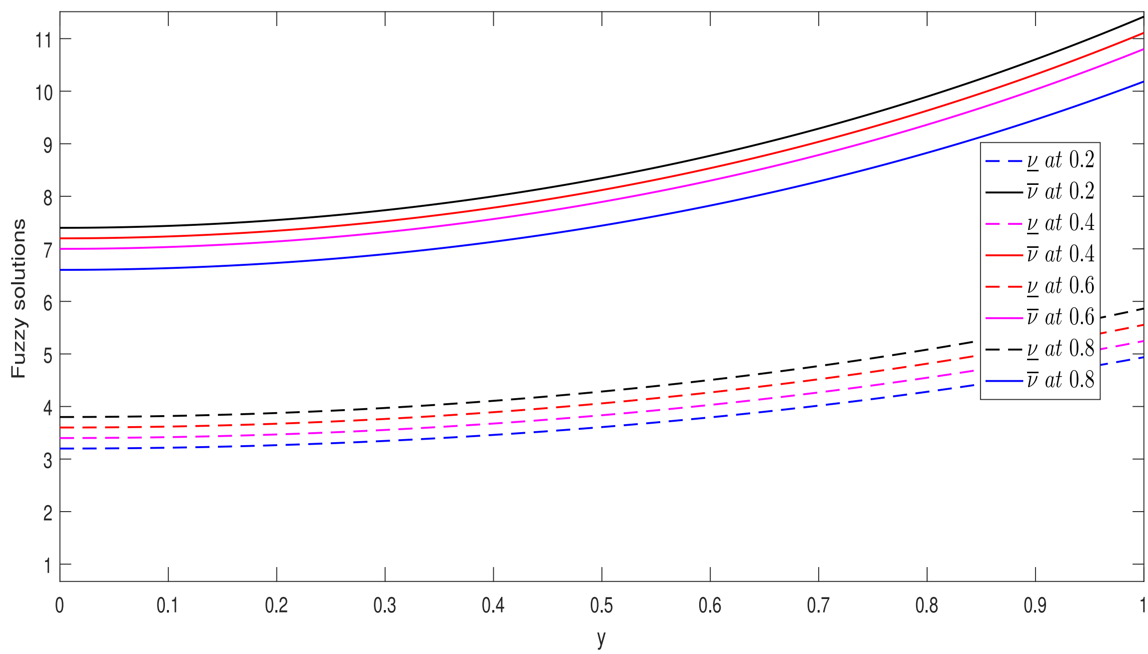

Fig. 1. Graphical presentation of fuzzy solutions at the given values of α_1 for Example 1.

Putting Eq. (22) in Eq. (21), one has

$$\begin{cases} \sum_{i=0}^{\infty}\underline{v}_i(y,\alpha_1) = \alpha_1\left(1-y-\frac{y^2}{2}\right) + L^{-1}\left[\frac{1}{s^2}L\left[\sum_{i=0}^{\infty}\underline{v}_i(y,\alpha_1)\right]\right], \\ \sum_{i=0}^{\infty}\overline{v}_i(y,\alpha_1) = (2-\alpha_1)\left(1-y-\frac{y^2}{2}\right) + L^{-1}\left[\frac{1}{s^2}L\left[\sum_{i=0}^{\infty}\overline{v}_i(y,\alpha_1)\right]\right]. \end{cases} \tag{23}$$

Comparing termwise of Eq. (23) both side respectively

$$\begin{cases} \underline{v}_0(y,\alpha_1) = \alpha_1\left(1-y-\frac{y^2}{2}\right), \\ \overline{v}_0(y,\alpha_1) = (2-\alpha_1)\left(1-y-\frac{y^2}{2}\right), \end{cases}$$

$$\begin{cases} \underline{v}_1(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\underline{v}_0(y,\alpha_1)]\right], \\ \overline{v}_1(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\overline{v}_0(y,\alpha_1)]\right], \end{cases}$$

$$\begin{cases} \underline{v}_2(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\underline{v}_1(y,\alpha_1)]\right], \\ \overline{v}_2(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\overline{v}_1(y,\alpha_1)]\right], \end{cases}$$

the remaining terms may be in same way computed. The general terms are written as

$$\begin{cases} \underline{v}_{n+1}(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\underline{v}_n(y,\alpha_1)]\right], \\ \overline{v}_{n+1}(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\overline{v}_n(y,\alpha_1)]\right], \ n \geqslant 0. \end{cases} \tag{24}$$

Taking first lower limit fuzzy solution of Eq. (19), and simplify, one gets

$$\begin{cases} \underline{v}_0(y,\alpha_1) = \alpha_1\left(1-y-\frac{y^2}{2!}\right), \\ \underline{v}_1(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\underline{v}_0(y,\alpha_1)]\right] = \alpha_1\left(\frac{y^2}{2!}-\frac{y^3}{3!}-\frac{y^4}{4!}\right), \\ \underline{v}_2(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\underline{v}_1(y,\alpha_1)]\right] = \alpha_1\left(\frac{y^4}{4!}-\frac{y^5}{5!}-\frac{y^6}{6!}\right), \\ \underline{v}_3(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\underline{v}_2(y,\alpha_1)]\right] = \alpha_1\left(\frac{y^6}{6!}-\frac{y^7}{7!}-\frac{y^8}{8!}\right) \end{cases} \tag{25}$$

and so on. Now taking upper limit fuzzy solution of Eq. (19), and simplify

$$\begin{cases} \overline{v}_0(y,\alpha_1) = (2-\alpha_1), \\ \overline{v}_1(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\overline{v}_0(y,\alpha_1)]\right] = (2-\alpha_1)\left(\frac{y^2}{2!}-\frac{y^3}{3!}-\frac{y^4}{4!}\right), \\ \overline{v}_2(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\overline{v}_1(y,\alpha_1)]\right] = (2-\alpha_1)\left(\frac{y^4}{4!}-\frac{y^5}{5!}-\frac{y^6}{6!}\right), \\ \overline{v}_3(y,\alpha_1) = L^{-1}\left[\frac{1}{s^2}L[\overline{v}_2(y,\alpha_1)]\right] = (2-\alpha_1)\left(\frac{y^6}{6!}-\frac{y^7}{7!}-\frac{y^8}{8!}\right) \end{cases} \tag{26}$$

and so on. Putting Eqs. (25) and (26)in Eq. (22) and simplify, we get

$$\begin{cases} \underline{v}(y,\alpha_1) = \alpha_1(1-\sinh y), \\ \overline{v}(y,\alpha_1) = (2-\alpha_1)(1-\sinh y), \end{cases}$$

which also can be written in closed form as

$$v(y,\alpha_1) = [\alpha_1, 2-\alpha_1](1-\sinh y).$$

hence exact solution of the problem. Next in Fig. 2, we provide graphical presentation of fuzzy solutions at the given values of uncertainty α_1 as

Example 3. Suppose the following fuzzy linear Volterra integral equation (Salahshour and Allahviranloo, 2013)

$$v(y,\alpha_1) = [\alpha_1-1, 1-\alpha_1]y + \int_0^y v(s,\alpha_1)ds, \tag{27}$$

$0 \leqslant y \leqslant 1$ and the exact solution is $v(y,\alpha_1) = [(\alpha_1-1),(1-\alpha_1)](\sinh y + \cosh y - 1)$. Let the parametric form of Eq. (27) is

$$\begin{cases} \underline{v}(y,\alpha_1) = (\alpha_1-1)y + \int_0^y \underline{v}(s,\alpha_1)ds, \\ \overline{v}(y,\alpha_1) = (1-\alpha_1)y + \int_0^y \overline{v}(s,\alpha_1)ds. \end{cases}$$

Applying the Laplace transform together with convolution Theorem 2 and the inverse Laplace transform, we obtain

$$\begin{cases} \underline{v}(y,\alpha_1) = (\alpha_1-1)y + L^{-1}\left[\frac{1}{s}L[\underline{v}(y,\alpha_1)]\right], \\ \overline{v}(y,\alpha_1) = (1-\alpha_1)y + L^{-1}\left[\frac{1}{s}L[\overline{v}(y,\alpha_1)]\right]. \end{cases} \tag{28}$$

Eq. (27) has solution in the form of series as

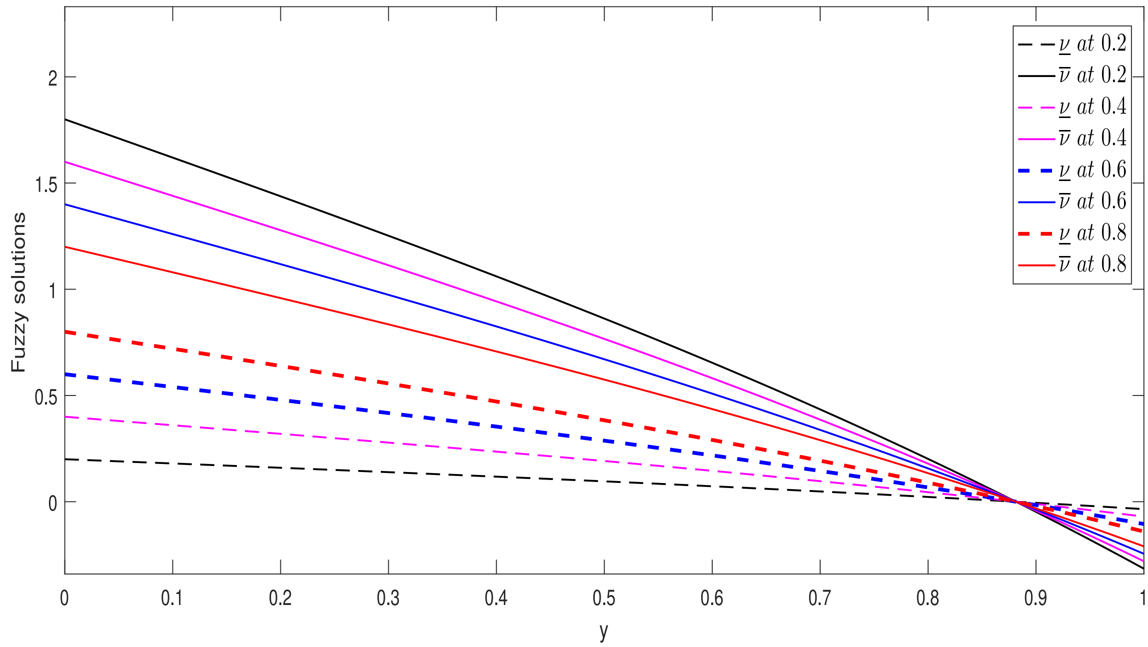

Fig. 2. Graphical presentation of fuzzy solutions at the given values of α_1 for Example 2.

$$\begin{cases} \underline{\nu}(y,\alpha_1) = \sum_{i=0}^{\infty} \underline{\nu}_i(y,\alpha_1), \\ \overline{\nu}(y,\alpha_1) = \sum_{i=0}^{\infty} \overline{\nu}_i(y,\alpha_1). \end{cases} \tag{29}$$

Putting Eq. (29) in Eq. (28),

$$\begin{cases} \sum_{i=0}^{\infty} \underline{\nu}_i(y,\alpha_1) = (\alpha_1 - 1)y + L^{-1}\left[\frac{1}{s}L\left[\sum_{i=0}^{\infty} \underline{\nu}_i(y,\alpha_1)\right]\right], \\ \sum_{i=0}^{\infty} \overline{\nu}_i(y,\alpha_1) = (1 - \alpha_1)y + L^{-1}\left[\frac{1}{s}L\left[\sum_{i=0}^{\infty} \overline{\nu}_i(y,\alpha_1)\right]\right]. \end{cases} \tag{30}$$

Comparing terms of Eq. (30) on both side respectively, we get

$$\begin{cases} \underline{\nu}_0(y,\alpha_1) = (\alpha_1 - 1)y, \\ \overline{\nu}_0(y,\alpha_1) = (1 - \alpha_1)y, \end{cases}$$

$$\begin{cases} \underline{\nu}_1(y,\alpha_1) = L^{-1}\left[\frac{1}{s}L[\underline{\nu}_0(y,\alpha_1)]\right], \\ \overline{\nu}_1(y,\alpha_1) = L^{-1}\left[\frac{1}{s}L[\overline{\nu}_0(y,\alpha_1)]\right], \end{cases}$$

$$\begin{cases} \underline{\nu}_2(y,\alpha_1) = L^{-1}\left[\frac{1}{s}L[\underline{\nu}_1(y,\alpha_1)]\right], \\ \overline{\nu}_2(y,\alpha_1) = L^{-1}\left[\frac{1}{s}L[\overline{\nu}_1(y,\alpha_1)]\right] \end{cases}$$

and so on.

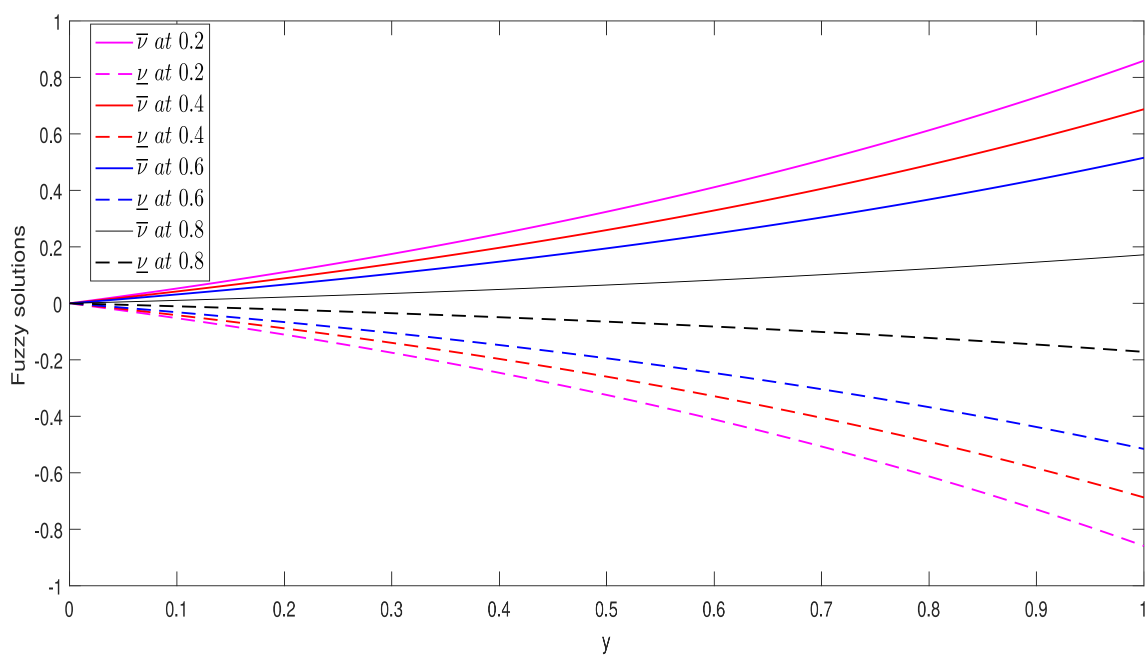

Fig. 3. Graphical presentation of fuzzy solutions at the given values of α_1 for Example 3.

$$\begin{cases} \underline{v}_{n+1}(y,\alpha_1) = L^{-1}\left[\frac{1}{s}L[\underline{v}_n(y,\alpha_1)]\right], \\ \overline{v}_{n+1}(y,\alpha_1) = L^{-1}\left[\frac{1}{s}L[\overline{v}_n(y,\alpha_1)]\right], \ n \geqslant 0. \end{cases} \tag{31}$$

Taking first lower limit solution of Eq. (27), and simplify

$$\begin{cases} \underline{v}_0(y,\alpha_1) = (\alpha_1 - 1), \\ \underline{v}_1(y,\alpha_1) = L^{-1}\left[\frac{1}{s}L[\underline{v}_0(y,\alpha_1)]\right] = (\alpha_1 - 1)\frac{y^2}{2!}, \\ \underline{v}_2(y,\alpha_1) = L^{-1}\left[\frac{1}{s}L[\underline{v}_1(y,\alpha_1)]\right] = (\alpha_1 - 1)\frac{y^3}{3!}, \\ \underline{v}_3(y,\alpha_1) = L^{-1}\left[\frac{1}{s}L[\underline{v}_2(y,\alpha_1)]\right] = (\alpha_1 - 1)\frac{y^4}{4!} \end{cases} \tag{32}$$

and so on. To find the upper limit solution of Eq. (27), so one has

$$\begin{cases} \overline{v}_0(y,\alpha_1) = (1 - \alpha_1), \\ \overline{v}_1(y,\alpha_1) = L^{-1}\left[\frac{1}{s}L[\overline{v}_0(y,\alpha_1)]\right] = (1 - \alpha_1)\frac{y^2}{2!}, \\ \overline{v}_2(y,\alpha_1) = L^{-1}\left[\frac{1}{s}L[\overline{v}_1(y,\alpha_1)]\right] = (1 - \alpha_1)\frac{y^3}{3!}, \\ \overline{v}_3(y,\alpha_1) = L^{-1}\left[\frac{1}{s}L[\overline{v}_2(y,\alpha_1)]\right] = (1 - \alpha_1)\frac{y^4}{4!}. \end{cases} \tag{33}$$

The remaining terms can be in similarly computed. Putting Eqs. (32) and (33) in Eq. (29) and simplify, we get

$$\begin{cases} \underline{v}(y,\alpha_1) = [\alpha_1 - 1](\sinh y + \cosh y - 1), \\ \overline{v}(y,\alpha_1) = [1 - \alpha_1](\sinh y + \cosh y - 1). \end{cases}$$

Further in more simplified form, we have

$$v(y,\alpha_1) = [\alpha_1 - 1, 1 - \alpha_1](\sinh y + \cosh y - 1).$$

Which is the exact solution of the give problem in closed form. Next in Fig. 3, we provide graphical presentation of fuzzy solutions at the given values of uncertainty α_1 as

Remark 1. Here we state that in the above test problems, Examples 1 and 2 have been solved numerically by least square method in Ameri and Nezhad (2017). The concerned method is purely a numerical procedure whose convergence is slower than LADM which rapidly converges to exact solution of the problem. Further the Example 3 has been solved in Salahshour and Allahviranloo (2013) by using fractional differential transform method (FDTM) and the same results were achieved as we have got by the proposed method. The one thing which makes LADM more papular is its simplicity in applications while FDTM is slightly complicated in using. Further the convergence rate of the proposed method is faster than differential FDTM (Naghipour and Manafian, 2015).

5. Conclusion

In this manuscript, we successfully handled series type analytical solutions to "fuzzy Volterra integral equations" corresponding to separable type kernels. For the required purposes we adopted "LADM" and developed two sequences of upper and lower limit solutions as general algorithm. Then we have tested our proposed scheme on three different problems. Also we stressed that the same solution may be computed in more easy way instead of using complicated method. From the results we concluded that LADM can be applied as a powerful tool in solving both linear and nonlinear problems of fuzzy integral equations. In future work, this method will be applied to investigate the solutions of fuzzy Fredholm and Volterra non-linear integral equations with different kind of crisp and fuzzy kernels.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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