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Exponential stabilization of swelling porous systems with thermoelastic damping

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ABSTRACT

We study two systems of swelling porous thermoelasticity with minimum damping. Employing the standard multiplier method, we stabilize the systems exponentially without imposing restrictions on the wave velocities of the systems. This result is contrary to those obtained for closely related systems (like Timoshenko and porous systems) with similar single damping. In such scenarios, the authors established that a single damping term is insufficient to exponentially stabilize the system unless the assumption of the equality of wave velocities is imposed.

1. Introduction

It is imperative to study the characteristics of engineering materials for energy conservation, safety, environmental sustainability, and the effective use of thermal insulation for mechanical and civil engineering applications (Thomas and Rees, 2009). Understanding the thermal properties of materials, such as thermal conductivity and diffusivity, could provide further details about their thermal performance under elastic loading conditions where a restrained strain will induce stresses. Several studies have addressed different methods to test these materials thermal related properties for engineering and energy conservation applications (Low et al., 2013; Sundberg, 1988; Upadhyay et al., 2011). It was established in Liu et al. (2020) and López-Acosta et al. (2021) that understanding the thermoelastic properties of the material enhances their effective utilization for foundation stability. Meanwhile, Ghorbani et al. (2017) proved that the presence of moisture could inversely affect the peak velocity under dynamic vibration at elevated temperatures. This could occur to the soil under structural foundations due to earthquakes which could induce more significant shear stress due to thermal changes of underline soil materials. Seasonal changes such as summer and winter could cause changes in the behavior of the substructure soil characteristics due to temperature gradient.

It is crucial to provide solutions towards understanding the complexity of swelling porous using mathematical formulations because it enables scientists and engineers to adequately comprehend the swelling soil's behavior under several loading and damping conditions. Mathematically, the fundamental system of equations governing the swelling porous thermoelastic soils (see Eringen, 1994; Quintanilla, 2002) is formulated as

$$
\begin{aligned}
&\rho_1 u_{tt} - a_1 u_{xx} - a_2 v_{xx} - \beta_1 \Theta_x + \xi(u_{xt} - v_{xt}) - \alpha u_{xxt} = 0,\\
&\rho_2 v_{tt} - \mu_1 v_{xx} - a_2 u_{xx} - \beta_2 \Theta_x - \xi(u_{xt} - v_{xt}) = 0,\\
&\rho \Theta_t - \mu_2 \Theta_{xx} - \beta_1 u_{xt} - \beta_2 v_{xt} = 0,
\end{aligned}
\tag{1.1}
$$

where the unknown variables $u, v, \Theta : (0, L) \times [0, \infty) \to \mathbb{R}$, respectively, symbolize the fluid displacement, the elastic solid material,

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and the temperature variation. The positive parameters ρ_1, ρ_2, designate the densities relate to z and u, respectively, and ρ denotes the heat capacity. The other parameters $\xi, \alpha, a_i, \mu_i, \beta_i, i = 1, 2$ are constitute constants satisfying

$$a_i > 0,\ \mu_i > 0,\ \beta_i > 0,\ \xi > 0,\ \alpha > 0,\ a_1\mu_1 > a_2^2.$$

Quintanilla (2002) considered (1.1) in the isothermal case, $\Theta = 0$, which necessitates that $\beta_1 = \beta_2 = 0$, that is

$$\begin{aligned} &\rho_1 u_{tt} - a_1 u_{xx} - a_2 v_{xx} + \xi(u_{xt} - v_{xt}) - \alpha u_{xxt} = 0, \\ &\rho_2 v_{tt} - \mu_1 v_{xx} - a_2 u_{xx} - \xi(u_{xt} - v_{xt}) = 0. \end{aligned} \tag{1.2}$$

Employing the standard energy method, he proved that system (1.2) decay exponentially. In addition, using Hurwitz theorem, he showed that system (1.2) with $\xi = 0$ also decay exponentially. It is obvious that system (1.1) in its general thermal case is exponentially stable. Further on the case $\xi = 0$, we mention the work of Wang and Guo (2006). They considered (1.2) with $\xi = 0$ and replaced the term $-\alpha u_{xxt}$ with $\alpha(x)u_t$, that is

$$\begin{aligned} &\rho_1 u_{tt} - a_1 u_{xx} - a_2 v_{xx} + \alpha(x)u_t = 0, \\ &\rho_2 v_{tt} - \mu_1 v_{xx} - a_2 u_{xx} = 0, \end{aligned} \tag{1.3}$$

and stabilized the system exponentially via the spectral method. Ramos et al. (2020) recently worked on system (1.2) with $\xi = \alpha = 0$ and a nonlinear feedback $\alpha(t)g(v_t)$ added to the second equation, that is

$$\begin{aligned} &\rho_1 u_{tt} - a_1 u_{xx} - a_2 v_{xx} = 0, \\ &\rho_2 v_{tt} - \mu_1 v_{xx} - a_2 u_{xx} + \alpha(t)g(v_t) = 0. \end{aligned} \tag{1.4}$$

They obtained an exponential stability result under the assumption of equality of wave velocities, that is

$$\frac{a_1}{\rho_1} = \frac{\mu_1}{\rho_2}. \tag{1.5}$$

Meanwhile, Apalara et al. (2021a) proved the same exponential stability result for system (1.4) regardless of assumption (1.5). In a similar work, Apalara (2020) investigated system (1.4) replacing the nonlinear feedback with a viscoelastic damping represented by $\int_0^t g(t-s)v_{xx}(s)ds$, that is,

$$\begin{aligned} &\rho_1 u_{tt} - a_1 u_{xx} - a_2 v_{xx} = 0, \\ &\rho_2 v_{tt} - \mu_1 v_{xx} - a_2 u_{xx} + \int_0^t g(t-s)v_{xx}(s)ds = 0, \end{aligned} \tag{1.6}$$

and achieved a general stability result without imposing the assumption of equality of the system's wave velocities. Readers are invited to consult (Feng et al., 2022; Quintanilla, 2004; Quintanilla, 2002; Ramos et al., 2021; Choucha et al., 2021; Ramos et al., 2022; Apalara et al., 2022a; Apalara et al., 2021b; Al-Mahdi et al., 2021; Al-Mahdi et al., 2022a; Al-Mahdi et al., 2022b; Al-Mahdi et al., 2022c; Youkana et al., 2022; Baibeche et al., 2022) and the references therein for more fascinating results.

In this article, we consider two independent swelling porous classical thermoelastic systems with minimum damping terms. The first system is

$$\begin{aligned} &\rho_1 u_{tt} - a_1 u_{xx} - a_2 v_{xx} = 0 && \text{in } (0,L)\times[0,\infty), \\ &\rho_2 v_{tt} - \mu_1 v_{xx} - a_2 u_{xx} - \beta_2\Theta_x = 0 && \text{in } (0,L)\times[0,\infty), \\ &\rho\Theta_t - \mu_2\Theta_{xx} - \beta_2 v_{xt} = 0 && \text{in } (0,L)\times[0,\infty) \end{aligned} \tag{1.7}$$

and the second system is

$$\begin{aligned} &\rho_1 u_{tt} - a_1 u_{xx} - a_2 v_{xx} - \beta_1\Theta_x = 0 && \text{in } (0,L)\times[0,\infty), \\ &\rho_2 v_{tt} - \mu_1 v_{xx} - a_2 u_{xx} = 0 && \text{in } (0,L)\times[0,\infty), \\ &\rho\Theta_t - \mu_2\Theta_{xx} - \beta_1 u_{xt} = 0 && \text{in } (0,L)\times[0,\infty). \end{aligned} \tag{1.8}$$

Each of the systems is supplemented with initial conditions:

$$\begin{aligned} &u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ v(x,0) = v_0(x), \\ &v_t(x,0) = v_1(x),\ \Theta(x,0) = \Theta_0(x), \qquad x \in (0,L) \end{aligned} \tag{1.9}$$

and boundary conditions:

$$\begin{aligned} &u(0,t) = u(L,t) = v(0,t) = v(L,t) = \Theta_x(0,t) = \Theta_x(L,t) = 0, \\ &t \in [0,\infty). \end{aligned} \tag{1.10}$$

In each case, we utilize the standard energy method (also known as the multiplier method) to demonstrate exponential stability results notwithstanding the system's wave velocities. The result is unexpected as opposed to the result obtained for similar systems like thermoelastic porous, where the exponential stability depends on the wave velocities. For example, in Casas and Quintanilla (2005), Casas and Quintanilla considered

$$\begin{aligned} &\rho u_{tt} - \mu_1 u_{xx} - b v_x + \beta\Theta_x = 0, \\ &J v_{tt} - \alpha v_{xx} + b u_x + \xi v - m\theta = 0, \\ &a\Theta_t - \mu_2\Theta_{xx} + \beta u_{xt} + m v_t = 0 \end{aligned} \tag{1.11}$$

and proved a slow decay of solutions. The same result was obtained by Pamplona et al. (2009) when γv_{xxt} was added to the fluid displacement equation in (1.11). However, Santos et al. (2019) extended the slow decay result obtained in Casas and Quintanilla (2005) to exponential stability under the equality of the system's wave velocities. Analogous result was achieved by Apalara in Apalara (2019). For the thermoelastic Timoshenko system, we mention the result of Rivera and Racke (2002), where they considered

$$\begin{aligned} &\rho_1 u_{tt} - \kappa(u_x + v)_x = 0, \\ &\rho_2 v_{tt} - b v_{xx} + \kappa(u_x + v) + \gamma\Theta_x = 0, \\ &\rho\Theta_t - \mu_2\Theta_{xx} + \gamma v_{xt} = 0 \end{aligned} \tag{1.12}$$

and demonstrated an exponential stability result owing to the assumption $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$. See (Júnior et al., 2014) for similar result. In all these systems, we see that the exponential stability results depend on the equality of wave velocities of the system. Interestingly, in our systems ((1.7) and (1.8)), we establish the exponential decay results without any restrictions on the wave velocities or any other relationship between the system's parameters. We must say that the result is similar to the result obtained for the thermoelastic Timoshenko system free of the second spectrum (Apalara et al., 2022b).

Using the boundary conditions (1.10), it follows from the last equation in (1.7)(which is also true for (1.8)) that

$$\frac{d}{dt}\left(\int_0^L \Theta(x,t)dx\right) = 0.$$

So, using the intial condition (1.9), we have $\int_0^L \Theta(x,t)dx = \int_0^L \Theta_0(x)dx$. By setting

$$\theta(x,t) = \Theta(x,t) - \frac{1}{L}\int_0^L \Theta_0(x)dx$$

we get $\int_0^L \theta(x,t)dx = 0$. Consequently, the usage of Poincaré's inequality is appropriate to the Neumann boundary conditions on Θ. Furthermore, it is obvious that $\theta_t = \Theta_t, \theta_x = \Theta_x$, and $\theta_{xx} = \Theta_{xx}$. Accordingly, systems (1.7), (1.9)–(1.10) and (1.8), (1.9)–(1.10), respectively, transform to

$$\begin{cases} \rho_1 u_{tt} - a_1 u_{xx} - a_2 v_{xx} = 0, \\ \rho_2 v_{tt} - \mu_1 v_{xx} - a_2 u_{xx} - \beta_2 \theta_x = 0, \\ \rho \theta_t - \mu_2 \theta_{xx} - \beta_2 v_{xt} = 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ v(x,0) = v_0(x), \\ v_t(x,0) = v_1(x),\ \theta(x,0) = \Theta_0(x) - \frac{1}{L}\int_0^L \Theta_0(x)dx, \\ u(0,t) = u(L,t) = v(0,t) = v(L,t) = \theta_x(0,t) = \theta_x(L,t) = 0 \end{cases} \tag{1.13}$$

and

$$\begin{cases} \rho_1 u_{tt} - a_1 u_{xx} - a_2 v_{xx} - \beta_1 \theta_x = 0, \\ \rho_2 v_{tt} - \mu_1 v_{xx} - a_2 u_{xx} = 0, \\ \rho \theta_t - \mu_2 \theta_{xx} - \beta_1 u_{xt} = 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ v(x,0) = v_0(x), \\ v_t(x,0) = v_1(x),\ \theta(x,0) = \Theta_0(x) - \frac{1}{L}\int_0^L \Theta_0(x)dx, \\ u(0,t) = u(L,t) = v(0,t) = v(L,t) = \theta_x(0,t) = \theta_x(L,t) = 0, \end{cases} \tag{1.14}$$

for $(x,t) \in (0,L) \times [0,\infty)$. Moving forward, we consider systems (1.13) and (1.14). The following is the outline of the remaining sections: In Section 2, we demonstrate an exponential stability result for (1.13). Section 3 deals with the proof of an exponential stability result for (1.14). We end the paper with conclusions in Section 4. Throughout this article, the letter c_p in the estimates denotes a Poincaré's constant.

2. Stability result for (1.7), (1.13)

Specific to our system, we have the following suitable Lyapunov functional

$$\mathscr{P}(t) := \mathscr{C}E(t) + \sum_{i=1}^{4} \mathscr{C}_i \mathscr{F}_i(t), \qquad \forall t \geq 0, \tag{2.1}$$

where $\mathscr{C} > 0$, $\mathscr{C}_i > 0, i = 1 \cdots 4$, are constants to be discreetly chosen later. $\mathscr{F}_i, i = 1 \cdots 4$, are auxiliary functionals given by

$$\mathscr{F}_1(t) := a_2 \rho_2 \int_0^L (v_t u - u_t v) dx, \tag{2.2}$$

$$\mathscr{F}_2(t) := \rho_2 \int_0^L v_t v dx - \frac{a_2}{a_1} \rho_1 \int_0^L u_t v dx, \tag{2.3}$$

$$\mathscr{F}_3(t) := -\rho \int_0^L v_t \int_0^x \theta(y) dy dx, \tag{2.4}$$

$$\mathscr{F}_4(t) := -\int_0^L u_t u dx, \tag{2.5}$$

and E represents the energy of the system (it is the same for both systems) defined by

$$E(t) = \frac{1}{2}\int_0^L \left[\rho_1 u_t^2 + a_1 u_x^2 + \rho_2 v_t^2 + \mu_1 v_x^2 + \rho \theta^2 + 2a_2 u_x v_x\right] dx, \qquad \forall t \geq 0. \tag{2.6}$$

The routine implementation of both Young's inequality as well as Cauchy–Schwarz inequality leads to the critical fact that the Lyapunov functional $\mathscr{P}$ is equivalent to the energy E provided that $\mathscr{C}$ is sufficiently large. Specifically, for large $\mathscr{C}$ and some constants $\ell_1,\ \ell_2 > 0$, we have

$$\ell_1 E(t) \leqslant \mathscr{P}(t) \leqslant \ell_2 E(t), \qquad \forall t \geq 0. \tag{2.7}$$

The next five lemmas capture the derivatives of the energy functional E and the estimate of the derivatives of functionals $\mathscr{F}_i, i = 1 \cdots 4$. Afterwards, we state and prove Theorem 2.6 which deals with the exponential stability result for system (1.13).

Lemma 2.1. *The energy E of system (1.13) defined by (2.6), satisfies*

$$E'(t) = -\mu_2 \int_0^L \theta_x^2 dx \leq 0, \qquad \forall t \geq 0. \tag{2.8}$$

Proof. By respectively multiplying $(1.13)_1$, $(1.13)_2$, and $(1.13)_3$ by u_t, v_t, and θ, then using integration by parts with respect to x on the product, we get

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^L u_t^2 dx + \frac{a_1}{2}\frac{d}{dt}\int_0^L u_x^2 dx + a_2 \frac{d}{dt}\int_0^L u_x v_x dx - a_2 \int_0^L u_x v_{xt} dx = 0, \quad \forall t \geq 0, \tag{2.9}$$

$$\frac{\rho_2}{2}\frac{d}{dt}\int_0^L v_t^2 dx + \frac{\mu_1}{2}\frac{d}{dt}\int_0^L v_x^2 dx + a_2 \int_0^L u_x v_{xt} dx + \beta_2 \int_0^L v_{xt}\theta dx = 0, \quad \forall t \geq 0, \tag{2.10}$$

$$\frac{\rho}{2}\frac{d}{dt}\int_0^L \theta^2 dx + \mu_2 \int_0^L \theta_x^2 dx - \beta_2 \int_0^L v_{xt}\theta dx = 0, \quad \forall t \geq 0. \tag{2.11}$$

Summing (2.9)–(2.11) and considering (2.6), yields (2.8). square

Lemma 2.2. *The functional $\mathscr{F}_1$, defined by (2.2), satisfies*

$$\mathscr{F}_1'(t) \leqslant -\frac{a_2^2}{2}\int_0^L u_x^2 dx + \left(\frac{a_2^2\rho_2}{\rho_1} + \left(\frac{a_1\rho_2}{\rho_1} - \mu_1\right)^2\right)\int_0^L v_x^2 dx + \beta_2^2 c_p \int_0^L \theta_x^2 dx, \qquad \forall t \geq 0. \tag{2.12}$$

Proof. Taking the derivative of (2.2) and using (1.13), we see that, for all $t \geq 0$,

$$\mathscr{F}_1'(t) = -a_2^2 \int_0^L u_x^2 dx + \frac{a_2^2\rho_2}{\rho_1}\int_0^L v_x^2 dx + a_2\left(\frac{a_1\rho_2}{\rho_1} - \mu_1\right)\int_0^L u_x v_x dx - a_2\beta_2 \int_0^L u_x \theta dx. \tag{2.13}$$

By first utilizing Young's inequality followed by Poincaré's inequality, we get,

$$a_2\left(\frac{a_1\rho_2}{\rho_1} - \mu_1\right)\int_0^L u_x v_x dx \leqslant \frac{a_2^2}{4}\int_0^L u_x^2 dx + \left(\frac{a_1\rho_2}{\rho_1} - \mu\right)^2 \int_0^L v_x^2 dx, \qquad \forall t \geq 0, \tag{2.14}$$

$$-a_2\beta_2 \int_0^L u_x \theta dx \leqslant \frac{a_2^2}{4}\int_0^L u_x^2 dx + \beta_2^2 c_p \int_0^L \theta_x^2 dx, \qquad \forall t \geq 0. \tag{2.15}$$

The substitution of estimates (2.14) and (2.15) into (2.13) gives (2.12). square

Lemma 2.3. *The functional $\mathscr{F}_2$, defined by (2.3), satisfies, for any $\varepsilon_1 > 0$ and some constant $m_0 > 0$,*

$$\mathscr{F}_2'(t) \leqslant -\frac{m_0}{2}\int_0^L v_x^2 dx + \varepsilon_1 \int_0^L u_t^2 dx + \left(\rho_2 + \frac{a_2^2\rho_1^2}{4a_1^2\varepsilon_1}\right)\int_0^L v_t^2 dx + \frac{\beta_2^2 c_p}{2m_0}\int_0^L \theta_x^2 dx, \qquad \forall t \geq 0. \tag{2.16}$$

Proof. Differentiating $\mathscr{F}_2$ and adapting $(1.13)_1$ and $(1.13)_2$ for terms u_{tt} and v_{tt}, respectively, we get

$$\mathscr{F}_2'(t) = -\left(\mu_1 - \frac{a_2^2}{a_1}\right)\int_0^L v_x^2 dx + \rho_2 \int_0^L v_t^2 dx - \frac{a_2\rho_1}{a_1}\int_0^L u_t v_t dx - \beta_2 \int_0^L v_x \theta dx, \qquad \forall t \geq 0. \tag{2.17}$$

Implementing Young's inequality for any $\sigma_1 > 0,\ \varepsilon_1 > 0$, followed by Poincaré's inequality, we achieve

$$-\frac{a_2\rho_1}{a_1}\int_0^L u_t v_t dx \leqslant \varepsilon_1\int_0^L u_t^2 dx + \frac{a_2^2\rho_1^2}{4a_1^2\varepsilon_1}\int_0^L v_t^2 dx, \tag{2.18}$$

$$-\beta_2\int_0^L v_x\theta dx \leqslant \sigma_1\int_0^L v_x^2 dx + \frac{\beta_2^2 c_p}{4\sigma_1}\int_0^L \theta_x^2 dx. \tag{2.19}$$

Direct substitution of (2.18) and (2.19) into (2.17) gives

$$\begin{aligned}\mathscr{F}'_2(t) \leqslant& -\left(\mu_1 - \frac{a_2^2}{a_1} - \sigma_1\right)\int_0^L v_x^2 dx + \varepsilon_1\int_0^L u_t^2 dx \\ &+ \left(\rho_2 + \frac{a_2^2\rho_1^2}{4a_1^2\varepsilon_1}\right)\int_0^L v_t^2 dx + \frac{\beta_2^2 c_p}{4\sigma_1}\int_0^L \theta_x^2 dx, \quad \forall t \geq 0.\end{aligned}$$

Using the fact that $a_1\mu_1 > a_2^2$, we have $m_0 = \mu_1 - \frac{a_2^2}{a_1} > 0$. So, by taking $\sigma_1 = \frac{m_0}{2}$, we end up with estimate (2.16). square

Lemma 2.4. *The functional $\mathscr{F}_3$, defined by (2.4), satisfies, for any $\varepsilon_2, \varepsilon_3 > 0$, the estimate*

$$\begin{aligned}\mathscr{F}'_3(t) \leqslant -& \tfrac{\beta_2}{2}\int_0^L v_t^2 dx + \varepsilon_2\int_0^L u_x^2 dx + \varepsilon_3\int_0^L v_x^2 dx \\ &+ \left(\tfrac{\rho\beta_2 c_p}{\rho_2} + \tfrac{\rho^2 a_2^2 c_p}{4\varepsilon_2\rho_2^2} + \tfrac{\rho^2\mu_1^2 c_p}{4\varepsilon_3\rho_2^2} + \tfrac{\mu_2^2}{2\beta_2}\right)\int_0^L \theta_x^2 dx.\end{aligned} \tag{2.20}$$

Proof. Employing (1.13), we obtain, for all $t \geq 0$,

$$\begin{aligned}\mathscr{F}'_3(t) = -& \beta_2\int_0^L v_t^2 dx + \tfrac{\rho\beta_2}{\rho_2}\int_0^L \theta^2 dx + \tfrac{\rho a_2}{\rho_2}\int_0^L u_x\theta dx \\ &+ \tfrac{\rho\mu_1}{\rho_2}\int_0^L v_x\theta dx - \mu_2\int_0^L v_t\theta_x dx.\end{aligned} \tag{2.21}$$

Similar to the proof of the previous two lemmas, by enforcing Young's inequality for any $\varepsilon_2, \varepsilon_3 > 0$, and Poincaré's inequality, the last three integrals of (2.21), give

$$\frac{\rho a_2}{\rho_2}\int_0^L u_x\theta dx \leqslant \varepsilon_2\int_0^L u_x^2 dx + \frac{\rho^2 a_2^2 c_p}{4\varepsilon_2\rho_2^2}\int_0^L \theta_x^2 dx, \tag{2.22}$$

$$\frac{\rho\mu_1}{\rho_2}\int_0^L v_x\theta dx \leqslant \varepsilon_3\int_0^L v_x^2 dx + \frac{\rho^2\mu_1^2 c_p}{4\varepsilon_3\rho_2^2}\int_0^L \theta_x^2 dx, \tag{2.23}$$

$$-\mu_2\int_0^L v_t\theta_x dx \leqslant \frac{\beta_2}{2}\int_0^L v_t^2 dx + \frac{\mu_2^2}{2\beta_2}\int_0^L \theta_x^2 dx. \tag{2.24}$$

The combination of (2.21)–(2.24) gives (2.20). square

Lemma 2.5. *The functional $\mathscr{F}_4$, defined by (2.5), satisfies*

$$\mathscr{F}'_4(t) \leqslant -\int_0^L u_t^2 dx + \frac{2a_1}{\rho_1}\int_0^L u_x^2 dx + \frac{\mu_1}{4\rho_1}\int_0^L v_x^2 dx, \qquad \forall t \geq 0. \tag{2.25}$$

Proof. Using (1.13), it is clear that

$$\mathscr{F}'_4(t) = -\int_0^L u_t^2 dx + \frac{a_1}{\rho_1}\int_0^L u_x^2 dx + \frac{a_2}{\rho_1}\int_0^L u_x v_x dx, \qquad \forall t \geq 0.$$

Using Young's inequality, we get

$$\begin{aligned}&\tfrac{a_2}{\rho_1}\int_0^L u_x v_x dx \leqslant \tfrac{a_1}{\rho_1}\int_0^L u_x^2 dx + \tfrac{a_2^2}{4a_1\rho_1}\int_0^L v_x^2 dx \\ \leqslant& \tfrac{a_1}{\rho_1}\int_0^L u_x^2 dx + \tfrac{\mu_1}{4\rho_1}\int_0^L v_x^2 dx \quad \text{using } a_1\mu_1 > a_2^2.\end{aligned}$$

Consequently, we end up with (2.25). square

Having achieved the necessary estimates, we now focus on the theorem which captures our first result.

Theorem 2.6. *The problem (1.13) is exponentially stable, that is, for some constants $k_0, k_1 > 0$, the energy of system (1.13) defined by (2.6) satisfies*

$$E(t) \leqslant k_0 e^{-k_1 t} \qquad \forall t \geq 0. \tag{2.26}$$

Proof. Recall the Lyapunov functional defined by (2.1), that is

$$\mathscr{P}(t) := \mathscr{C}E(t) + \sum_{i=1}^{4}\mathscr{C}_i\mathscr{F}_i(t), \qquad \forall t \geq 0,$$

and using the estimates (2.8), (2.12), (2.16)–(2.25), together with setting

$$\mathscr{C}_4 = 2, \quad \varepsilon_1 = \frac{1}{\mathscr{C}_2}, \qquad \varepsilon_2 = \frac{a_2^2\mathscr{C}_1}{4\mathscr{C}_3} \qquad \varepsilon_3 = \frac{m_0\mathscr{C}_2}{4\mathscr{C}_3},$$

we conclude that

$$\begin{aligned}\mathscr{P}'(t) \leqslant& -\int_0^L u_t^2 dx - \left[\tfrac{m_0}{4}\mathscr{C}_2 - \left(\tfrac{a_2^2\rho_2}{\rho_1} + \left(\tfrac{a_1\rho_2}{\rho_1} - \mu_1\right)^2\right)\mathscr{C}_1 - \tfrac{\mu_1}{2\rho_1}\right]\int_0^L v_x^2 dx \\ &-\left[\mu_2\mathscr{C} - \beta_2^2 c_p\mathscr{C}_1 - \tfrac{\beta_2^2 c_p}{2m_0}\mathscr{C}_2 - \left(\tfrac{\mu_2^2}{2\beta_2} + \tfrac{\rho\beta_2 c_p}{\rho_2} + \tfrac{\rho^2 c_p\mathscr{C}_3}{\rho_2^2\mathscr{C}_1} + \tfrac{\rho^2\mu_1^2 c_p\mathscr{C}_3}{m_0\rho_2^2\mathscr{C}_2}\right)\mathscr{C}_3\right]\int_0^L \theta_x^2 dx \\ &-\left[\tfrac{a_2^2}{4}\mathscr{C}_1 - \tfrac{4a_1}{\rho_1}\right]\int_0^L u_x^2 dx - \left[\tfrac{\beta_2}{2}\mathscr{C}_3 - \left(\rho_2 + \tfrac{a_2^2\rho_1^2}{4a_1^2}\right)\mathscr{C}_2\right]\int_0^L v_t^2 dx, \qquad \forall t \geq 0.\end{aligned}$$

By consecutively letting

$$\begin{cases}\mathscr{C}_1 = 4a_2^{-2}\left(\tfrac{4a_1}{\rho_1} + 1\right), \\ \mathscr{C}_2 = 4m_0^{-1}\left(\left(\tfrac{a_2^2\rho_2}{\rho_1} + \left(\tfrac{a_1\rho_2}{\rho_1} - \mu_1\right)^2\right)\mathscr{C}_1 + \tfrac{\mu_1}{2\rho_1} + 1\right), \\ \mathscr{C}_3 = 2\beta_2^{-1}\left(\left(\rho_2 + \tfrac{a_2^2\rho_1^2\mathscr{C}_2}{4a_1^2}\right)\mathscr{C}_2 + 1\right), \\ \mathscr{C} = \mu_2^{-1}\left(\beta_2^2 c_p\mathscr{C}_1 + \tfrac{\beta_2^2 c_p}{2m_0}\mathscr{C}_2 + \left(\tfrac{\mu_2^2}{2\beta_2} + \tfrac{\rho\beta_2 c_p}{\rho_2} + \tfrac{\rho^2 c_p\mathscr{C}_3}{\rho_2^2\mathscr{C}_1} + \tfrac{\rho^2\mu_1^2 c_p\mathscr{C}_3}{m_0\rho_2^2\mathscr{C}_2}\right)\mathscr{C}_3 + c_p\right)\end{cases}$$

and using Poincaré's inequality

$$-c_p\int_0^L \theta_x^2 dx \leqslant -\int_0^L \theta^2 dx, \qquad \forall t \geq 0,$$

it turns out that

$$\mathscr{P}'(t) \leqslant \quad -\int_0^L (u_t^2 + u_x^2 + v_t^2 + v_x^2 + \theta^2)dx, \qquad \forall t \geq 0. \tag{2.27}$$

Meanwhile, by recalling the energy functional defined by (2.6)

$$E(t) = \frac{1}{2}\int_0^L [\rho_1 u_t^2 + a_1 u_x^2 + \rho_2 v_t^2 + \mu_1 v_x^2 + \rho\theta^2 + 2a_2 u_x v_x]dx, \qquad \forall t \geq 0,$$

and using Young's inequality, we have, for some constant $c > 0$,

$$E(t) \leqslant c\int_0^L [u_t^2 + u_x^2 + v_t^2 + v_x^2 + \theta^2]dx, \qquad \forall t > 0$$

which implies

$$-\int_0^L [u_t^2 + u_x^2 + v_t^2 + v_x^2 + \theta^2]dx \leqslant -c_1 E(t), \qquad \forall t > 0, \tag{2.28}$$

where $c_1 = \frac{1}{c} > 0$. Consequently, by merging (2.27) and (2.28), we have

$$\mathscr{P}'(t) \leqslant -c_1 E(t).$$

Using (2.7), we get

$$\mathscr{P}'(t) \leqslant -\frac{c_1}{\ell_2}\mathscr{P}(t).$$

Solving the differential inequality, we arrive at

$$\mathscr{P}(t) \leqslant \mathscr{P}(0)\exp\left(-\frac{c_1}{\ell_2}\right).$$

Finally, using (2.7) once again, we obtain (2.26), with $k_0 = \frac{\ell_2}{\ell_1}E(0) > 0$ and $k_1 = \frac{c_1}{\ell_2} > 0$. Thus, we conclude the proof of Theorem 2.6. square

3. Stability result for (1.14)

Similar to Section 2, we specify the following Lyapunov functional

$$\mathscr{Q}(t) := \mathscr{M}E(t) + \sum_{i=1}^{4}\mathscr{M}_i\mathscr{G}_i(t), \qquad \forall t \geq 0, \tag{3.1}$$

where $\mathscr{M}$, $\mathscr{M}_i, i = 1\cdots 4$, are positive constants to be carefully selected subsequently. $\mathscr{G}_i, i = 1\cdots 4$, are auxiliary functionals given by, for all $t \geq 0$,

$$\mathscr{G}_1(t) := a_2\rho_1\int_0^L (u_t v - v_t u)dx, \tag{3.2}$$

$$\mathscr{G}_2(t) := \rho_1\int_0^L u_t u dx - \frac{a_2}{\mu_1}\rho_2\int_0^L v_t u dx, \tag{3.3}$$

$$\mathscr{G}_3(t) := -\rho\int_0^L u_t\int_0^x \theta(y)dydx, \tag{3.4}$$

$$\mathscr{G}_4(t) := -\int_0^L v_t v dx \tag{3.5}$$

and E remains the same as in Section 2. Using (1.14), the derivative of the functionals satisfy, for all $t \geq 0$,

$$\mathscr{G}_1'(t) \leqslant -\frac{a_2^2}{2}\int_0^L v_x^2dx + k\int_0^L u_x^2dx + kc_p\int_0^L \theta_x^2dx, \tag{3.6}$$

$$\mathscr{G}_2'(t) \leqslant -\frac{n_0}{2}\int_0^L u_x^2dx + \epsilon_1\int_0^L v_t^2dx + k\left(1+\frac{1}{\epsilon_1}\right)\int_0^L u_t^2dx + k\int_0^L \theta_x^2dx, \tag{3.7}$$

$$\mathscr{G}_3'(t) \leqslant -\frac{\beta_1}{2}\int_0^L u_t^2dx + \epsilon_2\int_0^L u_x^2dx + \epsilon_3\int_0^L v_x^2dx + k\left(1+\frac{1}{\epsilon_2}+\frac{1}{\epsilon_3}\right)\int_0^L \theta_x^2dx, \tag{3.8}$$

$$\mathscr{G}_4'(t) \leqslant -\int_0^L v_t^2dx + k\int_0^L v_x^2dx + k\int_0^L u_x^2dx, \tag{3.9}$$

where k represents a generic positive constant. Since $a_1\mu_1 > a_2^2$, we have $n_0 = a_1 - \frac{a_2^2}{\mu_1} > 0$. The main thorem of this section is:

Theorem 3.1. *The problem (1.14) is exponentially stable, that is, for some constants $\kappa_0, \kappa_1 > 0$, the energy of system (1.14) defined by (2.6) satisfies*

$$E(t) \leqslant \kappa_0 e^{-\kappa_1 t} \qquad \forall t \geq 0. \tag{3.10}$$

Proof. Using the Lyapunov functional defined by (3.1), the estimates (2.8), (3.6)–(3.9), and setting

$$\mathscr{M}_4 = 2, \quad \epsilon_1 = \frac{1}{\mathscr{M}_2}, \quad \epsilon_2 = \frac{n_0\mathscr{M}_2}{4\mathscr{M}_3}, \quad \epsilon_3 = \frac{a_2^2\mathscr{M}_1}{4\mathscr{M}_3},$$

we end up with

$$\begin{aligned}\mathscr{Q}'(t) \leqslant & -\left[\tfrac{\beta_1}{2}\mathscr{M}_3 - c\mathscr{M}_2(1+\mathscr{M}_2)\right]\int_0^L u_t^2dx \\ & -\left[\tfrac{n_0}{4}\mathscr{M}_2 - c\mathscr{M}_1 - c\right]\int_0^L u_x^2dx - \int_0^L v_t^2dx \\ & -\left[\tfrac{a_2^2}{4}\mathscr{M}_1 - c\right]\int_0^L v_x^2dx \\ & -\left[k\mathscr{M} - c\mathscr{M}_1 - c\mathscr{M}_2 - c\mathscr{M}_3\left(1+\tfrac{\mathscr{M}_1}{\mathscr{M}_3}+\tfrac{\mathscr{M}_2}{\mathscr{M}_3}\right)\right]\int_0^L \theta_x^2dx.\end{aligned}$$

Emulating the remaining steps in the proof of Theorem 2.6 yields (3.10). This marks the conclusion of our results. square

4. Conclusions

In this paper, we use the energy (also known as multiplier) method to achieve exponential decay results for two independent swelling porous thermoelastic systems where the heat conduction is controlled by the classical Fourier law. The results are obtained without imposing the well-known restrictions on the wave velocities or any other relationship between the coefficients of the system. So, our present results substantially contribute to the theory related to the asymptotic behaviors of swelling porous elastic media and improve earlier endeavors in the literature.

Availability of Data

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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Authors' contributions

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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