King Saud University

Journal of King Saud University – Science

www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

New exact solutions of sixth-order thin-film equation

Wafaa M. Taha *, M.S.M. Noorani, I. Hashim

School of Mathematical Sciences, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia

Abstract The $\left(\frac{G'}{G}\right)$-expansion method is used for the first time to find traveling-wave solutions for the sixth-order thin-film equation, where related balance numbers are not the usual positive integers. New types of exact traveling-wave solutions, such as – solitary wave solutions, are obtained the sixth-order thin-film equation, when parameters are taken at special values.

© 2013 Production and hosting by Elsevier B.V. on behalf of King Saud University.

1. Introduction

Higher-order nonlinear partial differential equations have considerable attention, because of their interesting mathematical structures and surprising properties. One of the most famous examples is the sixth-order thin-film equation (Flitton and King, 2004).

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(u^n u_{xxxxx}), n > 0, \tag{1}$$

which appears in flow modeling, and describes the spread of thin viscous droplets under different driving forces. The sixth-order thin-film equation has recently become more interesting for obtaining exact analytical solutions to NLPDEs, equations from the sixth-order thin-film wave phenomena, by using appropriate techniques. Several important techniques have been developed such as the tanh-method (Krisnangkura et al., 2012), sine–cosine method (Shi et al., 2012), tanh–coth

method (Jabbari and Kheiri, 2010), exp–function method (Parand and Rad, 2012), homogeneous–balance method (Elboree, 2012a), Jacobi–elliptic function method (Honga and Lub, 2012), and first-integral method (Taghizadeh et al., 2012). All methods have limitations in their applications. In fact, no unified method can be used to handle all types of NLPDEs.

One of the most effective direct methods to develop the traveling-wave solution of NLPDEs is the $\left(\frac{G'}{G}\right)$-expansion method, which was first proposed by Wang et al. (2008). The $\left(\frac{G'}{G}\right)$-expansion method has been successfully applied to obtain the exact solution for a variety of NLPDEs (Kim and Sakthivel, 2010; Kilicman and Abazari, 2012; Ebadi et al., 2012a,b; Ayhan and Bekir, 2012; Malik et al., 2012; Elboree, 2012b; Jafari et al., 2013; Taha and Noorani, 2013; Taha et al., 2013). In this paper, the $\left(\frac{G'}{G}\right)$-expansion method is used to study the sixth-order thin-film equation in fluid mechanics for the first time. Exact traveling-wave solutions are obtained when the choice of parameters are taken at special values. Moreover, the solution obtained via this method is in good agreement with the previously obtained solutions of other researchers. Our main objective in this study is to apply the $\left(\frac{G'}{G}\right)$ method to provide the closed-form traveling-wave solutions of the sixth-order thin-film equation. To the best of our knowledge, our study is the first to apply the $\left(\frac{G'}{G}\right)$-expansion method to the sixth-order thin-film equation. In solving

* Corresponding author. Tel.: + 60 17 322 6126; fax: + 60 38 925 4519.
E-mail address: wafaa_y2005@yahoo.com (W.M. Taha).
Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

these equations, we find an instance where the related balance numbers are not the usual positive integers (see Zhang, 2009; Zayed and EL-Malky, 2011). New solitary wave solutions are also for appropriate parameters. We compare our solutions with the solutions previously obtained by Flitton and King (2004). The closed-form solution obtained via this method is in good agreement with the solutions reported in Flitton and King (2004).

Our paper is organized as follows. Section 2, provides the summary of the $\left(\frac{G'}{G}\right)$-expansion method. In Section 3, describes the applications of the $\left(\frac{G'}{G}\right)$-expansion method for the sixth-order thin-film equation. Finally, Section 4, concludes.

2. Summary of the $\left(\frac{G'}{G}\right)$-expansion method

In this section, we describe the $\left(\frac{G'}{G}\right)$-expansion method for finding the traveling-wave solutions of NLPDEs. Suppose that a nonlinear partial differential equation, in two independent variables, x and t, is given by the following

$$p(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \ldots) = 0, \tag{2}$$

where $u = u(x,t)$ is an unknown function, p is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which highest order derivatives and nonlinear terms are involved.

The summary of the $\left(\frac{G'}{G}\right)$-expansion method, can be presented in the following six steps:

Step 1:
To find the traveling-wave solutions of Eq. (2), we introduce a wave variable

$$u(x,t) = u(\zeta), \ \zeta = (x - ct), \tag{3}$$

where in the constant c is the wave velocity. By substituting Eq. (3) into Eq. (2), we obtain the following ordinary differential equations (ODEs) in ζ (which illustrate a principal advantage of a traveling-wave solution, i.e., a partial differential equation is reduced to an ODE).

$$p(u, cu', u', cu'', c^2 u'', u'', \ldots) = 0. \tag{4}$$

Step 2:
If necessary, we integrate Eq. (4) as many times as possible and set the constants of integration as zero for simplicity.

Step 3:
We suppose the solution of nonlinear partial differential equation can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as the following:

$$u(\zeta) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{5}$$

where $G = G(\zeta)$ satisfies the second-order linear ordinary differential equation

$$G''(\zeta) + \lambda G'(\zeta) + \mu G(\zeta) = 0, \tag{6}$$

$G' = \frac{dG}{d\zeta}$ and $G'' = \frac{d^2 G}{d\zeta^2}$; a_i, λ, and μ are real constants with $a_m \neq 0$. Here, the prime denotes the derivative with respect to ζ. By using the general solutions of Eq. (6), we obtain the following expression:

$$\left(\frac{G'}{G}\right) = \begin{cases} \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \left(\frac{c_1 \sinh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\} + c_2 \cosh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\}}{c_1 \cosh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\} + c_2 \sinh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\}}\right), & \lambda^2 - 4\mu > 0, \\ \frac{-\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \left(\frac{-c_1 \sin\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\} + c_2 \cos\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\}}{c_1 \cos\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\} + c_2 \sin\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\}}\right), & \lambda^2 - 4\mu < 0, \\ \left(\frac{c_2}{c_1+c_2\zeta}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu = 0. \end{cases} \tag{7}$$

The above results can be written in simplified forms as follows:

$$\left(\frac{G'}{G}\right) = \begin{cases} \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \tanh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\}, & \lambda^2 - 4\mu > 0, \\ \frac{-\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \tan\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\}, & \lambda^2 - 4\mu < 0, \\ \left(\frac{c_2}{c_1+c_2\zeta}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu = 0. \end{cases} \tag{8}$$

Step 4:
The positive integer m can be accomplished by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (4). If we define the degree of $u(\zeta)$ as $D[u(\zeta)] = m$, the degree of other expressions is defined by the following:

$$D\left[\frac{d^q u}{d\zeta^q}\right] = m + q,$$

$$D\left[u^r \left(\frac{d^q u}{d\zeta^q}\right)^s\right] = mr + s(q + m).$$

Therefore, we obtain the value of m in Eq. (5).

Step 5:
Substitute Eq. (5) into Eq. (4), and use the general solutions of Eq. (6), and collect all terms with the same order of $\left(\frac{G'}{G}\right)$ together. Setting each coefficient of this polynomial to zero yields a set of algebraic equations for a_i, c, λ, and μ.

Step 6:
Substitute a_i, c, λ, and μ obtained in Step 5 and the general solutions of Eq. (6) into Eq. (5). Depending on the sign of the discriminant $(\lambda^2 - 4\mu)$, we can obtain the explicit solutions of Eq. (2) immediately.

3. Application of the $\left(\frac{G'}{G}\right)$-expansion method

3.1. Sixth-order thin-film equation

To find the solitary wave solution of (1), we use the following transformations.

$$u(x, t) = u(\zeta), \zeta = (x - ct). \tag{9}$$

Substituting (9) into (1) is conducted in the following:

$$(u^n u'''')' - cu' = 0. \tag{10}$$

By integrating (10) with respect to ζ and setting the integration constant equal to zero, we obtain the following:

$$u^n u'''' - cu = 0. \tag{11}$$

According to the previous steps, by using the balancing procedure between $u^n u''''''$ and u, we obtain $m = \frac{-5}{n}$. Suppose that (10) has the following formal solution:

$$u(\zeta) = E\left(\frac{G'}{G}\right)^{\frac{-5}{n}}, \tag{12}$$

where E is the unknown constant that needs to be determined later.

By substituting (12) along with (6) into (11) and collecting all terms with the same order of $\left(\frac{G'}{G}\right)$, the left hand side of (11) is converted into a polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of the resulting polynomials to zero yields a set of algebraic equations for E, λ, c, and μ.

$$\left(\frac{G'}{G}\right)^{\frac{5(n-1)}{n}} : cn^5 + 3125E^n - 6250E^n n + 4372E^n n^2 - 1250E^n n^3 + 120E^n n^4 = 0,$$

$$\left(\frac{G'}{G}\right)^{\frac{n-5}{n}} : 110E^n n^4 \lambda^2 \mu + 6250E^n n \lambda^4 \mu + 125E^n \lambda^4 \mu n^3 + 12500E^n n \mu^3 + 80E^n n^4 \mu^2$$
$$- 1000E^n n^3 \mu^2 + 1250E^n \lambda^4 n^2 + 6250E^n n^2 \mu^2 - 125E^n \lambda^4 n^3 + 15625E^n \lambda^4 \mu$$
$$- 6250E^n \lambda^4 n + 93750E^n \lambda^2 \mu^2 + 31250E^n \mu^2 + 15625E^n \lambda^4 + 31250E^n \mu^3$$
$$+ 5E^n \lambda^4 n^4 + 37500E^n \lambda^2 n \mu^2 + 110E^n n^4 \mu^2 \lambda^2 + 1875E^n n^3 \lambda^2 \mu^2$$
$$- 37500E^n \lambda^2 \mu n - 1875E^n \lambda^2 \mu n^3 + 1250E^n \mu \lambda^4 n^2$$
$$+ 13125E^n \lambda^2 \mu^2 n^2 + 13125E^n \lambda^2 \mu n^2 = 0,$$

$$\left(\frac{G'}{G}\right)^{-\frac{5}{n}} : 3125E^n \lambda^5 + 93750E^n \lambda \mu^2 + 3750E^n \lambda^3 \mu n^2 + 11250E^n \lambda \mu^2 n^2$$
$$+ 80E^n n^4 \lambda \mu^2 + 10E^n n^4 \lambda^3 \mu + 62500E^n \lambda^3 \mu = 0,$$

$$\left(\frac{G'}{G}\right)^{\frac{4n-5}{n}} : -3500E^n \lambda n^3 + 300E^n n^4 \lambda + 15625E^n \lambda \mu^4 + 14375E^n \lambda n^2 - 25000E^n \lambda n$$
$$+ 3500E^n \lambda \mu^4 n^3 + 14375E^n \lambda \mu^4 n^2 + 25000E^n \lambda \mu^4 n = 0,$$

$$\left(\frac{G'}{G}\right)^{\frac{3n-5}{n}} : 16875E^n n^2 \lambda^2 - 2250E^n \mu n^3 + 250E^n n^4 \lambda^2 - 18750E^n \mu n - 37500E^n \lambda^2 n$$
$$+ 31250E^n \lambda^2 \mu^3 + 200E^n n^4 \mu + 9375E^n \mu n^2 + 18750E^n \mu^4 n^3 + 9375E^n n^2 \mu^4$$
$$+ 2250E^n n^3 \mu^4 + 200E^n n^4 \mu^4 - 3375E^n n^3 \lambda^2 + 15625E^n \mu^4 + 31250E^n \lambda^2$$
$$+ 16875E^n \lambda^2 \mu^3 n^2 + 3375E^n \lambda^2 \mu^3 n^3 + 37500E^n \lambda^2 \mu^3 n$$
$$+ 250E^n n^4 \lambda^2 \mu^3 = 0,$$

$$\left(\frac{G'}{G}\right)^{\frac{2n-5}{n}} : 31250E^n \lambda^3 \mu^2 - 1250E^n \lambda^3 n^3 + 62500E^n \lambda \mu^3 - 25000E^n \lambda \mu$$
$$+ 8125E^n \lambda^3 \mu^2 n^2 + 4000E^n \lambda \mu^3 n^3 + 1250E^n \lambda^3 \mu^2 n^3 - 4000E^n \lambda \mu n^3$$
$$+ 25000E^n \lambda^3 \mu^2 n + 20000E^n \lambda \mu n^2 - 50000E^n \lambda \mu n + 75E^n \lambda^3 \mu^2 n^4$$
$$+ 300E^n n^4 \lambda \mu + 50000E^n n \lambda \mu^3 + 300E^n n^4 \lambda \mu^3 + 20000E^n \lambda \mu^3 n^2 = 0,$$

$$\left(\frac{G'}{G}\right)^{\frac{-5(1+n)}{n}} : 120E^n n^4 \mu^5 + 4375E^n n^2 \mu^5 + 1250E^n n^3 \mu^5 + 6250E^n \mu^5 n$$
$$+ 3125E^n \mu^5 = 0.$$

By solving the above set of algebraic equations by Maple, we obtain the following:

$$\lambda = 0, \ \mu = 0, \ c$$
$$= -\frac{5E^n(4n-5)(3n-5)(2n-5)(n+5)}{n^5}, \tag{13}$$

where E and n are arbitrary constants.

Consequently, we obtain the exact traveling-wave solution of (1) as follows:

$$u(\zeta) = E\left(\frac{c_2}{c_1 + c_2 \zeta}\right)^{\frac{-5}{n}}, \tag{14}$$

where c_1 and c_2 are arbitrary constants $\zeta = x - ct$
$= x + \frac{5E^n(4n-5)(3n-5)(2n-5)(n+5)}{n^5} t$.

If we set $c_1 = 0$ and $c_2 = 1$ in (14), we obtain the solitary wave solution:

$$u(x,t) = u(\zeta) = E(\zeta)^{\frac{5}{n}}, \tag{15}$$

where ζ is the same as above. This solution is the exact same solution obtained by Flitton and King (2004):

$$h(x) = A_0(t)(a-x)^{\frac{5}{n}} as \ x \to a^-, \tag{16}$$

where $A_0(t) = \left(\frac{n^5 s}{-5(4n-5)(3n-5)(2n-5)(n+5)}\right)^{\frac{1}{n}}$ and $s > 0$.

If we integrate $c = s$ as in (13) into (16) and set $x = a + s(t) - z$, we obtain the same result.

4. Conclusion

The applications of the $\left(\frac{G'}{G}\right)$-expansion method are still limited in fluid mechanics and nonlinear evolution equations, where the balance numbers are not positive integers (see Zhang, 2009 and Zayed and EL-Malky, 2011). This paper presents a wider applicability for handling nonlinear sixth-order thin-film equations by using the $\left(\frac{G'}{G}\right)$-expansion method. In the general solution (14), we obtain the additional arbitrary constants c_1 and c_2. The special case of $c_1 = 0$ and $c_2 = 1$ reproduces the results of Flitton and King (2004) with an appropriate choice of c. The new type of exact traveling-wave solution obtained in this paper for the sixth-order thin-film equation will be of beneficial to future studies.

References

Ayhan, B., Bekir, A., 2012. The $\left(\frac{G'}{G}\right)$-expansion method for the nonlinear lattice equations. Communications in Nonlinear Science and Numerical Simulation 17, 3490–3498.

Ebadi, G., Yousefzadeh, N., Triki, H., Yildirim, A., 2012a. Envelope solitons, periodic wave and other solutions to Boussinesq–Burgers equation. Romanian Reports in Physics 64, 915–932.

Ebadi, G., Kara, A.H., Petkovic, M.D., Yildirim, A., Biswas, A., 2012b. Solitons and conserved quantities of the Ito equation. Proceedings of the Romanian Academy, Series A 13, 215–224.

Elboree, M.K., 2012a. New soliton solutions for a Kadomtsev–Petviashvili (KP) like equation coupled to a Schrödinger equation. Applied Mathematics and Computation 218, 5966–5973.

Elboree, M.K., 2012b. Hyperbolic and trigonometric solutions for some nonlinear evolution equations. Communications in Nonlinear Science and Numerical Simulation 17, 4085–4096.

Flitton, J.C., King, J.R., 2004. Moving-boundary and fixed-domain problems for a sixth-order thin-film equation. European Journal of Applied Mathematics 15, 713–754.

Honga, B., Lub, D., 2012. New Jacobi elliptic function-like solutions for the general KdV equation with variable coefficients. Mathematical and Computer Modelling 55, 1594–1600.

Jabbari, A., Kheiri, H., 2010. New exact travelling wave solutions for the Kawahara and modified Kawahara equations by using modified tanh–coth method. Acta Universitatis Apulensis 23, 21–38.

Jafari, H., Kadkhoda, N., Biswas, A., 2013. The $\left(\frac{G'}{G}\right)$-expansion method for solutions of evolution equations from isothermal magnetostatic atmospheres. Journal of King Saud University Science 25, 57–62.

Kilicman, A., Abazari, R., 2012. Travelling wave solutions of the Schrödinger–Boussinesq system. Abstract and Applied Analysis. Article ID 198398, 11 pages.

Kim, H., Sakthivel, R., 2010. Travelling wave solutions for time-delayed nonlinear evolution equations. Applied Mathematics Letters 23, 527–532.

Krisnangkura, M., Chinviriyasit, S., Chinviriyasit, W., 2012. Analytic study of the generalized Burgers–Huxley equation by hyperbolic tangent method. Applied Mathematics and Computation 218, 10843–10847.

Malik, A., Chand, F., Kumar, H., Mishra, S.C., 2012. Exact solutions of the Bogoyavlenskii equation using the multiple $\left(\frac{G'}{G}\right)$-expansion method. Computers and Mathematics with Applications 64, 2850–2859.

Parand, K., Rad, J.A., 2012. Exp–function method for some nonlinear pde's and a nonlinear ode's. Journal of King Saud University-Science 24, 1–10.

Shi, Q., Xiao, Q., Liu, X., 2012. Extended wave solutions for a nonlinear Klein–Gordon–Zakharov system. Applied Mathematics and Computation 218, 9922–9929.

Taghizadeh, N., Mirzazadeh, M., Tascan, F., 2012. The first-integral method applied to the Eckhaus equation. Applied Mathematics Letters 25, 798–802.

Taha, W.M., Noorani, M.S.M., 2013. Exact solutions of equation generated by the Jaulent–Miodek Hierarchy by $\left(\frac{G'}{G}\right)$-expansion method. Mathematical Problems in Engineering. Article ID 392830, 7 pages.

Taha, W.M., Noorani, M.S.M., Hashim, I., 2013. New application of the $\left(\frac{G'}{G}\right)$-expansion method for thin film Equations. Abstract and Applied Analysis. Article ID535138, 6 pages.

Wang, M., Li, X., Zhang, J., 2008. The $\left(\frac{G'}{G}\right)$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Physics Letters A 372, 417–423.

Zayed, E.M.E., EL-Malky, M.A.S., 2011. Exact solutions of nonlinear partial differential equations in mathematical physics using the $\left(\frac{G'}{G}\right)$-expansion method. Advances in Theoretical and Applied Mechanics 4, 91–100.

Zhang, H., 2009. New application of the $\left(\frac{G'}{G}\right)$-expansion method. Communications in Nonlinear Science and Numerical Simulation 14, 3220–3225.